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SCALAR APPROXIMANTS OF QUADRATIC OPERATORS WITH APPLICATIONS

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Abstract. Among other results, we find the best scalar approximant of a quadratic operator with respect to the numerical radius and the operator norm. We use these results to give estimates for the numerical radii of products and commutators of quadratic operators.

1. Introduction

In what follows, all operators are bounded linear operators on a complex Hilbert space H. Let $W(A)$ and $w(A)$ denote, respectively, the *numerical range* and the *numerical radius* of A defined as

$$W(A) = \{\langle Ax, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

and

$$w(A) = \sup_{\lambda \in W(A)} |\lambda|,$$

where $\langle \cdot,\cdot\rangle$ and $\|\cdot\|$ in the definition of $W(A)$ are the inner product and its corresponding norm on H. It is well-known that $W(A)$ is a bounded convex subset of the complex plane, which contains the spectrum of A in its closure. Moreover, if H is finite dimensional, then $W(A)$ is compact. The numerical radius $w(\cdot)$ defines a norm on the space of all bounded linear operators on H, which is equivalent to the operator norm. In fact, the inequalities

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\| \tag{1}$$

hold. For proofs and more facts about the numerical range and the numerical radius, we refer the reader to [8] and [10]. Recall that A is *quadratic* if $A^2 + \alpha A + \beta I = 0$ for some scalars α and β. A is *square-zero* if $A^2 = 0$, *idempotent* if $A^2 = A$ and *involution* if $A^2 = I$.

In Section 2, we prove some results regarding quadratic operators. Among other results, we show that if A is a quadratic operator with spectrum $\sigma(A) = \{a, b\}$, then

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$d(A) = w\left(A - \frac{a+b}{2}I\right)$ and $D(A) = \left\|A - \frac{a+b}{2}I\right\|$. Here, $d(A)$ and $D(A)$ denote the distances of A from scalar operators, with respect to the numerical radius and the operator norm, respectively, that is,

$$d(A) = \inf_{\lambda \in \mathbb{C}} w(A - \lambda I)$$

and

$$D(A) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|.$$

In Section 3, the results of Section 2 are employed to give estimates for the numerical radii of the products AB and the commutators $AB \pm BA$ when A is quadratic.

2. Scalar approximants of quadratic operators

In order to achieve our goals, we need the following lemmas. The first lemma is well-known. The second lemma, which can be found in [11], gives estimates for the operator norms of the 2×2 operator matrices. The third lemma describes the spectrum, the canonical form and the numerical range of a quadratic operator. It is from [17].

LEMMA 1. *For every A,*

$$w(A) = \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}(e^{i\theta}A)\|.$$

Replacing A by iA in the previous identity, we have

$$w(A) = \sup_{\theta \in \mathbb{R}} \|\operatorname{Im}(e^{i\theta}A)\|.$$

LEMMA 2. *Let $A \in B(H_1, H_1)$, $B \in B(H_2, H_1)$, $C \in B(H_1, H_2)$ and $D \in B(H_2, H_2)$. Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

Here $B(H_j, H_i)$ is the space of all bounded linear operators from H_j to H_i.

LEMMA 3. *Let A be a quadratic operator satisfying $A^2 + \alpha A + \beta I = 0$ for some scalars α and β. Then*

(a) $\sigma(A) = \{a, b\}$, where a and b are the roots of the quadratic equation $z^2 + \alpha z + \beta = 0$,

(b) A is unitarily equivalent to an operator of the form

$$aI_1 \oplus bI_2 \oplus \begin{bmatrix} aI_3 & T \\ 0 & bI_3 \end{bmatrix},$$

where T is positive definite, and

(c) $W(A)$ is the elliptic disc with foci a and b, major axis of length $\|A - aI\|$ and minor axis of length $\sqrt{\|A - aI\|^2 - |a - b|^2}$.

The follwing lemma is a useful tool used in this paper.

