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ON THE BLOCK NUMERICAL RANGE OF OPERATORS ON ARBITRARY BANACH SPACES

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To the memory of our esteemed colleague Karl-Heinz Förster

(*Communicated by M. Embree*)

Abstract. We investigate the block numerical range of bounded linear operators on arbitrary Banach spaces. We show that the spectrum of an operator is always contained in the closure of its block numerical range. The inclusion between block numerical ranges for refined block decompositions hold only in special cases which we characterize completely. Thereby we achieve a new characterization of L^p-spaces. Finally we obtain an estimate of the resolvent in terms of the block numerical range. All our results are new even for $n \times n$-matrices.

1. Introduction and notation

The *block numerical range* of a bounded linear operator A on a Hilbert space with respect to an *orthogonal* decomposition was introduced and analyzed by C. Tretter and M. Wagenhofer in [20], see also the book [21, Chapter 1.11]. This notion was generalized to decompositions of arbitrary Banach spaces by P. Kallus [12, p. 8], where it is completely analyzed in the case of p-direct sums of Banach spaces. Unfortunately, this paper has not yet been published.

In the present paper we investigate the block numerical range of bounded linear operators on a complex Banach space X in the case of an *arbitrary* decomposition of X.

1.1. Decomposition of Banach spaces

Our starting point is a complex Banach space $(X, \|\cdot\|)$ which is decomposed into the direct sum of $n \geqslant 1$ closed subspaces $X_1, \ldots, X_n$ of X:

$$X = X_1 \oplus \cdots \oplus X_n.$$

We abbreviate this decomposition as $\mathscr{D}$. The number $n = n(\mathscr{D})$ is called the *order* of $\mathscr{D}$. There exist n uniquely determined continuous projections $P_1, \ldots, P_n$ satisfying

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(i) $X_k = P_k(X)$.

(ii) $P_i P_k = 0$ for $i \neq k$ and

(iii) $\sum_1^n P_k = I$ (the identity on X).

We set $\mathscr{P}_{\mathscr{D}} := \{P_1, \ldots, P_n\}$.

The linear map

$$T : X \to \prod_1^n X_k, \quad Tx = (P_1 x, \ldots, P_n x) =: (x_1, \ldots, x_n)$$

is an isometric isomorphism, where $\prod_1^n X_k$ is equipped with the norm $\|(x_1, \ldots, x_n)\| := \|x_1 + \cdots + x_n\|$. For the sake of convenience and in agreement with the preceding fundamental papers on our topic, e.g. [20, 12] we shall use henceforth the isometrically isomorphic Cartesian product $X_1 \times \cdots \times X_n$ in place of $X_1 \oplus \cdots \oplus X_n$.

A decomposition of X induces a corresponding decomposition $\mathscr{D}'$ of the dual space X', explicitly

$$X' = \prod_1^n P_k'(X') = X_1' \times \cdots \times X_n',$$

and by

$$X_k' \ni \varphi \mapsto J\varphi = \varphi \circ P_k$$

the space X_k' is canonically embedded into X'. This embedding is isometric if and only if P_k is a contraction. A decomposition is called *contractive* if all projections P_k are contractions.

Typical examples of this latter kind of decompositions are orthogonal decompositions of Hilbert spaces, as well as decompositions of Banach lattices into the direct sum of projection bands. More precisely a direct sum decomposition $X = \prod_1^n X_k$ is called a *band decomposition* if all X_k are projection bands, i. e. P_k is a positive contractive projection of X onto the band X_k for each k. For more details see [22, Definition 2.8 on p. 61].

Let us point out that the decompositions of the canonical Hilbert space $\mathbb{C}^n$, considered in [8], are nothing else than band decompositions of $\mathbb{C}^n$ equipped with its canonical order.

Another class of contractive decompositions is the following one: Let $\mathbb{C}^n$ be equipped with a lattice norm ρ satisfying $\rho(e_j) = 1$ for the canonical basis $\{e_j : j = 1, \ldots, n\}$ ($e_j = (\delta_{j,k})$, $\delta_{j,k}$ denotes the Kronecker symbol). Sometimes ρ is called a *normalized monotone norm*, sometimes *absolute norm* (see [2, Theorem 2] for the equivalence of these notions). The decomposition $\mathscr{D}$ is then required to satisfy

$$\|x\| = \rho\left(\begin{pmatrix} \|P_1 x\| \\ \vdots \\ \|P_n x\| \end{pmatrix}\right).$$

Since ρ is monotone and

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \|P_k x\| \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leqslant \begin{pmatrix} \|P_1 x\| \\ \vdots \\ \|P_n x\| \end{pmatrix}, \tag{1}$$

$\mathscr{D}$ is contractive. This kind of direct sums becomes more and more important e. g. under the name ψ-direct sum, see [6, 19] . We shall call it a ρ*-normed decomposition*. The ℓ_p norm on $\mathbb{C}^n$ yields the p-direct sum as a special case. In particular, orthogonal decompositions of a Hilbert space fall within this definition.

Our setting, however, comprises also decompositions of Hilbert spaces which are *not* contractive, i. e. *not* orthogonal, thus including e. g. the spectral decomposition of non-normal $n \times n$ matrices.

1.2. The block numerical range

In the following, $\mathscr{L}(X)$ denotes the set of bounded linear operators on the complex Banach space X. Moreover we denote the unit sphere of the Banach space X by $S_X = \{x \in X : \|x\| = 1\}$.

Let $X = \prod_1^n X_k$ be an arbitrary decomposition of X, abbreviated as $\mathscr{D}$. For $A \in \mathscr{L}(X)$ and $i, j \in \{1, \ldots, n\}$ we define bounded linear operators

$$A_{ij} : X_j \to X_i, \quad X_j \ni x \mapsto P_i A x.$$

Obviously $A_{ij} = P_i A P_j|_{X_j}$. Clearly, A can be written as an $n \times n$ operator matrix $(A_{ij})_{i,j=1}^n$ where

$$(A_{ij})_{i,j=1}^n : X \to X, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \sum_{k=1}^n A_{1k} x_k \\ \vdots \\ \sum_{k=1}^n A_{nk} x_k \end{pmatrix}.$$

For an arbitrary complex Banach space Y with dual space Y' we define

$$S_{att}(Y) := \{(x, \varphi) \in S_Y \times S_{Y'} : \varphi(x) = 1\}.$$

Then for a given decomposition $\mathscr{D}$ of order n we set

$$S_{\mathscr{D}} := S_{att}(X_1) \times \cdots \times S_{att}(X_n).$$

Let $d = ((u_1, \varphi_1), \ldots, (u_n, \varphi_n)) = \prod_{k=1}^n (u_k, \varphi_k) \in S_{\mathscr{D}}$. We define the projection P_d from X onto $X_d := \operatorname{span}\{u_1, \ldots, u_n\}$ by

$$\begin{aligned} P_d x &= (\varphi_1(P_1 x) u_1, \ldots, \varphi_n(P_n x) u_n) \\ &= (J_1 \varphi_1(x) u_1, \ldots, J_n \varphi_n(x) u_n) \\ &= (\varphi_1(x_1) u_1, \ldots, \varphi_n(x_n) u_n), \end{aligned}$$

where $J_k : X_k' \mapsto X'$, $J_k\varphi_k = \varphi_k \circ P_k$ and $x_k = P_k x$ (see Section 1.1).

Let $A \in \mathscr{L}(X)$ and $d \in S_{\mathscr{D}}$ be arbitrary. Then we define $A_d : X_d \to X_d$ by

$$A_d = P_d A|_{X_d}, \quad X_d \ni x \mapsto P_d(A_{ij})_{i,j=1}^n x.$$

The matrix representation of A_d with respect to the basis $\{u_1, \dots, u_n\}$ is

$$B_d = (\varphi_i(A_{ij}u_j))_{i,j=1}^n = (\varphi_i(P_i A u_j))_{i,j=1}^n.$$

Denoting the spectrum of an operator T by the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}$$

we have $\sigma(A_d) = \sigma(B_d)$.

Now we are ready to define the block numerical range.

DEFINITION 1.1. ([12, p. 8]) Let $\mathscr{D}$ be a decomposition of the complex Banach space X and let $A \in \mathscr{L}(X)$. The *block numerical range of A with respect to the decomposition $\mathscr{D}$* is the set

$$V_{\mathscr{D}}(A) := \bigcup_{d \in S_{\mathscr{D}}} \sigma(A_d).$$

Observe that for the trivial decomposition $X = X$, denoted by $\mathscr{D}_0$ the block numerical range $V_{\mathscr{D}_0}(A)$ is nothing else than the well known spatial numerical range (see [4, 5, 9])

$$V(A) = \{\varphi(Au) : (u, \varphi) \in S_{att}(X)\}.$$

Moreover, if H is a Hilbert space and $\mathscr{D}$ an orthogonal decomposition, then $V_{\mathscr{D}}(A)$ coincides with the block numerical range introduced in [20]. More precisely in that case $S_{att}(H) = \{(x, x') : x \in S_H\}$, where x' denotes the linear form $y \mapsto (x|y)$ (the scalar product on H). Then $d = ((x_j, x_j'))_{j=1 \cdots n}$ and the corresponding matrix B_d is $B_d = ((x_i | A_{ij} x_j))_{i,j=1}^n$.

As already the research of C. Tretter and M. Wagenhofer (see [20]) shows the block numerical range need not be closed nor convex. Moreover it depends heavily on the choice of the decomposition $\mathscr{D}$, even if two decompositions are of the same order (see [20, p. 1007] as well as [21, Chapter 1.11] for striking examples).

