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AN EXTENSION OF THE CHEN–BEURLING–HELSON–LOWDENSLAGER THEOREM

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Abstract. Yanni Chen [3] extended the classical Beurling-Helson-Lowdenslager theorem for Hardy spaces on the unit circle $\mathbb{T}$ defined in terms of continuous gauge norms on L^∞ that dominate $\|\cdot\|_1$. We extend Chen's result to a much larger class of continuous gauge norms. A key ingredient is our result that if α is a continuous normalized gauge norm on L^∞, then there is a probability measure λ, mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$, such that $\alpha \geqslant c\|\cdot\|_{1,\lambda}$ for some $0 < c \leqslant 1$.

1. Introduction

Let $\mathbb{T}$ be the unit circle, i.e., $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and let μ be Haar measure (i.e., normalized arc length) on $\mathbb{T}$. The classical and influential Beurling-Helson-Lowdenslager theorem (see [1], [7]) states that if W is a closed $H^\infty(\mathbb{T},\mu)$-invariant subspace (or, equivalently, $zW \subseteq W$) of $L^2(\mathbb{T},\mu)$, then $W = \varphi H^2$ for some $\varphi \in L^\infty(\mathbb{T},\mu)$, with $|\varphi| = 1$ a.e.(μ) or $W = \chi_E L^2(\mathbb{T},\mu)$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^2(\mathbb{T},\mu)$, then $W = \varphi H^2(\mathbb{T},\mu)$ for some $\varphi \in H^\infty(\mathbb{T},\mu)$ with $|\varphi| = 1$ a.e. (μ). Later, the Beurling's theorem was extended to $L^p(\mathbb{T},\mu)$ and $H^p(\mathbb{T},\mu)$ with $1 \leqslant p \leqslant \infty$, with the assumption that W is weak*-closed when $p = \infty$ (see [5], [6], [7], [8]). In [3], Yanni Chen extended the Helson-Lowdenslager-Beurling theorem for all continuous $\|\cdot\|_{1,\mu}$-dominating normalized gauge norms on $\mathbb{T}$.

In this paper we extend the Helson-Lowdenslager-Beurling theorem for a much larger class of norms. We first extend Chen's results to the case of $c\|\cdot\|_{1,\mu}$-dominating continuous gauge norms. We then prove that for any continuous gauge norm α, there is a probability measure λ that is mutually absolutely continuous with respect to μ such that α is $c\|\cdot\|_{1,\lambda}$-dominating. We use this result to extend Chen's theorem. Our extension depends on Radon-Nikodym derivative $d\lambda/d\mu$. In particular, Chen's theorem extends exactly whenever $\log(d\lambda/d\mu) \in L^1(\mathbb{T},\mu)$.

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2. Continuous gauge norms on Ω

Suppose (Ω,Σ,ν) is a probability space. A norm α on $L^{\infty}(\Omega,\nu)$ is a *normalized gauge norm* if

1. $\alpha(1)=1$,
2. $\alpha(|f|)=\alpha(f)$ for every $f\in L^{\infty}(\Omega,\nu)$.

In addition we say α is *continuous* (ν*-continuous*) if

$$\lim_{\nu(E)\to 0}\alpha(\chi_E)=0,$$

that is, whenever $\{E_n\}$ is a sequence in Σ and $\nu(E_n)\to 0$, we have $\alpha(\chi_{E_n})\to 0$.

We say that a *normalized gauge norm* α is $c\|\cdot\|_{1,\nu}$*-dominating* for some $c>0$ if

$$\alpha(f)\geqslant c\|f\|_{1,\nu},\ \text{for every } f\in L^{\infty}(\Omega,\nu).$$

It is easily to see the following fact that

(1) The common norm $\|\cdot\|_{p,\nu}$ is a α norm for $1\leqslant p\leqslant\infty$.

(2) If ν and λ are mutually absolutely continuous probability measures, then $L^{\infty}(\Omega,\nu)=L^{\infty}(\Omega,\lambda)$ and a normalized gauge norm is ν-continuous if and only if it is λ-continuous.

We can extend the normalized gauge norm α from $L^{\infty}(\Omega,\nu)$ to the set of all measurable functions, and define α for all measurable functions f on Ω by

$$\alpha(f)=\sup\{\alpha(s): s \text{ is a simple function },0\leqslant s\leqslant|f|\}.$$

It is clear that $\alpha(f)=\alpha(|f|)$ still holds.

Define

$$\mathscr{L}^{\alpha}(\Omega,\nu)=\{f: f \text{ is a measurable function on } \Omega \text{ with } \alpha(f)<\infty\},$$

$$L^{\alpha}(\Omega,\nu)=\overline{L^{\infty}(\nu)}^{\alpha},\ \text{i.e., the } \alpha\text{-closure of } L^{\infty}(\nu) \text{ in } \mathscr{L}^{\alpha}.$$

Since $L^{\infty}(\Omega,\nu)$ with the norm α is dense in $L^{\alpha}(\Omega,\nu)$, they have the same dual spaces. We prove in the next lemma that the normed dual $(L^{\alpha}(\Omega,\nu),\alpha)^{\#}=(L^{\infty}(\Omega,\nu),\alpha)^{\#}$ can be viewed as a vector subspace of $L^{1}(\Omega,\nu)$. Suppose $w\in L^{1}(\Omega,\nu)$, we define the functional $\varphi_w: L^{\infty}(\Omega,\nu)\to\mathbb{C}$ by

$$\varphi_w(f)=\int_{\Omega} fwd\nu.$$

LEMMA 2.1. *Suppose (Ω,Σ,ν) is a probability space and α is a continuous normalized gauge norm on $L^{\infty}(\Omega,\nu)$. Then*

(1) if $\varphi: L^{\infty}(\Omega,\nu)\to\mathbb{C}$ is an α-continuous linear functional, then there is a $w\in L^{1}(\Omega,\nu)$ such that $\varphi=\varphi_w$,

(2) if φ_w is α-continuous on $L^{\infty}(\Omega,\nu)$, then

(a) $\|w\|_{1,\mu} \leqslant \|\varphi_w\| = \left\|\varphi_{|w|}\right\|$,

(b) given φ *in the dual of* $L^\alpha(\Omega,\lambda)$, *i.e.,* $\varphi \in (L^\alpha(\Omega,\lambda))^\#$, *there exists a* $w \in L^1(\Omega,\lambda)$, *such that*

$$\forall f \in L^\infty(\Omega,\lambda), \quad \varphi(f) = \int_\Omega fwd\lambda \quad \text{and} \quad wL^\alpha(\Omega,\lambda) \subseteq L^1(\Omega,\lambda).$$