LEMMA 4. *Let* $A = aI_1 \oplus bI_2 \oplus \begin{bmatrix} aI_3 & T \\ 0 & bI_3 \end{bmatrix}$ *be a quadratic operator and let* $A' = \begin{bmatrix} a & \|T\| \\ 0 & b \end{bmatrix}$. *Then*

(a) $\|A\| = \|A'\|$,

(b) $\|A^*A + AA^*\| = \|A'^*A' + A'A'^*\|$,

(c) $\overline{W(A)} = W(A')$ *and* $w(A) = w(A')$.

Proof. (a) For a proof, see, e.g., [17, Lemma 2.2] and [1, Lemma 3.2].

(b) Without loss of generality, we prove the assertion for $A = \begin{bmatrix} aI & T \\ 0 & bI \end{bmatrix}$ on $H \oplus H$. First, notice that

$$\begin{aligned} \|A'^*A' + A'A'^*\| &= \left\| \begin{bmatrix} 2|a|^2 + \|T\|^2 & \overline{(a+b)}\|T\| \\ (a+b)\|T\| & 2|b|^2 + \|T\|^2 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 2|a|^2 + \|T\|^2 & |a+b|\,\|T\| \\ |a+b|\,\|T\| & 2|b|^2 + \|T\|^2 \end{bmatrix} \right\|. \end{aligned}$$

Hence,

$$\|A'^*A' + A'A'^*\| = \left\langle \begin{bmatrix} 2|a|^2 + \|T\|^2 & |a+b|\,\|T\| \\ |a+b|\,\|T\| & 2|b|^2 + \|T\|^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle$$

for some scalars α and β with $|\alpha|^2 + |\beta|^2 = 1$. By Lemma 2,

$$\begin{aligned} \|A^*A + AA^*\| &= \left\| \begin{bmatrix} 2|a|^2 I + TT^* & \overline{(a+b)}T \\ (a+b)T^* & 2|b|^2 I + T^*T \end{bmatrix} \right\| \\ &\leqslant \left\| \begin{bmatrix} 2|a|^2 + \|T\|^2 & |a+b|\,\|T\| \\ |a+b|\,\|T\| & 2|b|^2 + \|T\|^2 \end{bmatrix} \right\| \qquad (2) \\ &= \|A'^*A' + A'A'^*\|. \end{aligned}$$

Now, let $\{x_n\}$ and $\{y_n\}$ be sequences of unit vectors in H such that $|\langle Ty_n, x_n\rangle| \to \|T\|$ (as $n \to \infty$) and let θ_n be a real number satisfying $\overline{(a+b)}\langle Ty_n, x_n\rangle = |a+b|\,|\langle Ty_n, x_n\rangle|\,e^{i\theta_n}$. Consider the sequence $\{z_n\} = \{\alpha e^{i\theta_n} x_n \oplus \beta y_n\}$. It is easy to see that $\{z_n\}$ is a sequence of unit vectors in $H \oplus H$ and

$$\begin{aligned} &\langle (A^*A + AA^*) z_n, z_n\rangle \\ &= (2|a|^2 + \|T^*x_n\|^2)|\alpha|^2 + 2|a+b|\,|\langle Ty_n, x_n\rangle|\,\mathrm{Re}(\overline{\alpha}\beta) + (2|b|^2 + \|Ty_n\|^2)|\beta|^2 \\ &\to (2|a|^2 + \|T\|^2)|\alpha|^2 + 2|a+b|\,\|T\|\,\mathrm{Re}(\overline{\alpha}\beta) + (2|b|^2 + \|T\|^2)|\beta|^2 \quad (\text{as } n \to \infty) \\ &= \left\langle \begin{bmatrix} 2|a|^2 + \|T\|^2 & |a+b|\,\|T\| \\ |a+b|\,\|T\| & 2|b|^2 + \|T\|^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \\ &= \|A'^*A' + A'A'^*\|. \end{aligned}$$

Thus,

$$\|A^*A+AA^*\| \geqslant \|A^{'*}A^{'}+A^{'}A^{'*}\|. \tag{3}$$

By the inequalities (2) and (3), we deduce that

$$\|A^*A+AA^*\| = \|A^{'*}A^{'}+A^{'}A^{'*}\|,$$

as required.