The block numerical range is always bounded. This is a consequence of the following lemma:

LEMMA 1.2. *Let $\mathscr{D}$ be an arbitrary decomposition of order n of the Banach space X. Then*

$$\|\mathscr{D}\| := \sup\{\|P_d\| : d \in S_{\mathscr{D}}\} \leqslant \sum_1^n \|P_j\|.$$

Proof. Let $d = \prod_1^n (u_j, \varphi_j) \in S_{\mathscr{D}}$ be arbitrary. For $x \in S_X$ we obtain

$$\|P_d x\| = \left\| \sum_1^n \varphi_j(P_j x) u_j \right\| \leqslant \sum_1^n |\varphi_j(P_j x)| \leqslant \sum_1^n \|P_j\|,$$

and the assertion follows. $\square$

We call $\|\mathscr{D}\|$ the *norm* of the decomposition $\mathscr{D}$.

PROPOSITION 1.3. *$V_{\mathscr{D}}(A)$ is bounded by $\|\mathscr{D}\| \cdot \|A\|$.*

Proof. For $\lambda \in V_{\mathscr{D}}(A)$ there exists $d \in S_{\mathscr{D}}$ such that $\lambda \in \sigma(A_d)$. Then

$$|\lambda| \leqslant \|A_d\| \leqslant \|P_d\|\|A\| \leqslant \|\mathscr{D}\| \cdot \|A\|. \quad \square$$

1.3. Organization of the paper

Our paper is organized as follows:

Section 2 is devoted to the spectral inclusion $\sigma(A) \subset \overline{V_{\mathscr{D}}(A)}$.

In Section 3 we address the question of whether $V_{\mathscr{D}}(A) \subset V(A)$ holds. In contrast to orthogonal decompositions of a Hilbert space, and, more generally to p-direct sums, this inclusion is not true in general. The main result of this section (Theorem 3.6) is a characterization of those decompositions for which the inclusion is satisfied. An application to Banach lattices leads to a new characterization of L^p-spaces.

Finally in Section 4 we give an estimate of the norm of the resolvent of an operator in terms of the block numerical range thereby generalizing [20, Theorem 4.2]. This estimate is new even in the case of p-direct sums.

2. Spectral inclusion

In the following let X be a complex Banach space and let $A \in \mathscr{L}(X)$ be arbitrary. Moreover let $\mathscr{D}$ be an arbitrary decomposition of X. The main result of this section is the inclusion $\sigma(A) \subset \overline{V_{\mathscr{D}}(A)}$. This inclusion was proved first in the case of orthogonal decompositions of a Hilbert space in [20] and generalized to p-direct sums in [12, Theorem 1.13].

2.1. Ultrapowers

For our proof we need the theory of ultrapowers, see e. g. [10].

Let X be a complex Banach space and let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. Consider the space $\ell_\infty(X)$ of all bounded sequences on X, equipped with the sup-norm. Then $c_{0,\mathscr{U}}(X) = \{(x_n)_n \in \ell_\infty(X) : \lim_{\mathscr{U}} x_n = 0\}$ is a closed subspace, and the ultrapower $\hat{X}$ is the quotient space $\ell_\infty(X)/c_{0,\mathscr{U}}(X)$.

If (x_n) is a representative of $\hat{x}$, then we also use the notation $\widehat{(x_n)} := \hat{x}$.

The norm on $\hat{X}$ is given by $\|\hat{x}\| = \lim_{\mathscr{U}} \|x_n\|$ for some and hence all representing sequences $(x_n)_n$ of $\hat{x}$.

The space X is always isometrically embedded into $\hat{X}$ by $X \ni x \mapsto \widehat{(x,x,\dots)}$ where $(x,x,\dots)$ denotes the constant sequence with members x. We denote such an element by x thus viewing X as a subspace of $\hat{X}$.

Let Y be another Banach space. An operator $A \in \mathscr{L}(X,Y)$ has a natural extension to a bounded linear operator from $\hat{X}$ to $\hat{Y}$ which is in the following denoted by $\hat{A}$. It is given by

$$\hat{A}\hat{x} = \widehat{(Ax_n)_n}. \tag{2}$$

The ultrapower $\widehat{(X')}$ of X' is canonically embedded into the dual space $(\hat{X})'$ of $\hat{X}$ by

$$\hat{\varphi}(\hat{x}) = \lim_{\mathscr{U}}(\varphi_n(x_n)),$$

where $(\varphi_n)_n$, $(x_n)_n$ are arbitrary representing sequences of $\hat{\varphi}$, $\hat{x}$, respectively. Note that $\widehat{(X')} = (\hat{X})'$ holds if and only if X is super-reflexive, see [10, Cor. 7.2]. For our purposes, however, it suffices to work only with $\widehat{(X')}$.

Let Y be another Banach space and let A be a bounded linear operator from X to Y. Considering representing sequences we obtain at once the following relation for $\hat{\psi} \in \widehat{(Y')}$:

$$(\widehat{A'}\hat{\psi})(\hat{x}) = \hat{\psi}(\hat{A}\hat{x}).$$

Thus

$$(\hat{A})'|_{\widehat{(Y')}} = \widehat{A'} \tag{3}$$

holds. In particular identifying Y' with its canonical image in $\widehat{(Y')}$ we obtain the following useful formula for all $\varphi \in Y'$ (notice $\varphi = \hat{\varphi}$):

$$\begin{aligned}(A'\varphi)(\hat{x}) &= (A'\varphi)(\widehat{x_n}) && (4)\\ &= \varphi(\widehat{(Ax_n)}) && (5)\\ &= \varphi(\hat{A}\hat{x}) && (6)\\ &= \hat{\varphi}(\hat{A}\hat{x}) = (\widehat{A'}\hat{\varphi})(\hat{x}). && (7)\end{aligned}$$

Now we define

$$S^r_{att}(\hat{X}) := S_{att}(\hat{X}) \cap (\hat{X} \times \widehat{(X')}).$$

The superscript r in the definition above means *reduced to the pair* $(\hat{X}, \widehat{(X')})$ in place of $(\hat{X}, (\hat{X})')$.

DEFINITION 2.1. Let X be a complex Banach space and $A \in \mathscr{L}(X)$. Then we call

$$V^r(\hat{A}) := \{\hat{\varphi}(\hat{A}\hat{x}) : (\hat{x}, \hat{\varphi}) \in S^r_{att}(\hat{X})\}$$

the *reduced numerical range* of $\hat{A}$.

If $\mathscr{P}_{\mathscr{D}} = \{P_1, \ldots, P_n\}$ is the set of projections of a decomposition $\mathscr{D}$ of X, then $\mathscr{P}_{\hat{\mathscr{D}}} = (\hat{P}_1, \ldots, \hat{P}_n)$ is the set of projections of the induced decomposition $\hat{\mathscr{D}}$ of $\hat{X}$ given by $\hat{P}_j(\hat{X}) = \hat{X}_j$. Hence, for a given decomposition $\mathscr{D}$ of X we can define

$$S^r_{\hat{\mathscr{D}}} := \prod_{k=1}^{n} S^r_{att}(\hat{X}_k).$$

This leads to the following definition.

DEFINITION 2.2. If X is a complex Banach space, $A \in \mathscr{L}(X)$ and $\mathscr{D}$ a decomposition of X, then we call

$$V^r_{\hat{\mathscr{D}}}(\hat{A}) := \bigcup_{\hat{d} \in S^r_{\hat{\mathscr{D}}}} \sigma(\hat{A}_{\hat{d}})$$

the *reduced block numerical range* of $\hat{A}$.

Obviously it is bounded by $\|\mathscr{D}\| \cdot \|A\|$, cf. Proposition 1.3.

We now show that to an arbitrary element $(\hat{x}, \hat{\varphi}) \in S^r_{att}(\hat{X})$ there exists a sequence $(z_n, \rho_n)_n$ in $S_{att}(X)$ representing it. Our main tool for the proof is the Bishop-Phelps-Bollobás theorem, see e.g. [5, Theorem 16.1].

LEMMA 2.3. *For each $(\hat{x}, \hat{\varphi}) \in S^r_{att}(\hat{X})$ there are representatives (z_n) of $\hat{x}$ and (ρ_n) of $\hat{\varphi}$ such that $(z_n, \rho_n) \in S_{att}(X)$, i. e.*

$$\|z_n\| = \|\rho_n\| = \rho_n(z_n) = 1 \quad \textit{holds for all } n \in \mathbb{N}.$$

Proof. Let (y_n) be an arbitrary representative of $\hat{x} \in \hat{X}, \|\hat{x}\| = 1$. Then $U := \{n \in \mathbb{N} : y_n \neq 0\} \in \mathscr{U}$. Let x be an arbitrary normalized element in X. For $n \in \mathbb{N}$ define

$$x_n := \begin{cases} x, & n \notin U, \\ y_n / \|y_n\|, & n \in U. \end{cases}$$

Clearly, (x_n) is also a representative of $\hat{x}$ with the property that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Similarly, we find a representative (φ_n) of $\hat{\varphi}$ such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$. Consider $\eta_n := |1 - \varphi_n(x_n)|$. By hypothesis, the sequence $(\eta_n)_n$ converges to 0 along $\mathscr{U}$. Let $\varepsilon_n := 2\sqrt{\eta_n}$. By the Bishop-Phelps-Bollobás theorem, for each $n \in \mathbb{N}$ there exist $z_n \in X$ and $\rho_n \in X'$ such that

$$\|z_n\| = \|\rho_n\| = \rho_n(z_n) = 1, \quad \|x_n - z_n\| < \varepsilon_n \quad \text{and} \quad \|\rho_n - \varphi_n\| < \varepsilon_n.$$

Thus, (z_n) and (ρ_n) are representatives of $\hat{x}$ and $\hat{\varphi}$, respectively, with the desired properties. □

The next lemma is a direct consequence of the *principle of local reflexivity*, see [11, p. 493]:

LEMMA 2.4. *Let X be a complex Banach space. Let $G \subset X''$, $H \subset X'$ be arbitrary finite-dimensional subspaces. Then there exists an isometric embedding R from G to $\hat{X}$ with the following properties:*

(i)
$$Rx = x \ \textit{for all} \ x \in G \cap X. \tag{8}$$

(ii)
$$\xi(\varphi) = \varphi(R\xi) \ \textit{for all} \ \ \varphi \in H, \xi \in G, \tag{9}$$

where φ is identified with its canonical image in $\widehat{(X')}$.