Proof. (1) If α is continuous, it follows that, whenever $\{E_n\}$ is a disjoint sequence of measurable sets,

$$\lim_{N\to\infty} \alpha\left(\chi_{\cup_{n=1}^\infty E_n} - \sum_{k=1}^N \chi_{E_k}\right) = \lim_{N\to\infty} \alpha\left(\chi_{\cup_{k=N+1}^\infty E_k}\right) = 0,$$

since $\lim_{N\to\infty} \nu\left(\cup_{k=N+1}^\infty E_k\right) = 0$. It follows that

$$\rho(E) = \varphi(\chi_E)$$

defines a measure ρ and $\rho \ll \nu$. It follows that if $w = d\rho/d\nu$, then

$$\begin{aligned}\|w\|_{1,\nu} &= \sup\left\{\left|\int_\Omega wsd\nu\right| : s \text{ is simple, } \|s\|_\infty \leqslant 1\right\}\\ &= \sup\{|\varphi(s)| : s \text{ simple, } \|s\|_\infty \leqslant 1\} \leqslant \|\varphi\|.\end{aligned}$$

Hence $w \in L^1(\Omega,\nu)$. Also, since, for every $f \in L^\infty(\Omega,\nu)$

$$|\varphi(f)| \leqslant \|\varphi\|\,\alpha(f) \leqslant \|\varphi\|\,\|f\|_\infty,$$

we see that φ is $\|\cdot\|_\infty$-continuous on $L^\infty(\Omega,\nu)$, so it follows that $\varphi = \varphi_w$.

(2a) From (1) we will see $\|w\|_{1,\nu} \leqslant \|\varphi\|$.

(2b) For any measurable set $E \subseteq \Omega$, and for all $\varphi \in (L^\alpha(\lambda))^\#$, define $\rho(E) = \varphi(\chi_E)$. we can prove ρ is a measure as in Theorem 2.2, and $\rho \ll \lambda$. By Radon-Nikodym theorem, there exists a function $w \in L^1(\lambda)$ such that, for every measurable set $E \subseteq \Omega$, $\varphi(\chi_E) = \rho(E) = \int_\Omega \chi_E wd\lambda$. Thus $\forall f \in L^\infty(\Omega,\lambda)$, $\varphi(f) = \int_\Omega fwd\lambda = \int_\Omega fwgd\mu = \int_\Omega fw|h|d\mu = \int_\Omega fwuhd\mu = \int_\Omega f\widetilde{w}hd\mu$, where $\widetilde{w} = wu, |\widetilde{w}| = |w|$, here $\widetilde{w} \in L^1(\Omega,\lambda)$ and g,h as in Theorem 2.2, so $\widetilde{w}h \in L^1(\mu)$. Therefore, $\varphi(f) = \int_\Omega f\widetilde{w}hd\mu$ for all $f \in L^\alpha(\Omega,\lambda)$.

Suppose $f \in L^\alpha(\Omega,\lambda)$, $f = u|f|$, $|u| = 1$. $|f| \in L^\alpha(\Omega,\lambda)$. There exists an increasing positive sequence s_n such that $s_n \to |f|$ a.e. (μ), thus $us_n \to u|f|$ a.e. (μ). $\forall w \in L^1(\Omega,\lambda)$, $w = v|w|$, where $|v| = 1$, so we have $\overline{v}s_n \to \overline{v}|f|$ a.e. (μ), where $\overline{v}$ is the conjugate of v and $\alpha(\overline{v}s_n - \overline{v}|f|) \to 0$. Thus $\varphi(\overline{v}s_n) \to \varphi(\overline{v}|f|)$. On the other hand, we also have $\varphi(\overline{v}s_n) = \int_\Omega \overline{v}s_n wd\lambda \to \int_\Omega \overline{v}|f|wd\lambda = \int_\Omega |f||w|d\lambda$ by monotone convergence theorem. Thus $\int_\Omega |f||w|d\lambda = \int_\Omega |f|\overline{v}wd\lambda = \varphi(\overline{v}|f|) < \infty$, therefore $fw \in L^1(\Omega,\lambda)$, i.e., $wL^\alpha(\Omega,\lambda) \subseteq L^1(\Omega,\lambda)$, where $w \in L^1(\Omega,\lambda)$. $\square$

THEOREM 2.2. *Suppose* (Ω,Σ,ν) *is a probability space,* α *is a continuous normalized gauge norm on* $L^\infty(\Omega,\nu)$ *and* $\varepsilon > 0$. *Then there exists a constant* c *with* $1-\varepsilon < c \leqslant 1$ *and a probability measure* λ *on* Σ *that is mutually absolutely continuous with respect to* ν *such that* α *is* $c\|\cdot\|_{1,\lambda}$*-dominating.*

Proof. Let $M=\{\nu\left(h^{-1}((0,\infty))\right): h\in L^1(\Omega,\nu), h\geqslant 0, \varphi_h$ is α-continuous$\}$. It follows from Lemma 2.1 that $M\neq\varnothing$. Choose $\{h_n\}$ in $L^1(\Omega,\nu)$ such that $h_n\geqslant 0$, φ_{h_n} is α-continuous, and such that

$$\nu\left(h_n^{-1}((0,\infty))\right)\to\sup M.$$

Let

$$h_0=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\|\varphi_{h_n}\|}h_n.$$

Since $\|h_n\|_{1,\nu}\leqslant\|\varphi_{h_n}\|$, we see that $\|h_0\|_{1,\nu}\leqslant 1$. Also

$$\varphi_{h_0}=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\|\varphi_{h_n}\|}\varphi_{h_n},$$

so φ_{h_0} is α-bounded and $\left\|\varphi_{h_0}\right\|\leqslant 1$. On the other hand $h_n^{-1}((0,\infty))\subset h_0^{-1}((0,\infty))$ for $n\geqslant 1$, so we have

$$\nu\left(h_0^{-1}((0,\infty))\right)=\sup M.$$

Let $E=\Omega\setminus h_0^{-1}((0,\infty))$ and assume, via contradiction, that $\nu(E)>0$. Then $\alpha(\chi_E)>0$. Hence, by the Hahn-Banach theorem, there is a $g\in L^1(\Omega,\nu)$ such that $\left\|\varphi_g\right\|=1$ and

$$\alpha(\chi_E)=\varphi_g(\chi_E)=\int_\Omega g\chi_E d\nu=\varphi_{g\chi_E}(\chi_E)\leqslant\varphi_{|g|\chi_E}(\chi_E).$$