(c) The assertion follows from Lemma 3 (c) by recalling that

$$\|A-aI\| = \left\| \begin{bmatrix} 0 & T \\ 0 & (b-a)I_3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & \|T\| \\ 0 & b-a \end{bmatrix} \right\| = \|A^{'}-aI^{'}\|,$$

where $I^{'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

REMARK 1. Let $A = aI_1 \oplus bI_2 \oplus \begin{bmatrix} aI_3 & T \\ 0 & bI_3 \end{bmatrix}$ be a quadratic operator and let $A^{'} = \begin{bmatrix} a & \|T\| \\ 0 & b \end{bmatrix}$.

(a) It follows from Lemma 4 (a) and (b) that

$$\|A\| = \frac{1}{\sqrt{2}}\sqrt{|a|^2+|b|^2+\|T\|^2+\sqrt{\left(|a|^2+|b|^2+\|T\|^2\right)^2-4\,|a|^2\,|b|^2}} \tag{4}$$

and

$$\|A^*A+AA^*\| = |a|^2+|b|^2+\|T\|^2+\sqrt{\left(|a|^2-|b|^2\right)^2+|a+b|^2\,\|T\|^2}. \tag{5}$$

Using a similar argument as in the proof of Lemma 4 (b), we can show that

$$\|\mathrm{Re}A\| = \|\,\mathrm{Re}A^{'}\| = \frac{1}{2}\left(\mathrm{Re}\,(a+b)+\sqrt{\mathrm{Re}^2\,(a-b)+\|T\|^2}\right) \tag{6}$$

and

$$\|\mathrm{Im}A\| = \|\,\mathrm{Im}A^{'}\| = \frac{1}{2}\left(\mathrm{Im}\,(a+b)+\sqrt{\mathrm{Im}^2\,(a-b)+\|T\|^2}\right). \tag{7}$$

(b) By Lemma 4 (c), in order to give a formula for $w(A)$, we need to calculate $w(A^{'})$. In fact, no explicit formula for the numerical radius of a general 2×2 matrix is known. In [12, Theorem 1.1], a general formula was given. However, its use is limited since it involves calculating a specific root of a quartic equation. For instance, it follows from the formula (1.4) in [12, Theorem 1.1] that if $a\bar{b}$ is real, then

$$w(A) = \frac{1}{2}\left(|a+b|+\sqrt{|a-b|^2+\|T\|^2}\right). \tag{8}$$

Formula (8) can also be proved as follows: When $a\overline{b}$ is real, then a and b lie on a line passing through the origin. By Lemma 3 (c), $W(A)$ is an elliptic disc with center $\frac{a+b}{2}$ and major axis of length $\|A-aI\|$. Hence,

$$\begin{aligned} w(A) &= \frac{1}{2}|a+b| + \frac{1}{2}\|A-aI\| \\ &= \frac{1}{2}\left(|a+b| + \sqrt{|a-b|^2 + \|T\|^2}\right). \end{aligned}$$

(c) It follows from the formula (4) of $\|A\|$, using simple computation, that

$$\|T\|^2 = \|A\|^2 + |a|^2|b|^2\|A\|^{-2} - (|a|^2+|b|^2). \tag{9}$$

Hence, the formulas in Lemma 5 and the formulas (5), (6), (7) and (8) can be written in terms of a,b and $\|A\|$ (rather than $\|T\|$).

(d) $A^2 = a^2I_1 \oplus b^2I_2 \oplus \begin{bmatrix} a^2I_3 & (a+b)T \\ 0 & b^2I_3 \end{bmatrix}$ is quadratic and (by Remark 1 (c))

$$|a+b|^2\|T\|^2 = \left\|A^2\right\|^2 + |a|^4|b|^4\left\|A^2\right\|^{-2} - (|a|^4+|b|^4). \tag{10}$$

The following proposition is a consequence of the formulas (5), (9) and (10).