(iii) *Let Y be another Banach space, let $T \in \mathscr{L}(X,Y)$, and let $\varphi \in (T')^{-1}(H)$ be arbitrary. Then for all $\xi \in G$ we have*

$$\xi(T'\varphi) = (T'\varphi)(R\xi) = \varphi(\hat{T}R\xi). \tag{10}$$

Proof. By the cited principle to each $n \in \mathbb{N}$ there exists a continuous linear injection T_n from G onto a finite-dimensional subspace of X satisfying

1. $\|T_n\| \cdot \|T_n^{-1}\| < 1 + 1/n$.
2. $T_n x = x$ for $x \in G \cap X$.
3. $\varphi(T_n(\xi)) = \xi(\varphi)$ for all $\xi \in G$, $\varphi \in H$.

Set $R\xi = \widehat{(T_n\xi)}$. Then equation (8) and (9) follow. The first part of equation (10) follows from equation (9). The second part follows from $(T'\varphi)(T_n\xi) = \varphi(TT_n\xi)$ for all n and from $\hat{T}(R\xi) = \widehat{(TT_n\xi)}$, see equation (2). $\square$

REMARK. There is a deep theorem stating the embeddability of X'' into $\hat{X}$ with properties similar to those above, due to C. W. Henson and L. C. Moore Jr. but it requires either nonstandard analysis (see [23, Prop. 4.3.17]) or a more sophisticated theory of ultraproducts (see e. g. [10, Theorem 6.7]).

2.2. The inclusion results

The usefulness of ultrapower techniques lies in the following proposition:

PROPOSITION 2.5. (cf. [17, Prop. 1.3.6]) *Let X be a complex Banach space and $A \in \mathscr{L}(X)$. Then for each direct sum decomposition $\mathscr{D}$ of X we have*

$$V^r_{\hat{\mathscr{D}}}(\hat{A}) = \overline{V_{\mathscr{D}}(A)}.$$

Proof. Let n be the order of the decomposition $\mathscr{D}$. We assume that $n \geqslant 2$. (The case $n = 1$ is analogous and is therefore omitted.)

"$\supset$" Let $\lambda \in \overline{V_{\mathscr{D}}(A)}$. Then there exist sequences

$$(d_m)_{m\in\mathbb{N}} = ((u_{1,m}, \varphi_{1,m}), \ldots, (u_{n,m}, \varphi_{n,m}))_{m\in\mathbb{N}} \subset S_{\mathscr{D}}$$

and $(\lambda_m)_{m\in\mathbb{N}}$ such that $\lambda_m \in \sigma(A_{d_m})$ and $\lambda = \lim_{m\to\infty} \lambda_m$. Moreover, we have

$$\hat{d} := ((\widehat{(u_{1,m})}, \widehat{(\varphi_{1,m})}), \ldots, (\widehat{(u_{n,m})}, \widehat{(\varphi_{n,m})}))_{m\in\mathbb{N}} \in S^r_{\hat{\mathscr{D}}}.$$

For each $m \in \mathbb{N}$ let $x_m = (x_{1,m}, \ldots, x_{n,m}) \in X_{d_m}$ be a normalized eigenvector of A_{d_m} corresponding to the eigenvalue λ_m. Then $\widehat{(x_m)} := (\widehat{(x_{1,m})}, \ldots, \widehat{(x_{n,m})}) \in \hat{X}_{\hat{d}}$ is an eigenvector of $\hat{A}_{\hat{d}}$ corresponding to the eigenvalue λ, since

$$\hat{A}_{\hat{d}}\widehat{(x_m)} = \widehat{(A_{d_m}x_m)} = \widehat{(\lambda_m x_m)} = \lambda\widehat{(x_m)}.$$

Hence, $\lambda \in V^r_{\hat{\mathscr{D}}}(\hat{A})$.

"$\subset$" Let $\lambda \in V^r_{\hat{\mathscr{D}}}(\hat{A})$. Then there is a $\hat{d} \in S^r_{\hat{\mathscr{D}}}$ such that $\lambda \in \sigma(\hat{A}_{\hat{d}}) = \sigma(B_{\hat{d}})$. By Lemma 2.3 applied to the components of $\hat{d}$, there is a sequence $(d_m) \subset S_{\mathscr{D}}$ such that $\widehat{(d_m)} = \hat{d}$. The entries of the $n \times n$ matrices B_{d_m} converge along $\mathscr{U}$ to the entries of $B_{\hat{d}}$. Hence, the Hausdorff distance between $\sigma(B_{d_m}) \subset V_{\mathscr{D}}(A)$ and $\sigma(B_{\hat{d}})$ converges along $\mathscr{U}$ to 0, see [3, Formula (VIII.3)], and thus $\lambda \in \overline{V_{\mathscr{D}}(A)}$. $\square$

The following proposition states inclusions between the block numerical ranges of an operator $A \in \mathscr{L}(X)$ and of its adjoint A'. The result is interesting in itself and will play a crucial role for the proof of the spectral inclusion in Theorem 2.8 below.

For a decomposition $\mathscr{D}$ let $\mathscr{D}'$ be the corresponding decomposition $X' = \prod_1^n P_j'(X) = X_1' \times \cdots \times X_n'$ of X' (see the introduction). Note that if $(A_{ij})_{i,j=1}^n$ is the block operator representation of A with respect to $\mathscr{D}$, then $(A'_{ji})_{i,j=1}^n$ is the block operator representation of A' with respect to $\mathscr{D}'$, where A'_{ij} is the adjoint of A_{ij}, which is a bounded linear operator from X_i' to X_j'.

PROPOSITION 2.6. *Let X be a complex Banach space and $A \in \mathscr{L}(X)$. Then for each decomposition $\mathscr{D}$ of X of order $n \geqslant 1$ the following inclusions hold:*

$$V_{\mathscr{D}}(A) \subset V_{\mathscr{D}'}(A') \subset \overline{V_{\mathscr{D}}(A)}.$$

Proof. First inclusion: Let $\lambda \in V_{\mathscr{D}}(A)$. There is a $d = \prod_{k=1}^n (u_k, \varphi_k) \in S_{\mathscr{D}}$ such that $\lambda \in \sigma(A_d)$. In the following we will identify X with its canonical image in the bidual X''. Then $d' := \prod_{k=1}^n (\varphi_k, u_k) \in S_{\mathscr{D}'}$ and $A'_{ij} = P_j' A'|_{X_i'}$ hence

$$B_{d'} = \big(u_i(A'_{ji}\varphi_j)\big)_{i,j=1}^n = \big((A'_{ji}\varphi_j)(u_i)\big)_{i,j=1}^n = (\varphi_j(A_{ji}u_i))_{i,j=1}^n = (B_d)^T,$$

where ^{T} denotes the operator of transposition. Hence, $\lambda \in \sigma(A'_{d'}) \subset V_{\mathscr{D}'}(A')$.

Second inclusion: Let $\lambda \in V_{\mathscr{D}'}(A')$. Then $\lambda \in \sigma(A'_{d'})$ for some $d' = \prod_{k=1}^n (\varphi_k, \xi_k) \in S_{\mathscr{D}'}$. Let $G = \operatorname{span}\{\xi_1, \ldots, \xi_n\}$ and $H = \operatorname{span}\{\varphi_1, \ldots, \varphi_n, A'_{ij}\varphi_i, i,j = 1, \ldots, n\}$. Moreover, let R be the mapping according to Lemma 2.4 and define $\hat{x}_k := R\xi_k$. We identify X_k' with its canonical image in $\widehat{(X_k')}$. By equation (3) and Lemma 2.4, equation (10),

$$\xi_j(A'_{ij}\varphi_i) \underbrace{=}_{eq.(9)} (A'_{ij}\varphi_i)(\hat{x}_j) \underbrace{=}_{eq.(3)} \varphi_i(\widehat{A_{ij}}\hat{x}_j) \tag{11}$$

holds for all $i,j \leqslant n$, $\varphi_i \in X_i'$ and $\hat{x}_j \in \widehat{X}_j$. This equation gives

$$\begin{aligned} B_{d'} &= \big(\xi_i(A'_{ki}\varphi_k)\big)_{i,k=1}^n \\ &= \big((A'_{ki}\varphi_k)(\hat{x}_i)\big)_{i,k=1}^n \\ &= \Big(\varphi_k(\widehat{A_{ik}}\hat{x}_i)\Big)_{i,k=1}^n = B_{\hat{d}}^T \end{aligned}$$

for $\hat{d} = \prod_1^n (\hat{x}_k, \varphi_k) \in S^r_{\mathscr{D}^r}$, and the assertion follows from Proposition 2.5. $\square$

REMARK. From $V_{\mathscr{D}}(A) \subset V_{\mathscr{D}}(A')$ we obtain immediately that $V_{\mathscr{D}}(A) = V_{\mathscr{D}}(A')$ holds for reflexive Banach spaces.