It follows that $\nu\left(\left(|g|\chi_E\right)^{-1}(0,\infty)\right)=\eta>0$, and that if $h_1=h_0+|g|\chi_E$, then

$$\sup M\geqslant\nu\left(h_1^{-1}((0,\infty))\right)=\nu\left(h^{-1}((0,\infty))\right)+\eta=\sup M+\eta.$$

This contradiction shows that $\nu(E)=0$, so we can assume that $h_0(\omega)>0$ a.e. (ν). By replacing h_0 with $h_0/\int_\Omega h_0 d\nu$, we can assume that $\int_\Omega h_0 d\nu=1$.

If we define a probability measure $\lambda:\Sigma\to[0,1]$ by

$$\lambda(E)=\int_E h_0 d\nu,$$

then λ is a measure, $\lambda\ll\nu$ and $\nu\ll\lambda$ since $0<h_0$ a.e. (ν). Also, we have for every $f\in L^\infty(\Omega,\nu)$,

$$\|f\|_{1,\lambda}=\int_\Omega|f|\,d\lambda=\int_\Omega|f|\,h_0 d\nu=\varphi_{h_0}(|f|)\leqslant\left\|\varphi_{h_0}\right\|\alpha(f).$$

Since $\varphi_{h_0}(1)=1$, we know $\left\|\varphi_{h_0}\right\|\geqslant 1$. Hence, $0<c_0=1/\left\|\varphi_{h_0}\right\|\leqslant 1$, and we see that α is $c_0\|\cdot\|_{1,\lambda}$-dominating on E. If we apply the Hahn-Banach theorem as above with $E=\Omega$, we can find a nonnegative function $k\in L^1(\Omega,\nu)$ such that

$$\|\varphi_k\|=1=\alpha(1)=\varphi_k(1)=\int_\Omega k1d\nu.$$

For $0<t<1$ let $h_t=(1-t)k+th_0$. Then $\varphi_{h_t}=(1-t)\varphi_k+t\varphi_{h_0}$. Thus

$$\lim_{t\to 0^+}\|\varphi_{h_t}\|=\|\varphi_k\|=1.$$

Choose t so that $\|\varphi_{h_t}\|<1/(1-\varepsilon)$, so $1-\varepsilon<c=1/\|\varphi_{h_t}\|\leqslant 1$. If we define a probability measure $\lambda_t:\Sigma\to[0,1]$ by

$$\lambda_t(E)=\int_E h_t d\nu,$$

we see that $\lambda_t\ll\mu\nu$ and since $h_t\geqslant th_0>0$, we see $\nu\ll\lambda_t$. As above we see, for every $f\in L^\infty(\Omega,\mu)$ we have

$$c\|f\|_{1,\lambda_t}\leqslant\frac{1}{\|\varphi_{h_t}\|}\int_\Omega|f|h_t d\nu=\frac{1}{\|\varphi_{h_t}\|}\varphi_{h_t}(|f|)\leqslant\alpha(f).$$

Therefore, α is $c\|\cdot\|_{1,\lambda_t}$-dominating on Ω. □

If we take $\Omega=\mathbb{T}$, Theorem 2.2 holds for the probability space $(\Omega,\nu)=(\mathbb{T},\mu)$. The L^p-version of the Helson-Lowdenslager theorem also holds, in a sense, on the circle $\mathbb{T}$ when μ is replaced with a mutually absolutely continuous probability measure λ. Here the role of $H^p(\mathbb{T},\lambda)$ is replaced with $\left(1/g^{\frac{1}{p}}\right)H^p(\mathbb{T},\mu)$. This result is well-known, we include a proof for completeness as the following corollary.

COROLLARY 2.3. *Suppose λ is a probability measure on $\mathbb{T}$ and $\mu\ll\lambda$ and $\lambda\ll\mu$. Let $g=d\lambda/d\mu$ and suppose $1\leqslant p<\infty$. Suppose W is a closed subspace of $L^p(\mathbb{T},\lambda)$, and $zW\subset W$. Then $g^{\frac{1}{p}}W=\chi_E L^1(\mathbb{T},\mu)$ for some Borel subset E of $\mathbb{T}$ or $g^{\frac{1}{p}}W=\varphi H^p(\mathbb{T},\mu)$ for some unimodular function φ.*

Proof. Define $U:L^p(\mathbb{T},\lambda)\longrightarrow L^p(\mathbb{T},\mu)$ by $Uf=fg^{\frac{1}{p}}$, for $f\in L^p(\mathbb{T},\lambda)$. Clearly U is a surjective isometry, since

$$\|Uf\|_{p,\mu}^p=\int_\mathbb{T}\left|fg^{\frac{1}{p}}\right|^p d\mu=\int_\mathbb{T}|f|^p g d\mu=\int_\mathbb{T}|f|^p d\lambda=\|f\|_{p,\lambda}.$$

Define

$$M_{z,\mu}:L^p(\mathbb{T},\mu)\longrightarrow L^p(\mathbb{T},\mu)\ \text{ by }\ M_{z,\mu}f=zf$$

and

$$M_{z,\lambda}:L^p(\mathbb{T},\lambda)\longrightarrow L^p(\mathbb{T},\lambda)\ \text{ by }\ M_{z,\lambda}f=zf.$$

Then

$$UM_{z,\lambda}f=U(zf)=g^{\frac{1}{p}}zf=zg^{\frac{1}{p}}f=M_{z,\mu}g^{\frac{1}{p}}f=M_{z,\mu}Uf,$$

so $UM_{z,\lambda}=M_{z,\mu}U$. It follows that W is a closed z-invariant subspace of $L^p(\mathbb{T},\lambda)$ if and only if $g^{\frac{1}{p}}W=U(W)$ is a z-invariant closed linear subspace of $L^p(\mathbb{T},\mu)$. The conclusion now follows from the classical Beurling theorem for $L^p(\mathbb{T},\mu)$. □

3. Continuous gauge norms on the unit circle

Suppose α is a continuous normalized gauge norm on $L^\infty(\mathbb{T},\mu)$, suppose that $c>0$ and λ is a probability measure on $\mathbb{T}$ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and such that α is $c\|\cdot\|_{1,\lambda}$-dominating. We let $g = d\lambda/d\mu$ and $g>0$. We consider two cases

(1) $\int |\log g|\, d\mu < \infty$,

(2) $\int |\log g|\, d\mu = \infty$.