PROPOSITION 1. *Let A be a quadratic operator with $\sigma(A) = \{a,b\}$.*
(a) If A is square-zero (i.e., $a=b=0$), then

$$\|A^*A + AA^*\| = \|A\|^2.$$

(b) Otherwise,

$$\|A^*A + AA^*\| = \|A\|^2 + \left\|A^2\right\| - |a|^2|b|^2\left(\left\|A^2\right\|^{-1} - \|A\|^{-2}\right).$$

REMARK 2. It follows from Proposition 1 that if A is quadratic, then

$$\|A^*A + AA^*\| \leqslant \|A\|^2 + \left\|A^2\right\|. \tag{11}$$

In fact, the inequality (11) holds for every A. See, e.g., [13, Lemma 7].

In the following lemma, we show that the operator $\frac{a+b}{2}I$ is the best scalar approximant of A with respect to the numerical radius and the operator norm.

LEMMA 5. *Let $A = aI_1 \oplus bI_2 \oplus \begin{bmatrix} aI_3 & T \\ 0 & bI_3 \end{bmatrix}$ be a quadratic operator. Then*
(a) $d(A) = w\left(A - \frac{a+b}{2}I\right) = \frac{1}{2}\sqrt{|a-b|^2 + \|T\|^2}$ and
(b) $D(A) = \left\|A - \frac{a+b}{2}I\right\| = \frac{1}{2}\left(\sqrt{|a-b|^2 + \|T\|^2} + \|T\|\right)$.

Proof. (a) It is not hard to see that $d(A)$ equals the radius of the smallest closed circular disc containing $W(A)$. The center of this disc is the scalar λ_0 for which $d(A) = w(A-\lambda_0 I)$. Hence, by Lemma 3 (c), we have

$$\begin{aligned} d(A) &= w\left(A - \tfrac{a+b}{2} I\right) \\ &= \frac{1}{2}\|A - aI\| = \frac{1}{2}\sqrt{|a-b|^2 + \|T\|^2}. \end{aligned}$$

(b) Let λ be an arbitrary scalar. Using Lemma 4 (a), it is sufficient to show that $\|A' - \frac{a+b}{2} I'\| \leqslant \|A' - \lambda I'\|$, where $A' = \begin{bmatrix} a & \|T\| \\ 0 & b \end{bmatrix}$ and $I' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Observe that

$$\begin{aligned} 2\|A' - \lambda I'\|^2 &= |a-\lambda|^2 + |b-\lambda|^2 + \|T\|^2 \\ &\quad + \sqrt{\left(|a-\lambda|^2 + |b-\lambda|^2 + \|T\|^2\right)^2 - 4|a-\lambda|^2|b-\lambda|^2} \\ &= |a-\lambda|^2 + |b-\lambda|^2 + \|T\|^2 \\ &\quad + \sqrt{\left(|a-\lambda|^2 - |b-\lambda|^2\right)^2 + 2\left(|a-\lambda|^2 + |b-\lambda|^2\right)\|T\|^2 + \|T\|^4} \\ &\geqslant |a-\lambda|^2 + |b-\lambda|^2 + \|T\|^2 + \sqrt{2\left(|a-\lambda|^2 + |b-\lambda|^2\right)\|T\|^2 + \|T\|^4}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2}|a-b|^2 &= \frac{1}{2}|a-\lambda-(b-\lambda)|^2 \\ &\leqslant \frac{1}{2}(|a-\lambda| + |b-\lambda|)^2 \leqslant |a-\lambda|^2 + |b-\lambda|^2, \end{aligned}$$

we have

$$\begin{aligned} 2\|A' - \lambda I'\|^2 &\geqslant \frac{1}{2}|a-b|^2 + \|T\|^2 + \sqrt{|a-b|^2\|T\|^2 + \|T\|^4} \\ &= \frac{1}{2}\left(\sqrt{|a-b|^2 + \|T\|^2} + \|T\|\right)^2 = 2\|A' - \tfrac{a+b}{2} I'\|^2, \end{aligned}$$

as required. $\square$

REMARK 3. (a) In connection with Lemma 5, it is worth mentioning that, for every A, if $\|A - \lambda_0 I\| = D(A)$ or $w(A - \lambda_0 I) = d(A)$, then $\lambda_0 \in \overline{W(A)}$. For the case of the operator norm, see [16, Theorem 4] or, for finite matrices, [5, Lemma 5]. The proof for the numerical radius case is easy.