LEMMA 2.7. *Let $\mathscr{D}$ be a direct sum decomposition of the Banach space X. Then the following assertions hold:*

(i) *for all $x \in X$ there is a $d \in S_{\mathscr{D}}$ such that $P_d x = x$;*

(ii) *for all $\hat{x} \in \hat{X}$ there is a $\hat{d} \in S^r_{\hat{\mathscr{D}}}$ such that $P_{\hat{d}}\hat{x} = \hat{x}$.*

Proof. (i) Let n be the order of the decomposition $\mathscr{D}$ and assume that $n \geqslant 2$. (The case $n = 1$ is analogous and is therefore omitted.) Let $x \in X$ be arbitrary. If $P_k x \neq 0$ we set $u_k = P_k x / \|P_k x\|$. If $P_k x = 0$ we choose an arbitrary element of norm 1 in X_k and call it u_k. For each $k \in \{1, \ldots, n\}$ we choose $\varphi_k \in X_k'$, $\|\varphi_k\| = 1$, such that $\varphi_k(u_k) = 1$. If $d := \prod_{k=1}^n (u_k, \varphi_k)$, then $d \in S_{\mathscr{D}}$. Moreover, $x = (\|P_k x\|\, u_k)_{k=1}^n$ and hence $P_d x = x$.

(ii) Let $(x_m)_{m\in\mathbb{N}}$ be a representative of $\hat{x} \in \hat{X}$. For each x_m let P_{d_m} be the projection defined in (i) such that $P_{d_m} x_m = x_m$. Then for $\hat{d} := \prod_{k=1}^n (\widehat{(u_{k,m})_m}, \widehat{(\varphi_{k,m})_m})$ we have $P_{\hat{d}}\hat{x} = \hat{x}$. $\square$

We are now ready to prove that the spectrum of an operator is contained in the closure of its block numerical range.

To this end we use the following subsets of the spectrum. The *point spectrum* of A is

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective}\}$$

while the *approximate point spectrum* of A is

$$\sigma_a(A) = \{\lambda \in \mathbb{C} : \inf\{\|(\lambda - A)x\| : \|x\| = 1\} = 0\}.$$

Clearly, the inclusion $\sigma_p(A) \subset \sigma_a(A)$ holds. Finally, the *residual spectrum* is

$$\sigma_r(A) = \sigma(A) \setminus \sigma_a(A).$$

As is easily seen the residual spectrum is contained in the point spectrum of the adjoint, i. e. $\sigma_r(A) \subset \sigma_p(A')$.

THEOREM 2.8. *Let X be a complex Banach space and $A \in \mathscr{L}(X)$. Then the following inclusions hold for each decomposition $\mathscr{D}$ of X of order $n \geqslant 1$:*

(i) $\sigma_p(A) \subset V_{\mathscr{D}}(A)$,

(ii) $\sigma_r(A) \subset V_{\mathscr{D}'}(A')$,

(iii) $\sigma(A) \subset \overline{V_{\mathscr{D}}(A)}$.

Proof. (i) Let λ be an eigenvalue of A corresponding to the normalized eigenvector x. According to Lemma 2.7, there is a $d \in S_{\mathscr{D}}$ such that $P_d x = x$. Then we have $A_d x = \lambda x$, and hence $\lambda \in V_{\mathscr{D}}(A)$.

(ii) As we have mentioned above, the inclusion $\sigma_r(A) \subset \sigma_p(A')$ is always satisfied. The assertion now follows by applying part (i) to A'.

(iii) If $\lambda \in \sigma_a(A)$, then there exists a normalized vector $\hat{x}$ with $\hat{A}\hat{x} = \lambda\hat{x}$ (see e.g. [22, Theorem V.1.4]). By Lemma 2.7 $\lambda \in V^r_{\hat{\mathscr{D}}}$ (cf. the proof of part (i)). But $V^r_{\hat{\mathscr{D}}}(A) = \overline{V_{\mathscr{D}}(A)}$ by Proposition 2.5 and the assertion follows. $\square$

3. The block numerical range of refinements

3.1. Refinements and coarsenings

The philosophy behind the notion of the block numerical range in [20] was that the finer the decomposition is, the nearer the block numerical range is to the spectrum. In order to state this philosophy more precisely we need the following definition:

DEFINITION 3.1. Let $\mathscr{D} : X = X_1 \times X_2 \times \cdots \times X_n$ be a direct sum decomposition. For each j let $\mathscr{D}_j : X_j = Y_{j1} \times \cdots \times Y_{jr_j}$ be a decomposition of X_j (the trivial decomposition $X_j = Y_{j1}$ is allowed). Then the decomposition

$$\mathscr{D}^* : X = \prod_{j=1}^{n} \prod_{k=1}^{r_j} Y_{jk}$$

is called a *refinement* of $\mathscr{D}$ and $\mathscr{D}$ itself is a called a *coarsening* of $\mathscr{D}^*$.

Let $\mathscr{D}^*$ be a refinement of $\mathscr{D}$. If X is a Hilbert space and the decomposition is an orthogonal sum, or, more generally, if X is a p-direct sum of Banach spaces, then the inclusion

$$V_{\mathscr{D}^*}(A) \subset V_{\mathscr{D}}(A)$$

always holds, see [20, Theorem 3.5] for Hilbert spaces and [12, Theorem 1.16] for p-direct decompositions.

Unfortunately, however, this is no longer true for an arbitrary direct sum decomposition of X, as easy examples of *non orthogonal*, i. e. *non contractive* decompositions of the Hilbert space $\mathbb{C}^3$ show. In the following example the decomposition is *contractive* and the norm is *uniformly convex and smooth*. Nevertheless there is an operator A such that $V_{\mathscr{D}}(A)$ is not contained in $\overline{V_{\mathscr{D}_0}(A)} = \overline{V(A)}$.

EXAMPLE. Let $X = \mathbb{C}^3$. Consider the canonical basis $\{e_1, e_2, e_3\}$ where $e_j = (\delta_{kj})_{k=1,2,3}$ and δ_{kj} denotes the Kronecker symbol. Let $B_j = \mathrm{span}\{e_j\}$. Obviously, the decomposition $\mathscr{D}^* :\ X = B_1 \times B_2 \times B_3$ satisfies $V_{\mathscr{D}^*}(A) = \sigma(A)$ independently of the norm on X.

Let $4 \leqslant p < \infty$ and $q = p/(p-1)$. We choose the norm

$$\|x\| = (|x_1|^p + [(|x_2|^q + |x_3|^q)^{1/q})]^p)^{1/p} = (|x_1|^p + (|x_2|^q + |x_3|^q)^{p-1})^{1/p}.$$

Now consider the contractive coarsening $\mathscr{D} : X = \operatorname{span}\{e_1, e_2\} \times B_3$. It is a band decomposition and as such one it fits into the framework of [8]. Set

$$u_p = ((3/4)^{1/p}, (1/4)^{1/p}, 0)^T \tag{12}$$

$$\varphi_p = ((3/4)^{1/q}, (1/4)^{1/q}, 0) \tag{13}$$

$$z_p = ((3/4)^{1/p}, 0, (1/4)^{1/p})^T. \tag{14}$$

Here, the superscript ^{T} means transposition. Note that $\|u_p\| = \|\varphi_p\| = \|z_p\| = 1 = \varphi_p(u_p)$. Then for $d := ((u_p, \varphi_p), (e_3, e_3')) \in S_{\mathscr{D}}$ with $e_3' = (0,0,1)$, the projection P_d satisfies $\|P_d z_p\| = 1.1074\ldots$ if $p = 4$. Moreover, the norm is monotone with respect to p, so $\|P_d z_p\| > 1$ for all $4 \leqslant p < \infty$.

The norm is smooth, so there exists exactly one normalized linear form ρ with $\rho(P_d z_p) = \|P_d z_p\|$ (> 1). Clearly, ρ is positive. For $A = \rho \otimes z_p$ we obtain that $B_d = \begin{pmatrix} \rho(u_p)\varphi_p(z_p) & \rho(e_3)\varphi_p(z_p) \\ \rho(u_p)e_3'(z_p) & \rho(e_3)e_3'(z_p) \end{pmatrix}$ is positive and of rank 1 hence its trace is in $\sigma(B_d)$. This in turn implies $V_{\mathscr{D}}(A) \ni \operatorname{trace}(B_d) = \rho(P_d z_p) > 1$ whereas $V(A) \subset \{\lambda : |\lambda| \leqslant 1\}$, since A is a contraction. Thus $V_{\mathscr{D}}(A) \not\subset \overline{V(A)}$.

In fact this example suggests that all P_d should be contractions in order to ensure the inclusion $V_{\mathscr{D}}(A) \subset V(A)$. This leads to the following definition.

DEFINITION 3.2. The direct sum decomposition $\mathscr{D}$ of X is called *ideal* if every projection P_d, $d \in S_{\mathscr{D}}$, is a contraction.

It turns out that every p-direct decomposition is ideal. More generally, every ρ-normed direct sum (see the introduction) is ideal. This is a consequence of the following proposition.