We define $L^p(\mathbb{T},\lambda)$ to be the $\|\cdot\|_{p,\lambda}$-closure of $L^\infty(\mathbb{T},\lambda)$ and define $H^p(\mathbb{T},\lambda)$ to be $\|\cdot\|_{p,\lambda}$-closure of the polynomials for $1 \leqslant p < \infty$. Denote $L^\infty(\mathbb{T},\mu) = L^\infty(\mu)$, $L^p(\mathbb{T},\mu) = L^p(\mu)$ and $H^p(\mathbb{T},\mu) = H^p(\mu)$.

LEMMA 3.1. *The following are true:*

(1) $\int |\log g|\, d\mu < \infty \Leftrightarrow$ *there is an outer function* $h \in H^1(\mu)$ *with* $|h| = g$,

(2) $\int |\log g|\, d\mu = \infty \Leftrightarrow H^1(\lambda) = L^1(\lambda)$.

Proof. Clearly $H^1(\lambda)$ is a closed z-invariant subspace of $L^1(\lambda)$. Thus, by corollary 2.3, either $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ or $gH^1(\lambda) = \chi_E L^1(\mu)$ for some Borel set $E \subset \mathbb{T}$.

For (1), if $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ, and $0 < g \in gH^1(\lambda)$, then $0 \neq \overline{\varphi} g \in H^1(\mu)$ which implies $\log g = \log|\overline{\varphi} g| \in L^1(\mu)$. It is a standard fact that if $g > 0$ and $\log g$ are in $L^1(\mu)$, then there exists an outer function $h \in H^1(\mu)$ with the same modulus as g,(i.e., $|h| = g$). Therefore, (1) is proved by Lemma 3.2 in [3].

For (2), since $gH^1(\lambda) = \varphi H^1(\mu)$ if and only if $\int |\log g|\, d\mu < \infty$. Suppose $\int |\log g|\, d\mu = \infty$. Then $gH^1(\lambda) = \chi_E L^1(\mu)$. We have $g = \chi_E f$ for some $f \in L^1(\mu)$, which implies $\chi_E = 1$ since $g > 0$. Thus $gH^1(\lambda) = L^1(\mu) = gL^1(\mu)$, which implies $H^1(\lambda) = L^1(\lambda)$. Conversely, if $H^1(\lambda) = L^1(\lambda)$, then $gH^1(\lambda) = gL^1(\lambda) = L^1(\mu) = \chi_{\mathbb{T}} L^1(\mu)$, which means $gH^1(\lambda) \neq \varphi H^1(\mu)$, i.e., $\int |\log g|\, d\mu = \infty$. $\square$

There is an important characterization of outer functions in $H^1(\mu)$.

LEMMA 3.2. *A function* f *is an outer function in* $H^1(\mu)$ *if and only there is a real harmonic function* u *with harmonic conjugate* $\overline{u}$ *such that*

(1) $u \in L^1(\mu)$,

(2) $f = e^{u + i\overline{u}}$,

(3) $f \in L^1(\mu)$.

Through the remainder of following sections we assume

1. α is a continuous normalized gauge norm on $L^\infty(\mu)$.
2. and that $c > 0$ and λ is a probability measure on $\mathbb{T}$ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and such that α is $c\|\cdot\|_{1,\lambda}$-dominating.
3. $h \in H^1(\mu)$ is an outer function, η is unimodular and $\bar{\eta} h = g = d\lambda/d\mu$.

Since λ and μ are mutually absolutely continuous we have $L^\infty(\mu) = L^\infty(\lambda)$, $L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$, we will use L^∞ to denote $L^\infty(\mu)$ and $L^\infty(\lambda)$, use L^α to denote $L^\alpha(\mu)$ and $L^\alpha(\lambda)$, use H^α to denote $H^\alpha(\mu)$ and $H^\alpha(\lambda)$. It follows that $L^\alpha, L^\infty, H^\alpha$ do not depend on λ or μ. However, this notation slightly conflicts with the classical notation for $L^1(\mu) = L^{\|\cdot\|_{1,\mu}}$ or $H^1(\mu) = H^{\|\cdot\|_{1,\mu}}$, so we will add the measure to the notation when we are talking about L^p or H^p.

THEOREM 3.3. *We have $hL^1(\lambda) = L^1(\mu)$ and $hH^1(\lambda) = H^1(\mu)$.*

Proof. We know from our assumption (3) that $hL^1(\lambda) = g\eta L^1(\lambda) = gL^1(\lambda) = L^1(\mu)$. By Lemma 3.1(1), we have $gH^1(\lambda) = \eta H^1(\mu)$, so

$$hH^1(\lambda) = \eta g H^1(\lambda) = \eta\eta H^1(\mu) = H^1(\mu). \quad \square$$

COROLLARY 3.4. *$gH^1(\lambda) = \gamma H^1(\mu)$ for some unimodular $\gamma \Leftrightarrow \int_{\mathbb{T}} |\log g|\, d\mu < \infty$.*

Proof. Assume $gH^1(\lambda) = \gamma H^1(\mu)$, Since $1 \in H^1(\lambda), g \in gH^1(\lambda)$, $\exists \phi \in H^1(\mu)$ such that $g = \gamma\phi$. Since $\phi \in H^1(\mu), \phi = \psi h$, where ψ is an inner function and h is an outer function. Thus, $\int_{\mathbb{T}} |\log g|\, d\mu = \int_{\mathbb{T}} \log|g|\, d\mu = \int_{\mathbb{T}} \log|h|\, d\mu < \infty$, since h is an outer function.