(b) Lemma 5 (b) for 2×2 matrices has been known years ago. For example, it was noted in [9, Section 13].

3. Estimates for the numerical radii of products and commutators of quadratic operators

The numerical radius is not submultiplicative. In fact, the inequality

$$w(AB) \leqslant w(A)\, w(B) \tag{12}$$

is not true even for commuting operators A and B. By the inequalities in (1), we have

$$w(AB) \leqslant 2\,\|A\|\, w(B)\,. \tag{13}$$

The constant 2 in the inequality (13) is best possible (see, e.g., [10, p. 118]). In [14], it has been shown, using a 12-dimensional example, that the inequality

$$w(AB) \leqslant \|A\|\, w(B) \tag{14}$$

(which is weaker than the inequality (12)) is also not true even if A and B commute. One of the known conditions so that the inequality (14) holds is the double commutativity of A and B (i.e., $AB = BA$ and $AB^* = B^*A$). Recently, it has been shown in [18, Theorem 5] that if A is quadratic and $AB = BA$, then $w(AB) \leqslant w(A)\,\|B\|$. The question whether, under the same condition, $w(AB) \leqslant \|A\|\, w(B)$ holds is still open, although it is known to be true if A is square-zero, idempotent (see [7, Proposition 2 and Theorem 3, respectively]) or involution (see [15]). Also, it is known (see [6, Theorem 7 and Theorem 11]) that for every A and B,

$$w(AB \pm BA) \leqslant 4w(A)w(B) \tag{15}$$

and

$$w(AB \pm BA) \leqslant 2\sqrt{2}\,\|A\|\, w(B). \tag{16}$$

The constants 4 and $2\sqrt{2}$ in the previous inequalities are best possible. In [2], [3] and [4], many refinements of the inequalities (13), (15) and (16) have been established. Among other inequalities, the following were given.

LEMMA 6. *For every A and B,*

$$w(AB) \leqslant (\|A\| + D(A))\, w(B), \tag{17}$$

$$w(AB + BA) \leqslant 2\,(w(A) + d(A))\, w(B) \tag{18}$$

and

$$w(AB + BA) \leqslant \sqrt{\|A^*A + AA^*\|}\sqrt{\|B^*B + BB^*\|}. \tag{19}$$

REMARK 4. It is obvious that the inequalities (17) and (18) refine the inequalities (13) and (15), respectively. The inequality (19) refine both inequalities (15) and (16). To see this, it is sufficient to show that for every A,

$$\|A^*A + AA^*\| \leqslant 4w^2(A)\,. \tag{20}$$

Notice that $A^*A+AA^* = 2(\mathrm{Re}^2 A + \mathrm{Im}^2 A)$. Hence, by the triangle inequality and Lemma 1, we have

$$\|A^*A+AA^*\| \leqslant 2(\|\mathrm{Re}A\|^2 + \|\mathrm{Im}A\|^2) \leqslant 4w^2(A).$$

Since

$$\|A\|^2 = \|A^*A\| \leqslant \|A^*A+AA^*\|,$$

we have

$$\|A\|^2 \leqslant \|A^*A+AA^*\| \leqslant 4w^2(A). \tag{21}$$

This proves that the inequality (20) is a refined version of the inequality

$$\|A\| \leqslant 2w(A). \tag{22}$$

The inequalities in (21) are sharp. In fact, if A is square-zero, then

$$\|A\|^2 = \|A^*A+AA^*\| = 4w^2(A).$$

In [1, Theorem 2.4], the following refinement of the inequality (20) has been given. For every A,

$$\|A^*A+AA^*\| \leqslant 4w^2(A) - 2m(A^2), \tag{23}$$

where $m(A^2) = \inf\limits_{\|x\|=1} \left|\left\langle A^2x, x\right\rangle\right|$. A refinement of the inequality (20) (different from (23)) and a refinment of (22) are given in the following proposition.