PROPOSITION 3.3. *Let $\mathscr{D}$ be a ρ-normed decomposition of order n of the Banach space X. Then $\mathscr{D}$ is contractive and ideal.*

Proof. We know already from the introduction that $\mathscr{D}$ is contractive.
Let $d = ((u_1, \varphi_1), \ldots, (u_n, \varphi_n)) \in S_{\mathscr{D}}$. Then

$$P_d x = (\varphi_1(x_1)u_1, \ldots, \varphi_n(x_n)u_n)$$

for

$$x = (P_1 x, \ldots, P_n x) = (x_1, \ldots, x_n).$$

Since $\|\varphi_k\| = 1 = \|u_k\|$, $k = 1, \ldots, n$, and since all P_j are contractions we obtain

$$\|\varphi_k(x_k)u_k\| \leqslant \|x_k\|.$$

Using that ρ is a monotone norm (see the introduction), we finally have

$$\begin{aligned} \|P_d x\| &= \|(\varphi_1(x_1)u_1, \ldots, \varphi_n(x_n)u_n)\| \\ &= \rho((\|\varphi_1(x_1)u_1\|, \ldots, \|\varphi_n(x_n)u_n\|)^T) \\ &\leqslant \rho((\|x_1\|, \ldots, \|x_n\|)^T) \\ &= \|(x_1, \ldots, x_n)\| = \|x\|, \end{aligned}$$

where ^{T} denotes the transposition. □

REMARK. An ideal decomposition need not be contractive. For example, every decomposition of the 2-dimensional Hilbert space is ideal (P_d is always the identity), but it is contractive if and only if it is orthogonal.

The next two lemmata are crucial for the proof of Theorem 3.6 below.

LEMMA 3.4. *Let Y be a Banach space and let $\mathscr{E} : Y = \prod_1^r Y_j$ be an arbitrary decomposition. Let $x \in Y$ satisfy $P_e x = x$ for some $e \in S_{\mathscr{E}}$. Then if P_e is a contraction, there exists $(v, \varphi) \in S_{att}(Y)$ such that $\varphi \otimes v(x) = x$ and $\varphi \otimes v = (\varphi \otimes v) \circ P_e$ hold.*

Proof. If $x = 0$ then choose an arbitrary $v \in S_{P_e(Y)}$, otherwise set $v = \frac{x}{\|x\|}$. To v there exists ψ such that $1 = \psi(v) = \|\psi\|$ holds. Set $\varphi = P_e'\psi$. Then

$$1 = \psi(v) = \psi(P_e v) = (P_e'\psi)(v) = \varphi(v) \leqslant \|\varphi\| \leqslant \|\psi\| = 1,$$

since P_e' is a contraction by hypothesis. Moreover $\varphi \otimes v(x) = x$ and $\varphi \otimes v = \varphi \otimes v \circ P_e$ hold. □

LEMMA 3.5. *Let X be a complex Banach space. Let $\mathscr{D}$ be an arbitrary decomposition of the Banach space X and let $\mathscr{D}^*$ be an ideal refinement of $\mathscr{D}$. Then for every $d^* \in S_{\mathscr{D}^*}$ and $x \in X$ satisfying $P_{d^*}x = x$ there exists $d \in S_{\mathscr{D}}$ such that $P_d P_{d^*} = P_d$ as well as $P_d x = x$ hold.*

Proof. Let $\mathscr{D} : X = X_1 \times X_2 \times \cdots \times X_n$ and $\mathscr{D}^* : X = \prod_{j=1}^n \prod_{k=1}^{r_j} Y_{jk}$ be given. Let P_{e_j} be the restriction of P_{d^*} to X_j. It is a contraction as well. Applying Lemma 3.4 to X_j, P_{e_j} and $x_j = P_j(x)$, where P_j is the projection of X onto X_j, we obtain $(u_j, \varphi_j) \in S_{att}(X_j)$ with $\varphi_j \otimes u_j(x_j) = x_j$ and $\varphi_j \otimes u_j \circ P_{e_j} = \varphi_j \otimes u_j$. The projection P_d for $d = ((u_1, \varphi_1), \ldots, (u_n, \varphi_n))$ is the desired one. □

THEOREM 3.6. *Let $\mathscr{D}$ be an arbitrary decomposition of order n of the complex Banach space X. Then the following assertions are equivalent.*

(i) *For every $A \in \mathscr{L}(X)$ and for every coarsening $\tilde{\mathscr{D}}$ of $\mathscr{D}$ we have*

$$V_{\mathscr{D}}(A) \subset V_{\tilde{\mathscr{D}}}(A).$$

(ii) *For every $A \in \mathscr{L}(X)$ we have*

$$V_{\mathscr{D}}(A) \subset \overline{V(A)}.$$

(iii) *$\mathscr{D}$ is ideal.*

Proof. $(i) \Rightarrow (ii)$: This is obvious because $V(A) = V_{\mathscr{D}_0}(A)$ for the trivial decomposition $X = X$, abbreviated as $\mathscr{D}_0$ (see the introduction).

$(ii) \Rightarrow (iii)$: (cf. the example at the beginning of this section)

Suppose that $\mathscr{D}$ is not ideal. Then there exists $d = \prod_1^n (u_j, \varphi_j) \in S_{\mathscr{D}}$ with $\|P_d\| > 1$. Hence there exists z of norm 1 satisfying $\|P_d z\| = 1 + \delta > 1$. To $P_d(z)$ there exists a linear functional φ of norm 1 such that $\varphi(P_d z) = 1 + \delta$. The operator $\varphi \otimes z =: A$ is a contraction of rank 1. Hence, the $n \times n$-matrix B_d corresponding to A_d is of rank 1. This in turn implies that $\sigma(B_d) = \{0, \mathrm{trace}(B_d)\} \subset V_{\mathscr{D}}(A)$. But $\mathrm{trace}(B_d) = \varphi(P_d z) = 1 + \delta$, whereas $V(A) \subset \{\lambda : |\lambda| \leqslant 1\}$ since A is a contraction. This contradiction proves the stated implication.

$(iii) \Rightarrow (i)$: Let A be an arbitrary bounded linear operator and let $\tilde{\mathscr{D}}$ be an arbitrary coarsening of $\mathscr{D}$. Finally, let $\lambda \in V_{\mathscr{D}}(A)$ be arbitrary. Then there exists a $d \in S_{\mathscr{D}}$ and an $x \neq 0$ in $P_d(X) =: X_d$ satisfying $P_d A x = \lambda x$. By Lemma 3.5 there exists $\tilde{d} \in S_{\tilde{\mathscr{D}}}$ such that $P_{\tilde{d}} P_d = P_d$ as well as $P_{\tilde{d}} x = x$ hold. But then

$$P_{\tilde{d}} A x = P_{\tilde{d}} P_d A x = P_{\tilde{d}}(\lambda x) = \lambda P_{\tilde{d}} x = \lambda x,$$

from which $\lambda \in V_{\tilde{\mathscr{D}}}(A)$ follows. $\square$

This theorem together with Proposition 3.3 yields

COROLLARY 3.7. *Let $\mathscr{D}$ be a ρ-normed decomposition. Then $V_{\mathscr{D}}(A) \subset V_{\tilde{\mathscr{D}}}(A)$ for every coarsening $\tilde{\mathscr{D}}$ of $\mathscr{D}$.*

In the special case of p-direct sums this Corollary is due to P. Kallus, [12, Theorem 1.16].

Let now X be a complex Banach lattice. For notions not explained here we refer to [22, 16, 18]. We assume that X is order complete. Then all bands are projection bands. This is the case, for example, if the norm is order continuous. For an $x \in X$ we define $x^\perp = \{y : \inf(|x|, |y|) = 0\}$. This set is a band. Moreover for an arbitrary set $M \subset X$ we set $M^\perp = \bigcap_{x \in M} x^\perp$. Notice that $M \subset M^{\perp\perp}$. In the following we consider the band decomposition $X = x^\perp \times x^{\perp\perp}$.

The following proposition rests heavily on a deep result of T. Ando [1, Lemma 1].

PROPOSITION 3.8. *Let X be a complex Banach lattice of dimension $\dim(X) \geqslant 3$ and with order continuous norm. Then the following assertions are equivalent:*

(i) *For each bounded linear operator A on X and for each band decomposition $\mathscr{D}$ of order 2, we have the inclusion*

$$V_{\mathscr{D}}(A) \subset \overline{V(A)}.$$

(ii) *For each positive linear operator A and for each band decomposition $\mathscr{D}$ of order 2, we have the inclusion*

$$V_{\mathscr{D}}(A) \subset \overline{V(A)}.$$

(iii) *X is order isometrically isomorphic to an L^p-space for some $p \in [1, \infty)$ or to $c_0(\Gamma)$ for some discrete index set Γ.*

Proof. $(iii) \Rightarrow (i)$: This follows from Corollary 3.7 since every band decomposition is a p-direct sum.

$(i) \Rightarrow (ii)$: obvious.

$(ii) \Rightarrow (iii)$: Let F be an arbitrary sublattice of dimension 2. We show, that F is the range of a positive contraction. Then (iii) follows from [1, Lemma 1], and (in the case of $p = \infty$) in addition from our hypothesis that the norm is order continuous.

Since F is a 2-dimensional lattice there exist two positive unit vectors $u, v \in F$ and normalized positive linear forms φ and ψ such that $\inf\{u, v\} = 0$, $\varphi(u) = 1 = \psi(v)$, $\varphi(v) = 0 = \psi(u)$, and $F = \operatorname{span}\{u, v\}$. We consider the band decomposition $X = u^{\perp} \times u^{\perp\perp}$. The projection P_d with $d = ((v, \psi), (u, \varphi))$ is positive and satisfies $P_d(X) = F$.

Claim: P_d is contractive.