Assume $\int_{\mathbb{T}} |\log g|\, d\mu < \infty, g$ and $\log g \in L^1(\mu), g > 0$. Thus there exists an outer function $h \in H^1(\mu)$, such that $|h| = |g| = g, |h| = \phi h, |\phi| = 1, g = \eta h$, Define $V : L^1(\lambda) \longrightarrow L^1(\mu)$ by $Vf = hf$, as in Theorem 3.3, we have $hH^1(\lambda) = H^1(\mu)$, so $gH^1(\lambda) = \eta h H^1(\lambda) = \eta H^1(\mu)$. Let $\gamma = \eta$, then $gH^1(\lambda) = \gamma H^1(\mu)$. $\square$

We now get a Helson-Lowdenslager theorem when $\alpha = \|\cdot\|_{p,\lambda}$ and $\log g \in L^1(\mu)$.

COROLLARY 3.5. *Suppose $1 \leqslant p < \infty$. If W is a closed subspace of $L^p(\lambda)$ and $zW \subseteq W$, then either $W = \gamma H^p(\lambda)$ for some unimodular function γ, or $W = \chi_E L^p(\lambda)$ for some Borel subset E of $\mathbb{T}$.*

The following theorem shows the relation between $H^\alpha, H^1(\lambda)$ and L^α. This result parallels a result of Y. Chen [3], which is a key ingredient in her proof of her general Beurling theorem. However, her result was for $H^1(\mu)$ instead of $H^1(\lambda)$.

THEOREM 3.6. *$H^\alpha = H^1(\lambda) \cap L^\alpha$.*

Proof. Since α is continuous $c\|\cdot\|_{1,\lambda}$-dominating, α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, thus

$$H^\alpha = \overline{H^\infty}^\alpha \subseteq \overline{H^\infty}^{\|\cdot\|_{1,\lambda}} = H^1(\lambda).$$

Also,

$$H^\alpha = \overline{H^\infty(\lambda)}^\alpha \subset \overline{L^\infty}^\alpha = L^\alpha.$$

Thus $H^{\alpha} \subseteq H^1(\lambda) \cap L^{\alpha}$.

Since α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, $H^1(\lambda) \cap L^{\alpha}$ is an α-closed subspace of L^{α}. Suppose $\varphi \in (L^{\alpha})^{\#}$ such that $\varphi|H^{\infty} = 0$. It follows from Lemma 2.1 that there is a $w \in L^1(\lambda)$ such that $wL^{\alpha} \subset L^1(\lambda)$ and such that, for every $f \in L^{\alpha}$,

$$\varphi(f) = \int f\bar{\eta}wd\lambda = \int fwhd\mu.$$

Since $wL^{\alpha} \subset L^1(\lambda)$, we know that $whL^{\alpha} \subset L^1(\mu)$. Since $\varphi|_{H^{\infty}} = 0$, we have

$$\int_{\mathbb{T}} z^n hwd\mu = \varphi(z^n) = 0$$

for every integer $n \geqslant 0$. Thus $hw \in H_0^1(\mu)$.

Now suppose $f \in H^1(\lambda) \cap L^{\alpha}$. Then $hf \in H^1(\mu)$. We know that every function in $H^1(\mu)$ has a unique inner-outer factorization. Thus we can write

$$hf = \gamma_1 h_1$$

with γ_1 inner and h_1 outer. Moreover, since $hw \in H_0^1(\mu)$, we can write

$$(hw)(z) = z\gamma_2(z)h_2(z)$$

with γ_2 inner and h_2 outer. By Lemma 3.2, we can find real harmonic functions $u, u_1, u_2 \in L^1(\mu)$ such that

$$h = e^{u+i\bar{u}}, \quad h_1 = e^{u_1+i\bar{u}_1}, \quad \text{and} \quad h_2 = e^{u_2+i\bar{u}_2}.$$

Thus

$$hfw = hfhw/h = z\gamma_1\gamma_2 e^{(u_1+u_2-u)+i(\bar{u}_1+\bar{u}_2-\bar{u})} \in H^1(\mu).$$

It follows from Lemma 3.2 that

$$\varphi(f) = \int_{\mathbb{T}} hfwd\mu = (hfw)(0) = 0.$$

Hence every continuous linear functional on L^{α} that annihilates H^{α} also annihilates $H^1(\lambda) \cap L^{\alpha}$. It follows from the Hahn-Banach theorem that $H^1(\lambda) \cap L^{\alpha} \subset H^{\alpha}$. $\square$

The following result is a factorization theorem for L^{α}.

THEOREM 3.7. *If $k \in L^{\infty}$, $k^{-1} \in L^{\alpha}$, then there is a unimodular function $u \in L^{\infty}$ and an outer function $s \in H^{\infty}$ such that $k = us$ and $s^{-1} \in H^{\alpha}$.*

Proof. Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Since $k^{-1} \in L^{\alpha} \subseteq L^1(\lambda)$, we know that $\|k\|_{\infty} > 0$. Thus $\log|k| \leqslant \log\|k\|_{\infty} \in \mathbb{R}$. Moreover, $k^{-1} \in L^{\alpha} \subseteq L^1(\lambda)$ implies $hk^{-1} \in L^1(\mu)$, so

$$\log|h| - \log|k| = \log\left(\left|hk^{-1}\right|\right) \leqslant \left|hk^{-1}\right|.$$

Hence

$$\log|h| - \left|hk^{-1}\right| \leqslant \log|k| \leqslant \log\|k\|_\infty,$$

and since $\log|h|$, $\left|hk^{-1}\right|$ and $\log\|k\|_\infty$ are in $L^1(\mu)$, we see that $\log|k| \in L^1(\mu)$. Therefore, by the first statement of Lemma 3.1, there is an outer function $s \in H^1(\mu)$ such that $|s| = |k|$. It follows that $s \in H^\infty$. Hence there is a unimodular function u such that $k = us$.