PROPOSITION 2. *For every A,*

$$\|A\| \leqslant w(A) + d(A) \tag{24}$$

and

$$\|A^*A+AA^*\| \leqslant 2\left(w^2(A) + d^2(A)\right). \tag{25}$$

Proof. Let λ_0 be the scalar satisfying $w(A-\lambda_0 I) = d(A)$ and let θ be a real number such that $\lambda_0 = |\lambda_0| e^{i\theta}$. Then, we have for every A,

$$\|A\| = \|\mathrm{Re}A + i\,\mathrm{Im}A\| \leqslant \|\mathrm{Re}A\| + \|\mathrm{Im}A\|.$$

Replacing A in the above inequality by $e^{-i\theta}A$, we get

$$\begin{aligned} \|A\| &= \|\mathrm{Re}(e^{-i\theta}A) + i\,\mathrm{Im}(e^{-i\theta}A)\| \\ &\leqslant \|\mathrm{Re}(e^{-i\theta}A)\| + \|\mathrm{Im}(e^{-i\theta}A)\| \\ &= \|\mathrm{Re}(e^{-i\theta}A)\| + \|\mathrm{Im}(e^{-i\theta}(A-\lambda_0 I)\|. \end{aligned}$$

Hence, by Lemma 1,

$$\|A\| \leqslant w(e^{-i\theta}A) + w(e^{-i\theta}(A-\lambda_0)) = w(A) + d(A).$$

The proof of the inequality (25) follows by recalling that $A^*A+AA^* = 2(\mathrm{Re}^2 A + \mathrm{Im}^2 A)$ and mimcking the proof of the inequality (24). $\square$

The following theorem is a direct consequence of Lemma 6, Lemma 5 and Remark 1 (c).

THEOREM 1. *Let A be a quadratic operator with $\sigma(A) = \{a, b\}$. Then for every B,*

$$w(AB) \leqslant \left(\|A\| + \frac{1}{2} \left(\sqrt{|a-b|^2 + c^2} + c \right) \right) w(B), \tag{26}$$

$$w(AB + BA) \leqslant 2 \left(w(A) + \frac{1}{2} \sqrt{|a-b|^2 + c^2} \right) w(B) \tag{27}$$

and

$$w(AB + BA) \leqslant \sqrt{|a|^2 + |b|^2 + c^2 + \sqrt{\left(|a|^2 - |b|^2\right)^2 + |a+b|^2 c^2}} \sqrt{\|B^*B + BB^*\|},$$

where $c = \sqrt{\|A\|^2 + |a|^2 |b|^2 \|A\|^{-2} - \left(|a|^2 + |b|^2\right)}$.

For every A and B, recall that

$$w(AB - BA) \leqslant 4w(A)w(B) \tag{28}$$

and

$$w(AB - BA) \leqslant 2\sqrt{2}\,\|A\|\, w(B). \tag{29}$$

By replacing A and B in the inequality (28), by $A - \lambda_0 I$ and $B - \lambda_1 I$, respectively, where λ_0 and λ_1 are scalars satisfying $w(A - \lambda_0 I) = d(A)$ and $w(B - \lambda_1 I) = d(B)$, we have

$$\begin{aligned} w(AB - BA) &= w((A - \lambda_0 I)(B - \lambda_1 I) - (B - \lambda_1 I)(A - \lambda_0 I)) \\ &\leqslant 4w(A - \lambda_0 I)\, w(B - \lambda_1 I). \end{aligned}$$

Hence,

$$w(AB - BA) \leqslant 4d(A)\, d(B).$$

Similarly, by considering the inequality (29), one can show that

$$w(AB - BA) \leqslant 2\sqrt{2} D(A) d(B).$$

Hence, we have the following theorem by Lemma 5 and Remark 1 (c).

THEOREM 2. *Let A be a quadratic operator with $\sigma(A) = \{a, b\}$. Then for every B,*

$$w(AB - BA) \leqslant 2\sqrt{|a-b|^2 + c^2}\, d(B)$$

and

$$w(AB - BA) \leqslant \sqrt{2} \left(\sqrt{|a-b|^2 + c^2} + c \right) d(B),$$

where $c = \sqrt{\|A\|^2 + |a|^2 |b|^2 \|A\|^{-2} - \left(|a|^2 + |b|^2\right)}$.

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