Proof: (we adapt the corresponding part of the proof of Theorem 3.6.) Assume that it is not contractive. Then there exists a positive normalized element z with $\|P_d z\| = 1 + \delta > 1$. Since $P_d z$ is positive there exists a positive linear form φ of norm 1 such that $\varphi(P_d z) = 1 + \delta$. $A = \varphi \otimes z$ is a *positive* contraction of rank 1. The same argument as in the proof of "$(ii) \Rightarrow (iii)$" of Theorem 3.6 shows $V_{\mathscr{D}}(A) \supset \sigma(B_d) \not\subset \overline{V(A)}$, a contradiction to (ii). This proves the claim and thereby the assertion. $\square$

4. Estimate of the resolvent

4.1. The main result

It is well known (see e. g. [9, Lemma 6.1-4]) that in the case of Hilbert spaces the resolvent of the operator A can be estimated in terms of the numerical range of A as follows:

$$\|(A - \lambda I)^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\lambda, V(A))}, \ \lambda \notin \overline{V(A)}.$$

C. Tretter and M. Wagenhofer [20, Theorem 4.2] gave the following generalization of the inequality above[1]:

THEOREM 4.1. *Let H be a Hilbert space, and let $\mathscr{D} : H = \prod_1^n H_j$ be an orthogonal decomposition of H. Let $A \in \mathscr{L}(H)$ be arbitrary. Then for $\lambda \notin \overline{V_{\mathscr{D}}(A)}$ the resolvent of A admits the estimate*

$$\|(A - \lambda)^{-1}\| \leqslant \frac{(\|A\| + |\lambda|)^{n-1}}{(\operatorname{dist}(\lambda, V_{\mathscr{D}}(A)))^n}. \tag{15}$$

It is our aim to prove a corresponding result for decompositions of arbitrary Banach spaces, see Theorem 4.3. The estimate we shall get differs from that one above only by a constant factor γ on the right hand side of the inequality depending solely on the decomposition $\mathscr{D}$, but not on the particular operator A.

Let $E = \mathbb{C}^n$. As before we denote by e_j the elements $e_j = (\delta_{j,k})$. Let $\mathscr{D}$ be a decomposition of X of order n and let $d = ((u_j, \varphi_j))_{1 \leqslant j \leqslant n} \in S_{\mathscr{D}}$ be arbitrary. We define

[1] We cite only that part of the Theorem which we want to generalize.

the mapping $T_d : P_d(X) =: X_d \to E$ by $T_d(\sum_1^n \xi_j u_j) = \sum_1^n \xi_j e_j$, and we set $\|\vec{\xi}\|_d := \|T_d^{-1}(\vec{\xi})\|$. Then all these norms satisfy $\|e_j\|_d = 1$ $(1 \leqslant j \leqslant n)$.

Unfortunately in general for $d \neq \tilde{d}$ the corresponding norms need not be equal as the following example, which is even contractive and ideal, shows:

EXAMPLE. Consider $X = \mathbb{C}^4$ with its canonical lattice structure and take the canonical band decomposition $X = \mathbb{C}^2 \times \mathbb{C}^2 = X_1 \times X_2$ Define the two seminorms $p_1(\vec{\xi},\vec{\eta}) = \max\{|\xi_1|,|\eta_1|\}$ and $p_2(\vec{\xi},\vec{\eta}) = |\xi_2| + |\eta_2|$. Then p, given by $p = \sqrt{p_1^2 + p_2^2}$ is a lattice norm on X which induces the usual Hilbert norm on each X_j. We parametrize a subset of $S_{\mathscr{D}}$ by the mapping $[0,1]^2 \ni (s,t) \mapsto ((x_s,x_s'),(y_t,y_t')) =: d_{s,t}$ where $x_s = (s,\sqrt{1-s^2})^T$ and $y_t = (t,\sqrt{1-t^2})^T$ (^{T} : transposition). Then $M := \{((x_s,x_s'),(y_t,y_t')) : s,t \in [0,1]\} \subset S_{\mathscr{D}}$ (see page 232). The corresponding norm $\|.\|_{d_{s,t}}$ on $\mathbb{C}^2$ is easily computed as

$$\|\vec{\xi}\|_{d_{s,t}}^2 = (\max\{|\xi_1|s,|\eta_1|t\})^2 + (|\xi_2|\sqrt{1-s^2} + |\eta_2|\sqrt{1-t^2})^2.$$

No two different subspaces $P_{d_{s,t}}(X)$, $P_{d_{u,v}}(X)$ are isometrically isomorphic.

Therefore the following notion is substantial for this section. At first let us introduce some helpful notations:

For $\vec{\xi} = \sum_1^n \xi_j e_j$ the norms $\|.\|_\infty$ and $\|.\|_1$ are defined by

$$\|\vec{\xi}\|_\infty = \max(|\xi_j|)_{1 \leqslant j \leqslant n},$$

$$\|\vec{\xi}\|_1 = \sum_1^n |\xi_j|.$$

DEFINITION 4.2. The decomposition $\mathscr{D}$ is called *bounded* if

$$\sup_{d \in S_{\mathscr{D}}} \sup_{0 \neq \vec{\xi}} \left(\frac{\|\vec{\xi}\|_\infty}{\|\vec{\xi}\|_d} \right) =: \gamma_{\mathscr{D}} < \infty.$$

The number $\gamma_{\mathscr{D}}$ is called the *bound of* $\mathscr{D}$.

First of all we show that many decompositions are bounded:

EXAMPLE.

1. Assume that all norms $\|.\|_d$ are monotone norms. Then $\gamma_{\mathscr{D}} = 1$. For let $\|.\|_d$ be such a monotone norm. Then $|\xi_j|e_j \leqslant |\vec{\xi}|$ $(1 \leqslant j \leqslant n)$ implies $|\xi_i| = \||\xi_i|e_i\|_d \leqslant \||\vec{\xi}|\|_d$, hence

$$\|\vec{\xi}\|_\infty \leqslant \||\vec{\xi}|\|_d = \|\vec{\xi}\|_d \leqslant \sum_1^n |\xi_j| \|e_j\|_d = \|\vec{\xi}\|_1.$$

2. For all p-direct sums and more generally for all ρ-normed decompositions we have $\|\vec{\xi}\|_d = \rho(\vec{\xi})$ for all d. In particular these decompositions have bound $\gamma_{\mathscr{D}} = 1$.

3. Let X be a Banach lattice and let $\mathscr{D}$ be a band decomposition of order n. For $d = \prod(u_j, \varphi_j) \in S_{\mathscr{D}}$ it follows that $|d| = \prod(|u_j|, |\varphi_j|) \in S_{\mathscr{D}}$. Then we have

$$\|(\xi_1 u_1, \dots, \xi_n u_n)\| = \||(\xi_1 u_1, \dots, \xi_n u_n)|\| = \|(|\xi_1||u_1|, \dots, |\xi_n||u_n|)\|$$

since $\inf(|u_i|, |u_j|) = 0$ for $i \neq j$. Thus $\|\cdot\|_d = \|\cdot\|_{|d|}$ holds and this latter norm is a monotone norm. So $\gamma_{\mathscr{D}} = 1$ by the first example.

Unfortunately not all decompositions are bounded as the following example shows:

EXAMPLE. First of all we define norms on $\mathbb{C}^2$ by

$$p_k(\vec{\xi})^2 = \left(1 - \frac{1}{2k^2}\right)|\xi_1 - \xi_2|^2 + \frac{1}{2k^2}|\xi_1 + \xi_2|^2.$$

Then

$$\sup_k \sup_{0 \neq \vec{\xi}} \left(\frac{\|\vec{\xi}\|_\infty}{p_k(\vec{\xi})}\right) = \infty.$$

To show this consider $\vec{\xi} = k \cdot 2^{-1/2}(e_1 + e_2)$. Then $p_k(\vec{\xi}) = 1$, but $\|\vec{\xi}\|_\infty = \frac{k}{\sqrt{2}}$.

Let $X_1 = X_2 = \ell_2(\mathbb{N})$ and equip $X = X_1 \times X_2$ with the norm p given by

$$p(x,y)^2 = \sum_1^\infty \left[p_k\left(\binom{x_k}{y_k}\right)\right]^2.$$

It is not hard to verify that this series converges for all $(x,y) \in X$, and moreover, that X is complete with respect to this norm, and that this norm induces the usual Hilbert space norm on each X_j.

Consider the decomposition $\mathscr{D} : X = X_1 \times X_2$. Let $e_k = (\delta_{j,k})_{j<\infty}$ and e'_k the corresponding element: $e'_k(x) = x_k$. Then $d_k = ((e_k, e'_k), (e_k, e'_k))$ is in $S_{\mathscr{D}}$ (see page 232) and the corresponding norm $\|.\|_{d_k}$ on $\mathbb{C}^2$ is nothing other than p_k. Thus $\mathscr{D}$ is unbounded.

The main theorem of this section, the announced generalization of Theorem 4.1, reads as follows:

THEOREM 4.3. *Let $\mathscr{D}$ be an arbitrary bounded decomposition of X of order $n \geqslant 2$ with bound $\gamma_{\mathscr{D}}$ and norm $\|\mathscr{D}\|$. Let A be an arbitrary bounded linear operator on X. Then for $\lambda \notin \overline{V_{\mathscr{D}}(A)}$ the resolvent of A admits the estimate*

$$\|(\lambda I - A)^{-1}\| \leqslant \gamma \cdot \frac{(\|A\| + |\lambda|)^{n-1}}{(\operatorname{dist}(\lambda, V_{\mathscr{D}}(A)))^n}, \tag{16}$$

where

$$\gamma := \gamma_{\mathscr{D}} \cdot n^2 \|\mathscr{D}\| \left(\sqrt{n-1} \cdot \max_i(\|P_i\|)\right)^{n-1} \tag{17}$$

depends only on the decomposition $\mathscr{D}$.

REMARKS.

1. It is an open problem whether there exists such an estimate for unbounded decompositions (obviously with another constant γ).