We also know that

$$\left|\log\left|hk^{-1}\right|\right| = |\log(|h|) - \log|k|| \leqslant |\log(|h|)| + |\log|k|| \in L^1(\mu),$$

so there exists an outer function $f \in H^1(\mu)$ such that $|k^{-1}h| = |f|$. Thus sf is outer in $H^1(\mu)$ and $|h| = |sf|$, so $h = e^{it}sf$ for some real number t. Since $H^1(\mu) = hH^1(\lambda)$, we see that there exists a function $f_1 \in H^1(\lambda)$ such that $hf_1 = f = h\left(e^{-it}s^{-1}\right)$. It follows that $s^{-1} = e^{it}f_1 \in H^1(\lambda)$. Also, $\left|s^{-1}\right| = \left|k^{-1}\right|$, so $s^{-1} \in L^\alpha$. It follows from Theorem 3.6 that $s^{-1} \in H^1(\lambda) \cap L^\alpha = H^\alpha$. $\square$

LEMMA 3.8. *If M is a closed subspace of L^α and $zM \subseteq M$, then $H^\infty M \subseteq M$.*

Proof. Suppose $\varphi \in (L^\alpha)^\#$ and $\varphi|_M = 0$. It follows from Lemma 2.1 that there is a $w \in L^1(\lambda)$ such that $wL^\alpha \subset L^1(\lambda)$ such that, for every $f \in L^\alpha$

$$\varphi(f) = \int_{\mathbb{T}} fw\bar{\eta}\,d\lambda = \int_T fwh\,d\mu.$$

Suppose $f \in M$. Then, for every integer $n \geqslant 0$, we have $z^n f \in M$, so

$$0 = \int_{\mathbb{T}} z^n fwh\,d\mu.$$

Since $fwh \in hL^1(\lambda) = L^1(\mu)$, it follows that $fwh \in H_0^1(\mu)$. Thus if $k \in H^\infty$, we have

$$0 = \int_{\mathbb{T}} kfwh\,d\mu = \varphi(kf).$$

Hence every $\varphi \in (L^\alpha)^\#$ that annihilates M must annihilate $H^\infty M$. It follows from the Hahn-Banach theorem that $H^\infty M \subset M$. $\square$

We let $\mathbb{B} = \{f \in L^\infty : \|f\|_\infty \leqslant 1\}$ denote the closed unit ball in $L^\infty(\lambda)$.

LEMMA 3.9. *Let α be a continuous norm on $L^\infty(\lambda)$, then*

(1) The α-topology, the $\|\cdot\|_{2,\lambda}$-topology, and the topology of convergence in λ-measure coincide on $\mathbb{B}$,

(2) $\mathbb{B} = \{f \in L^\infty(\lambda) : \|f\|_\infty \leqslant 1\}$ is α-closed.

Proof. For (1), since α is $c\|\cdot\|_{1,\lambda}$-dominating, α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, and $\|\cdot\|_{1,\lambda}$-convergence implies convergence in measure. Suppose $\{f_n\}$ is a sequence in $\mathbb{B}$, $f_n \to f$ in measure and $\varepsilon > 0$. If $E_n = \{z \in \mathbb{T} : |f(z) - f_n(z)| \geqslant \frac{\varepsilon}{2}\}$, then $\lim_{n\to\infty}\lambda(E_n)=0$. Since α is continuous, we have $\lim_{n\to\infty}\alpha(\chi_{E_n})=0$, which implies that

$$\begin{aligned}\alpha(f_n-f)&=\alpha((f-f_n)\chi_{E_n}+(f-f_n)\chi_{\mathbb{T}\setminus E_n})\\&\leqslant\alpha((f-f_n)\chi_{E_n})+\alpha((f-f_n)\chi_{\mathbb{T}\setminus E_n})\\&<\alpha((|f-f_n|)\chi_{E_n})+\frac{\varepsilon}{2}\leqslant\|f-f_n\|_\infty\alpha(\chi_{E_n})+\frac{\varepsilon}{2}\\&\leqslant 2\alpha(\chi_{E_n})+\frac{\varepsilon}{2}.\end{aligned}$$

Hence $\alpha(f_n-f)\to 0$ as $n\to\infty$. Therefore α-convergence is equivalent to convergence in measure on $\mathbb{B}$. Since α was arbitrary, letting $\alpha=\|\cdot\|_{2,\lambda}$, we see that $\|\cdot\|_{2,\lambda}$-convergence is also equivalent to convergence in measure. Therefore, the α-topology and the $\|\cdot\|_{2,\lambda}$-topology coincide on $\mathbb{B}$.

For (2), suppose $\{f_n\}$ is a sequence in $\mathbb{B}$, $f\in L^\alpha$ and $\alpha(f_n-f)\to 0$. Since $\|f\|_{1,\lambda}\leqslant\frac{1}{c}\alpha(f)$. it follows that $\|f_n-f\|_{1,\lambda}\to 0$, which implies that $f_n\to f$ in λ-measure. Then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}\to f$ a.e. (λ). Hence $f\in\mathbb{B}$. $\square$

The following theorem and its corollary relate the closed invariant subspaces of L^α to the weak*-closed invariant subspaces of L^∞.

THEOREM 3.10. *Let W be an α-closed linear subspace of L^α and M be a weak*-closed linear subspace of $L^\infty(\lambda)$ such that $zM\subseteq M$ and $zW\subseteq W$. Then*

(1) $M=\overline{M}^\alpha\cap L^\infty(\lambda)$,

(2) $W\cap L^\infty(\lambda)$ is weak-closed in $L^\infty(\lambda)$,*

(3) $W=\overline{W\cap L^\infty(\lambda)}^\alpha$.

Proof. For (1), it is clear that $M\subset\overline{M}^\alpha\cap L^\infty(\lambda)$. Assume, via contradiction, that $w\in\overline{M}^\alpha\cap L^\infty(\lambda)$ and $w\notin M$. Since M is weak*-closed, there is an $F\in L^1(\lambda)$ such that $\int_{\mathbb{T}}Fwd\lambda\neq 0$, but $\int_{\mathbb{T}}Frd\lambda=0$ for every $r\in M$. Since $k=\frac{1}{|F|+1}\in L^\infty(\lambda)$, $k^{-1}\in L^1(\lambda)$, it follows from Theorem 3.7, that there is an $s\in H^\infty(\lambda), s^{-1}\in H^1(\lambda)$ and a unimodular function u such that $k=us$. Choose a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\|s_n-s^{-1}\|_{1,\lambda}\to 0$. Since $sF=\bar{u}kF=\bar{u}\frac{F}{|F|+1}\in L^\infty(\lambda)$, we can conclude that $\|s_nsF-F\|_{1,\lambda}=\|s_nsF-s^{-1}sF\|_{1,\lambda}\leqslant\|s_n-s^{-1}\|_{1,\lambda}\|sF\|_\infty\to 0$. For each $n\in\mathbb{N}$. For every $r\in M$, from Lemma 3.8, we know that $s_nsr\in H^\infty(\lambda)M\subset M$. Hence

$$\int_{\mathbb{T}}rs_nsFd\lambda=\int_{\mathbb{T}}s_nsrFd\lambda=0,\forall r\in M.$$