2. In the case of a contractive and ideal decomposition we obtain

$$\gamma = \gamma_{\mathscr{D}} \cdot n^2 (n-1)^{(n-1)/2}.$$

But even stronger estimates are possible in special cases as the following corollary shows.

COROLLARY 4.4. *In the following cases the constant γ in equation (16) can be replaced by smaller constants:*

(a) If $\mathscr{D}$ is a p-direct sum $(1 \leqslant p \leqslant \infty,\ p \neq 2)$ then

$$\gamma = n \cdot (n-1)^{(n-1)/2}$$

suffices.

(b) If $\mathscr{D}$ is a 2-direct sum then $\gamma = 1$ suffices.

(c) If X is an order complete Banach lattice, and if $\mathscr{D}$ is a band decomposition of order n then $\gamma = n^2 \|\mathscr{D}\| \cdot (n-1)^{(n-1)/2}$ suffices.

REMARK. Assertion (b) implies that Theorem 4.1 from [20] holds not only for orthogonal decompositions of *Hilbert* spaces but more generally for all 2-direct sums of *arbitray* Banach spaces.

In order to prove our theorem we need several lemmata.

LEMMA 4.5. *Let $\mathscr{D}$ be an arbitrary decomposition of X and let A be a bounded linear operator on X. Then*

$$\inf\{\|P_d Ax\| : x \in S_{P_d(X)},\ d \in S_{\mathscr{D}}\} \leqslant \|\mathscr{D}\| \inf\{\|Ay\| : y \in S_X\}.$$

Proof. Set $\delta = \inf\{\|P_d Ax\| : x \in S_{P_d(X)},\ d \in S_{\mathscr{D}}\}$ and suppose the inequality does not hold. Then there exists $y \in S_X$ satisfying $\|\mathscr{D}\|\|Ay\| < \delta$. Now by Lemma 2.7 there exists $d \in S_{\mathscr{D}}$ with $P_d y = y$, in particular $y \in S_{P_d(X)}$. We obtain

$$\delta > \|\mathscr{D}\|\|Ay\| \geqslant \|P_d Ay\| \geqslant \delta,$$

a contradiction. □

In the following we identify $\mathscr{L}(\mathbb{C}^k)$ with the space of $k \times k$-matrices. For a given norm p on $\mathbb{C}^k$ and $D = (d_{ij})_{i,j=1}^k$ we consider the corresponding operator norm $\|D\|_{op}$ and in addition the norm $\|D\|_\infty = \max_{i,j}(|d_{ij}|)$.

The following inequality goes back to Hadamard. Its proof can be found in [7, Cor. 9.24].

LEMMA 4.6. *Let $D=(d_{ij})$ be a $k\times k$ matrix. Then*

$$|\det(D)| \leqslant (\sqrt{k}\|D\|_\infty)^k$$

holds.

We introduce the following two vectors in $\mathbb{C}^n$, $(\mathbb{C}^n)'$, respectively: $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\mathbf{1}' = (1,\dots,1)$. Obviously $|\mathbf{1}'(\vec{\xi})| \leqslant \|\vec{\xi}\|_\infty \cdot n$ and $\|\mathbf{1}\| \leqslant n$ for every norm $\|.\|$ on $\mathbb{C}^n$ satisfying $\|e_j\| = 1$ for all j.

LEMMA 4.7. *Let $B=(b_{ij})$ be an invertible $n\times n$-matrix and let $\|.\|$ be a norm on $E := \mathbb{C}^n$ satisfying $\|e_j\| = 1$ ($j = 1\dots n$) as well as $\|\vec{\xi}\|_\infty \leqslant \delta\|\vec{\xi}\|$ for some δ and all $\vec{\xi}$.*

(a) In general

$$\|B^{-1}\|_{op} \leqslant \frac{n^2\delta}{|\det(B)|} \cdot \left(\|B\|_\infty \cdot \sqrt{n-1}\right)^{n-1} \tag{18}$$

holds.

(b) If the norm is monotone then the following in general sharper estimate holds:

$$\|B^{-1}\|_{op} \leqslant \frac{\|\mathbf{1}'\|' \cdot \|\mathbf{1}\|}{|\det(B)|} \cdot \left(\|B\|_\infty \cdot \sqrt{n-1}\right)^{n-1}, \tag{19}$$

where $\|.\|'$ denotes the dual norm induced by $\|.\|$.

(c) If the norm is the usual l^2-norm, then

$$\|B^{-1}\|_{op} \leqslant \frac{\|B\|_{op}^{n-1}}{|\det(B)|} \tag{20}$$

holds.

Proof. (a) Let $\vec{\xi} \in S_E$ be arbitrary. Then

$$B^{-1}\vec{\xi} = \frac{1}{\det(B)} \sum_i \left(\sum_j (-1)^{i+j} \det(B_{ji})\xi_j \right) e_i, \tag{21}$$

where B_{ij} is obtained from B by deleting the ith row and the jth column. Hence $\|B_{ij}\|_\infty \leqslant \|B\|_\infty$. Using Lemma 4.6 we obtain

$$\begin{aligned} |\sum_j (-1)^{i+j} \det(B_{ji})\xi_j| &\leqslant (\|B\|_\infty\sqrt{n-1})^{n-1} \sum_j |\xi_j| && (22) \\ &= (\|B\|_\infty\sqrt{n-1})^{n-1}\mathbf{1}'(|\vec{\xi}|). && (23) \end{aligned}$$

This inequality, the triangle inequality, and the fact that $\|e_i\| = 1$ holds for all $1 \leqslant i \leqslant n$, together imply

$$\|B^{-1}\vec{\xi}\| \leqslant \frac{n^2}{|\det(B)|}(\|B\|_\infty\sqrt{n-1})^{n-1}\|\vec{\xi}\|_\infty.$$

But $\|\vec{\xi}\|_\infty \leqslant \delta$ by hypothesis. So inequality (18) follows.

(b) Let the norm be monotone. At first we note that equation (21) together with inequality (22) imply

$$|B^{-1}\vec{\xi}| \leqslant \frac{1}{|\det(B)|}(\|B\|_\infty\sqrt{n-1})^{n-1}\mathbf{1}'(|\vec{\xi}|)\cdot\mathbf{1}. \tag{24}$$

Since $\|B^{-1}\vec{\xi}\| = \||B^{-1}\vec{\xi}|\|$ and $\|\vec{\xi}\| = \||\vec{\xi}|\| = 1$ the inequality follows.

(c) This inequality is due to T. Kato [13, footnote 2 on p. 28]. $\square$

Proof of Theorem 4.3. (I) Assume first of all $\lambda = 0$, i. e. $\operatorname{dist}(0, V_{\mathscr{D}}(A)) > 0$. Then by Theorem 2.8 we have $0 \notin \sigma(A)$. Moreover, by definition of $V_{\mathscr{D}}(A)$, $0 \notin \sigma(B_d)$ for every $d \in S_{\mathscr{D}}$. Lemma 4.5 implies: to every $\varepsilon > 0$ there exists $d \in S_{\mathscr{D}}$ such that $\|A^{-1}\| - \varepsilon < \|\mathscr{D}\|\|B_d^{-1}\|_{op}$ where $\|.\|_{op}$ denotes the operator norm induced by $\|.\|_d$. By Lemma 4.7, inequality (18), and by our hypothesis that $\mathscr{D}$ is bounded by $\gamma_{\mathscr{D}}$ we obtain

$$\|B_d^{-1}\|_{op} \leqslant \frac{n^2\gamma_{\mathscr{D}}}{|\det(B_d)|}\left(\|B_d\|_\infty \cdot \sqrt{n-1}\right)^{n-1}.$$

Now

$$\|B_d\|_\infty = \max_{i,j}(|\varphi_i(P_iAu_j)|) \leqslant \max_i(\|P_i\|)\|A\|$$

and

$$|\det(B_d)| \geqslant (\operatorname{dist}(0,\sigma(B_d))^n \geqslant (\operatorname{dist}(0,V_{\mathscr{D}}(A))^n$$

hold. Summing up our considerations we obtain

$$\|A^{-1}\| - \varepsilon \leqslant \frac{n^2\gamma_{\mathscr{D}}\cdot\|\mathscr{D}\|}{(\operatorname{dist}(0,V_{\mathscr{D}}(A)))^n}\cdot\left(\sqrt{n-1}\cdot\max_i(\|P_i\|)\|A\|\right)^{n-1}. \tag{25}$$

Since ε was arbitrary the theorem follows for the case $\lambda = 0$.

(II) In the general case replace A by $\lambda I - A$ and apply (I). $\square$

Proof of Corollary 4.4. (a) $\rho(\vec{\xi}) = (\sum_{i=1}^n |\xi_i|^p)^{1/p}$ is a monotone norm, hence all norms $\|.\|_d$ are equal to ρ. In particular $\gamma_{\mathscr{D}} = 1 = \|\mathscr{D}\| = \max_j \|P_j\|$ (for the first equality see the example following Definition 4.2). Moreover using equation (19) and the fact that $\|\mathbf{1}'\|_q \cdot \|\mathbf{1}\|_p = n$ we obtain the result.

(b) We start as in the proof of the theorem by the inequality $\|A^{-1}\| - \varepsilon < \|\mathscr{D}\|\|B_d^{-1}\|_{op}$ for given $\varepsilon > 0$. Now $\|B_d\|_{op} = \|P_dA|_{P_d(X)}\| \leqslant \|A\|$ holds since $\|P_d\| = 1$. Equation (20) yields

$$\|B_d^{-1}\|_{op} \leqslant \frac{\|A\|^{n-1}}{|\det(B_d)|} \leqslant \frac{\|A\|^{n-1}}{(\operatorname{dist}(0,V_{\mathscr{D}}(A)))^n}.$$

Inserting this into the inequality above we obtain

$$\|A^{-1}\| - \varepsilon \leqslant \frac{\|A\|^{n-1}}{(\operatorname{dist}(0, V_{\mathscr{D}}(A)))^n},$$

since $\|\mathscr{D}\| = 1$ (see p. 231), and the assertion follows.