Suppose $r\in\overline{M}^\alpha$. Then there is a sequence $\{r_m\}$ in M such that $\alpha(r_m-r)\to 0$ as $m\to\infty$. For each $n\in\mathbb{N}$, it follows from $s_nsF\in H^\infty(\lambda)L^\infty(\lambda)$ that

$$
\begin{aligned}
|\int_{\mathbb{T}} rs_n sF d\lambda - \int_{\mathbb{T}} r_m s_n sF d\lambda| &\leqslant \int_{\mathbb{T}} |(r-r_m)s_n sF| d\lambda \\
&\leqslant \|s_n sF\|_\infty \int_{\mathbb{T}} |r-r_m| d\lambda = \|s_n sF\|_\infty \|r-r_m\|_{1,\lambda} \\
&\leqslant \|s_n sF\|_\infty \alpha(r-r_m) \to 0.
\end{aligned}
$$

$$
\int_{\mathbb{T}} rs_n sF d\lambda = \lim_{m\to 0} \int_{\mathbb{T}} r_m s_n sF d\lambda = 0,\ \forall r \in \overline{M}^\alpha.
$$

In particular, $w \in \overline{M}^\alpha \cap L^\infty(\lambda)$ implies that

$$
\int_{\mathbb{T}} s_n sFw d\lambda = \int_{\mathbb{T}} ws_n sF d\lambda = 0.
$$

Hence,

$$
\begin{aligned}
0 \neq |\int_{\mathbb{T}} Fw d\lambda| &\leqslant \lim_{n\to\infty} |\int_{\mathbb{T}} Fw - s_n sFw d\lambda| + \lim_{n\to\infty} |\int_{\mathbb{T}} s_n sFw d\lambda| \\
&\leqslant \lim_{n\to\infty} \|F - s_n sF\|_{1,\lambda} \|w\|_\infty + 0 = 0.
\end{aligned}
$$

We get a contradiction. Hence $M = \overline{M}^\alpha \cap L^\infty(\lambda)$.

For (2), to prove $W \cap L^\infty(\lambda)$ is weak*-closed in $L^\infty(\lambda)$, using the Krein-Smulian theorem, we only need to show that $W \cap L^\infty(\lambda) \cap \mathbb{B}$, i.e., $W \cap \mathbb{B}$, is weak*-closed. By Lemma 3.9, $W \cap \mathbb{B}$ is α-closed. Since α is $c\|\cdot\|_{1,\lambda}$-dominating, it follows from the Lemma 3.9, $W \cap \mathbb{B}$ is $\|\cdot\|_{2,\lambda}$ closed. The fact that $W \cap \mathbb{B}$ is convex implies $W \cap \mathbb{B}$ is closed in the weak topology on $L^2(\lambda)$. If $\{f_\lambda\}$ is a net in $W \cap \mathbb{B}$ and $f_\lambda \to f$ weak* in $L^\infty(\lambda)$, then, for every $w \in L^1(\lambda)$, $\int_{\mathbb{T}} (f_\lambda - f) w d\lambda \to 0$. Since $L^2(\lambda) \subset L^1(\lambda)$, $f_\lambda \to f$ weakly in $L^2(\lambda)$, so $f \in W \cap \mathbb{B}$. Hence $W \cap \mathbb{B}$ is weak*-closed in $L^\infty(\lambda)$.

For (3), since W is α-closed in L^α, it is clear that $W \supset \overline{W \cap L^\infty(\lambda)}^\alpha$, suppose $f \in W$ and let $k = \frac{1}{|f|+1}$. Then $k \in L^\infty(\lambda)$, $k^{-1} \in L^\alpha$. It follows from Theorem 3.7 that there is an $s \in H^\infty(\lambda)$, $s^{-1} \in H^\alpha$ and an unimodular function u such that $k = us$, so $sf = \overline{u} kf = \overline{u} \frac{f}{|f|+1} \in L^\infty(\lambda)$. There is a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\alpha(s_n - s^{-1}) \to 0$. For each $n \in \mathbb{N}$, it follows from Lemma 3.8 that $s_n sf \in H^\infty(\lambda) H^\infty(\lambda) W \subset W$ and $s_n sf \in H^\infty(\lambda) L^\infty(\lambda) \subset L^\infty(\lambda)$, which implies that $\{s_n sf\}$ is a sequence in $W \cap L^\infty(\lambda)$, $\alpha(s_n sf - f) \leqslant \alpha(s_n - s^{-1}) \|sf\|_\infty \to 0$. Thus $f \in \overline{W \cap L^\infty(\lambda)}^\alpha$. Therefore $W = \overline{W \cap L^\infty(\lambda)}^\alpha$. $\square$

COROLLARY 3.11. *A weak*-closed linear subspace M of $L^\infty(\lambda)$ satisfies $zM \subset M$ if and only if $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$.*

Proof. If $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$, clearly, a weak*-closed linear subspace M of $L^\infty(\lambda)$ with $zM \subset M$. Conversely, since $zM \subset M$, and we have $z\overline{M}^{\|\cdot\|_{2,\lambda}} \subset \overline{M}^{\|\cdot\|_{2,\lambda}}$. Hence by Beurling-Helson-Lowdenslager theorem for $\|\cdot\|_{2,\lambda}$, we consider either $\overline{M}^{\|\cdot\|_{2,\lambda}} = \varphi H^2(\lambda)$ for some unimodular function φ, then $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \varphi H^2(\lambda) \cap L^\infty(\lambda)$; or $\overline{M}^{\|\cdot\|_{2,\lambda}} = \chi_E L^2(\lambda)$, for some Borel subset E of $\mathbb{T}$, in this case, $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \chi_E L^2(\lambda) \cap L^\infty(\lambda) = \chi_E L^\infty(\lambda)$, i.e., $M = \chi_E L^\infty(\lambda)$. $\square$

Now we obtain our main theorem, which extends the Chen-Beurling Helson-Lowdenslager theorem.