(c) In this case $\|\gamma_{\mathscr{D}}\| = \max(\|P_j\|) = 1$ (see Example 3 on p. 245) and the assertion follows from the theorem. Notice that $\|\mathscr{D}\| > 1$ is possible as the example on p. 239 shows. $\square$

4.2. The case $n = 2$

As the application of Corollary 4.4 to $A = I$ (the identity operator) shows, the estimate is still rather rough. For ρ-normed decompositions of order 2 a much sharper estimate is possible.

Let $B = (b_{ij})$ be a 2×2 matrix acting as an operator on the Banach lattice $E = \mathbb{C}^2$, equipped with the lattice norm ρ. Let $C = (c_{ij})$ satisfy $c_{ij} \geqslant |b_{ij}|$ for all i, j. Then

$$\|B\|_{op} \leqslant \|(|b_{ij}|)\|_{op} \leqslant \|C\|_{op} \tag{26}$$

holds, where $\|.\|_{op}$ denotes the operator norm.

Let us consider a ρ-normed decomposition $X = X_1 \times X_2$ of the Banach space X into the Banach spaces X_1 and X_2, abbreviated as $\mathscr{D}$. For $A \in \mathscr{L}(X)$, $A = (A_{ik})$ we set

$$M(A) = \begin{pmatrix} \|A_{22}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{11}\| \end{pmatrix}$$

and $\kappa(A) = \|M(A)\|_{op}$, where $M(A)$ operates on $(\mathbb{C}^2, \rho)$. (Please note that A_{22} is the first element in the matrix $M(A)$ and not A_{11}.) For $\lambda \in \mathbb{C}$ $\kappa(\lambda I - A) \leqslant \kappa(A) + |\lambda|$ holds on account of inequality (26).

Now we can formulate our supplementary result:

PROPOSITION 4.8. *Let $\mathscr{D}$ be a ρ-normed decomposition of order $n = 2$ and let $A \in \mathscr{L}(X)$ be arbitrary. Let $\lambda \notin \overline{V_{\mathscr{D}}(A)}$. Then*

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{\kappa(A) + |\lambda|}{(\operatorname{dist}(\lambda, V_{\mathscr{D}}(A)))^2}. \tag{27}$$

Proof. First of all let $\lambda = 0$. As in the proof of Theorem 4.3 to every $\varepsilon > 0$ we find B_d satisfying $\|A^{-1}\| - \varepsilon \leqslant \|B_d^{-1}\|_{op}$ for some $d = ((u_1, \varphi_1), (u_2, \varphi_2))$. Now

$$B_d^{-1} = \frac{1}{\det(B_d)} \begin{pmatrix} \varphi_2(A_{22}u_2) & -\varphi_1(A_{12}u_2) \\ -\varphi_2(A_{21}u_1) & \varphi_1(A_{11}u_1) \end{pmatrix}.$$

So by using inequality (26) for $C = M(A)/|\det(B_d)|$ we find

$$\|A^{-1}\| - \varepsilon \leqslant \|B_d^{-1}\|_{op} \leqslant \frac{\kappa(A)}{|\det(B_d)|} \leqslant \frac{\kappa(A)}{(\operatorname{dist}(0, V_{\mathscr{D}}(A)))^2}.$$

If $\lambda \neq 0$ then apply the preceding proof to $\lambda I - A$. $\square$

REMARKS.

1. Replacing $\|A\|$ by $\kappa(A)$ in inequality (15) by C. Tretter and M. Wagenhofer yields our inequality (27).

2. For $A = I$ and $\lambda \neq 1$ inequality (27) yields $|\lambda - 1|^{-1} \leqslant \frac{1+|\lambda|}{|\lambda-1|^2}$ showing that in the case of $n = 2$ this inequality is an improvement of Corollary 4.4 (a). Indeed for $\lambda = 0$ one obtains even equality.

3. For the norm $\rho = \|.\|_\infty$ we get

$$\kappa(A) = \max\left(\|A_{22}\| + \|A_{12}\|, \|A_{21}\| + \|A_{11}\|\right).$$

For $1 < p < \infty$ and $\rho = \|.\|_p$ it is rather difficult to calculate the operator norm. So one may take the following estimate using the so called $L_{q,p}$ norm ($q = \frac{p}{p-1}$) $\|M(A)\|_{q,p}$ of $M(A)$:

$$\kappa(A)^p \leqslant (\|A_{22}\|^q + \|A_{12}\|^q)^{p/q} + (\|A_{21}\|^q + \|A_{11}\|^q)^{p/q}.$$

The right hand side of this inequality is the so-called $L_{q,p}$ norm $\|M(A)\|_{q,p}$ of $M(A)$. For $p = 2$ it is the Hilbert-Schmidt norm.

To see this inequality, let C be an $n \times n$ matrix with rows $\vec{c}_j$, $j = 1, \ldots, n$ and let x be a vector with $\|x\|_p = 1$. Then using Hölder's inequality we obtain

$$\|Cx\|^p = \sum_j |\vec{c}_j \cdot x|^p \leqslant \sum_j (\|\vec{c}_j\|_q \cdot \|x\|_p)^p = \sum_j (\|\vec{c}_j\|_q)^p =: \beta,$$

and hence $\|C\|_{op}^p \leqslant \beta$.

4.3. Examples

1. Let $X = (\mathbb{C}^4, \|.\|_\infty) = \mathbb{C}^2 \times \mathbb{C}^2$ ($\|.\|_\infty$ the maximum norm). We consider $A = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}$ where $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $V_{\mathscr{D}}(A) = \mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Proof. $V_{\mathscr{D}}(A) \subseteq \mathbb{D}$ holds by Propositions 1.3 and 3.3.

In order to show equality choose $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ \delta \end{pmatrix}$, where $|\delta| = 1$. Moreover let $\varphi_1 = (1,0)$ and $\varphi_{2,t} = ((1-t), \bar{\delta}t)$, $0 \leqslant t \leqslant 1$. Then $d = ((u_1, \varphi_1), (u_2, \varphi_{2,t})) \in S_{\mathscr{D}}$ and $B_d = \begin{pmatrix} 0 & \delta \\ (1-t) + \bar{\delta}t & 0 \end{pmatrix}$. So $\lambda_{1,2}^2 = \delta(1-t) + 1 \cdot t$ holds for the eigenvalues $\lambda_{1,2}$ of B_d. Notice that the lines $\{\delta(1-t) + 1 \cdot t : 0 \leqslant t \leqslant 1, |\delta| = 1\}$ exhaust $\mathbb{D}$. Therefore for every $\lambda \in \mathbb{D}$ there exist δ, t such that $\lambda^2 = \delta(1-t) + 1 \cdot t$. $\square$

$A^2 = I$ implies $(\lambda I - A)^{-1} = \frac{1}{\lambda^2 - 1} \cdot (\lambda I + A)$ for $\lambda \notin \sigma(A)$, in particular for $|\lambda| > 1$. This implies $\|(\lambda I - A)^{-1}\| = \frac{1}{\lambda - 1}$ for $\lambda > 1$. Now $\kappa(A) = \|A\| = 1$, hence for $\lambda > 1$ inequality (27) yields

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{\lambda+1}{(\lambda-1)^2},$$

whereas Corollary 4.4, part (a) gives

$$\|(\lambda I - A)^{-1}\| \leqslant 2 \cdot \frac{\lambda+1}{(\lambda-1)^2}.$$

2. let $X = \ell^\infty(\mathbb{Z}) = \ell^\infty(\mathbb{Z}_-) \times \ell^\infty(\mathbb{Z}_+)$ where $\mathbb{Z}_+ = \{z \in \mathbb{Z} : z \geqslant 0\}$ and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$. Let A be the shift on X, given by $Af(z) = f(z+1)$, and let $S_\pm$ be its restriction to the corresponding subspaces. Then the matrix representation with respect to the decomposition $\mathscr{D}$ is $A = \begin{pmatrix} S_- & e_0' \otimes e_{-1} \\ 0 & S_+ \end{pmatrix}$. Here $e_k(l) = \delta_{kl}$ and $e_k'(f) = f(k)$. So every B_d is upper triangular hence $V_{\mathscr{D}}(A) = V(S_+) \cup V(S_-)$. We show that both sets are equal to $\mathbb{D}$. Because $\|S_\pm\| = 1$ $V(S_\pm) \subset \mathbb{D}$. Let $\lambda \in \mathbb{D}$ be arbitrary, $\lambda = \delta \cdot |\lambda|$, $|\lambda| = t$. Set $u = e_1 + \delta e_2$ and $\varphi = te_1' + \bar{\delta}(1-t)e_2'$. Then $\varphi(S_+u) = \varphi(e_0 + \delta e_1) = \delta t = \lambda$. Similarly we obtain the result for S_-.

Now $\|A\| = 1$. Let $\lambda > 1$ be arbitrary. Then $\|(\lambda I - A)^{-1}\| = \frac{1}{\lambda - 1}$, and $M(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, hence $\kappa(A) = 2$. So inequality (27) gives

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{\lambda+2}{(\lambda-1)^2},$$

whereas Corollary 4.4 (a) yields

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{2\lambda+2}{(\lambda-1)^2}.$$

FINAL REMARK. First applications based on a preliminary version of our theory may be found in [17, 18]. Further applications in particular for positive operators on Banach lattices are in preparation.

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