THEOREM 3.12. *Suppose μ is Haar measure on $\mathbb{T}$ and α is a continuous normalized gauge norm on $L^\infty(\mu)$. Suppose also that $c > 0$ and λ is a probability measure that is mutually absolutely continuous with respect to μ such that α is $c\|\|_{1,\lambda}$-dominating and* $\log|d\lambda/d\mu| \in L^1(\mu)$. *Then a closed linear subspace W of $L^\alpha(\mu)$ satisfies $zW \subset W$ if and only if either $W = \varphi H^\alpha(\mu)$ for some unimodular function φ, or $W = \chi_E L^\alpha(\mu)$, for some Borel subset E of $\mathbb{T}$. If $0 \neq W \subset H^\alpha(\mu)$, then $W = \varphi H^\alpha(\mu)$ for some inner function φ.*

Proof. Recall that $L^\infty(\mu) = L^\infty(\lambda)$, $L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$. The only if part is obvious. Let $M = W \cap L^\infty(\lambda)$, and in Theorem 2.2, we have proved that there exists a measure λ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and there exists $c > 0$, $\forall f \in L^\infty(\mu) = L^\infty(\lambda)$, $\alpha(f) \geqslant c\|f\|_{1,\lambda}$. i.e., α is a continuous $c\|\cdot\|_{1,\lambda}$-dominating normalized gauge norm on $L^\infty(\lambda)$. It follows from the (2) in Theorem 3.10 that M is weak* closed in $L^\infty(\lambda)$. Since $zW \subset W$, it is easy to check that $zM \subset M$. Then by Corollary 3.11, we can conclude that either $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$. By the (3) in Theorem 3.10, if $M = \varphi H^\infty(\lambda)$, $W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\varphi H^\infty(\lambda)}^\alpha = \varphi H^\alpha = \varphi H^\alpha(\mu)$, for some unimodular function φ. If $M = \chi_E L^\infty(\lambda)$, $W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\chi_E L^\infty(\lambda)}^\alpha = \chi_E L^\alpha = \chi_E L^\alpha(\mu)$, for some Borel subset E of $\mathbb{T}$. The proof is completed. $\square$

4. Which α's have a good λ?

In the preceding section we proved a version of Beurling's theorem for L^α when there is a probability measure λ on $\mathbb{T}$ that is mutually absolutely continuous with respect to μ, such that α is $c\|\cdot\|_{1,\lambda}$-dominating and $d\lambda/d\mu$ is log-integrable with respect to μ. How do we tell when such a good λ exists. Suppose ρ is a probability measure on $\mathbb{T}$ that is mutually absolutely continuous with respect to μ such that

$$\int_{\mathbb{T}} \log(d\rho/d\mu)\, d\mu = -\infty.$$

Here are some useful examples.

EXAMPLE 4.1. Let $\alpha = \frac{1}{2}\left(\|\cdot\|_{1,\mu} + \|\cdot\|_{1,\rho}\right)$. Then α is a continuous gauge norm. If we let $\lambda_1 = \rho$ and $\lambda_2 = \mu$ we see that $\alpha \geqslant \frac{1}{2}\lambda_k$ for $k = 1, 2$ and

$$\int_{\mathbb{T}} |\log(d\lambda_k/d\mu)|\, d\mu = \begin{cases} \infty & \text{if } k = 1 \\ 0 & \text{if } k = 2 \end{cases}.$$

Hence there is both a bad choice of λ and a good choice.

EXAMPLE 4.2. Suppose ρ is as in the preceding example and let $\alpha = \|\cdot\|_{1,\rho}$. Suppose λ is a probability measure that is mutually absolutely continuous with respect to μ and

$$\|\cdot\|_{1,\rho} = \alpha \geqslant c\|\cdot\|_{1,\lambda} \text{ for some constant } c.$$

It follows that $d\lambda/d\rho \leqslant c$ a.e., and thus

$$\int_{\mathbb{T}} \log(d\lambda/d\mu)\,d\mu = \int_{\mathbb{T}} \log(d\lambda/d\rho)\,d\mu + \int_{\mathbb{T}} \log(d\rho/d\mu)\,d\mu \leqslant \log\varepsilon + (-\infty) = -\infty.$$

In this case there is no good λ.

5. A special case

Suppose λ is any probability measure that is mutually absolutely continuous with respect to μ and $\alpha = \|\cdot\|_{p,\lambda}$ for some p with $1 \leqslant p < \infty$. Assume λ is bad, i.e., $\int_{\mathbb{T}} \left|\log \frac{d\lambda}{d\mu}\right| d\mu = \infty$. In this case, we define a bijective isometry mapping $U : L^p(\lambda) \to L^p(\mu)$ by $Uf = g^{\frac{1}{p}} f$. Let $H^p(\lambda)$ be the α-closure of all polynomials, then $H^p(\lambda)$ is a closed subspace of $L^p(\lambda)$ and $zH^p(\lambda) \subseteq H^p(\lambda)$. Therefore, $g^{\frac{1}{p}} H^p(\lambda)$ is a z-invariant closed subspace of $L^p(\mu)$. By Beurling-Helson-Lowdenslager theorem, we have

$$g^{\frac{1}{p}} H^p(\lambda) = \chi_E L^p(\mu) \text{ for some Borel set } E \subseteq \mathbb{T}, \text{ or } \varphi H^p(\mu), \text{ where } |\varphi| = 1.$$

If $g^{\frac{1}{p}} H^p(\lambda) = \chi_E L^p(\mu)$, then $H^p(\lambda) = L^p(\lambda)$, in this case, $\varphi H^p(\lambda) = \varphi L^p(\lambda)$, where $|\varphi| = 1$. If $M_0 = \frac{1}{g^{1/p}} H^p(\mu)$, then M_0 is a proper z-invariant closed subspace of $L^p(\lambda)$, and $M_0 \neq \chi_E L^p(\lambda)$. Therefore, Beurling-Helson-Lowdenslager theorem is not true for this case. However, we have the following theorem

THEOREM 5.1. *Suppose λ is any probability measure that is mutually absolutely continuous with respect to μ and $\alpha = \|\cdot\|_{p,\lambda}$ for some p with $1 \leqslant p < \infty$. Also assume $\int_{\mathbb{T}} \left|\log \frac{d\lambda}{d\mu}\right| d\mu = \infty$. If M is a closed subspace of $L^\alpha(\lambda)$, then $zM \subseteq M$ if and only if*

(1) $M = \varphi M_0$ for some unimodular function φ, where $M_0 = \frac{1}{g^{1/p}} H^p(\mu)$, or

(2) $M = \chi_E L^\alpha(\lambda)$ for some Borel subset E of $\mathbb{T}$.

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