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UC–MAJORIZATION AND ITS STRONGLY LINEAR PRESERVERS

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Abstract. In this paper we introduce the concept of semimajorization as a generalized form of majorization. After that we discuss uc-majorization. For $x,y \in \mathbb{R}^n$, we say x is uc-majorized by y (written as $x \prec_{uc} y$) if there exists an upper triangular column stochastic matrix A such that $x = Ay$. In our main theorem we characterize all linear maps that strongly preserve $\prec_{uc}$ on $\mathbb{R}^n$. Furthermore at the end, we characterize strong linear preservers for lc-majorization.

1. Introduction

Many researches on majorization and their linear preservers have been conducted recently. Assume that $\mathbb{R}^n$ is the vector space of all real $n \times 1$ real matrices called real vectors. For a relation $\sim$ on $\mathbb{R}^n$, a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a strong linear preserver of $\sim$, if for all $x,y \in \mathbb{R}^n$

$$x \sim y \Leftrightarrow Tx \sim Ty.$$

Let $M_n(\mathbb{R})$ be the space of all real $n \times n$ matrices. A nonnegative matrix $D \in M_n(\mathbb{R})$ is called column(row) stochastic if the sum of all entries in each column (row) is equal to 1, and D is called doubly stochastic if it is both column and row stochastic. For $x,y \in \mathbb{R}^n$, it is said that x is majorized by y (denoted by $x \prec y$) if there is a doubly stochastic matrix D such that $x = Dy$. It is well known that $x \prec y$ if and only if

$$\sum_{j=1}^{k} x_{[j]} \leqslant \sum_{j=1}^{k} y_{[j]}, \qquad \text{for} \quad k = 1,2,\ldots,n-1,$$

and

$$\sum_{j=1}^{n} x_{[j]} = \sum_{j=1}^{n} y_{[j]},$$

where $x_{[j]}$ is the j^{th} largest element of vector x. For more study see [7].

In [1, 5], all strong linear preservers of $\prec$ on $\mathbb{R}^n$ are characterized as follows:

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THEOREM 1. *A linear operator $T:\mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec$ if and only if there exist $\alpha,\beta \in \mathbb{R}$ and a permutation matrix P such that $Tx = \alpha Px + \beta \boldsymbol{J}$ and $\alpha(\alpha+n\beta) \neq 0$, where $\boldsymbol{J}$ denoted the matrix whose entries are all 1.*

Let A and B be real $n\times m$ matrices. A is said to be GUT-majorized by B, if there exists an $n\times n$ real row stochastic matrix R, $A = RB$. In [2] authors characterized all strong linear preservers of GUT-majorization as follows:

THEOREM 2. *A linear map $T: M_{n,m} \longrightarrow M_{n,m}$ strongly preserves GUT-majorization if and only if $TX = AXR + EXS$ for $R,S \in M_n$ and invertible real row stochastic matrix A, such that $R(R+S)$ is invertible.*

Moreover some kinds of linear preservers of special majorizations can be found in[3, 6].

In this paper, in the first place, we introduce the concept of semimajorization and uc-majorization and then we characterize all linear operators that strongly preserve this kind of semimajorization.

Throughout this paper we use the following notations: $M_{n,m}$ denotes the set of all $n\times m$ real matrices, $\mathbb{R}^n = M_{n,1}$ the set of all $n\times 1$ real vectors, $tr(x)$ the sum of all entries of vector $x\in\mathbb{R}^n$, $\mathscr{U}_n^{uc}$ the set of all upper triangular column stochastic matrices, $\{e_1,\cdots,e_n\}$ the standard basis of $\mathbb{R}^n$ and $GL_n(\mathbb{R}^n)$ the group of all linear maps on $\mathbb{R}^n$. Also for $x\in\mathbb{R}^n$ we use the notation $x \geqslant 0$, if all entries of x are nonnegative and we denote by $[T] = (a_{ij})$ the matrix representation of linear map $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

2. uc(lc)-majorization

We start this section with some preliminaries and the definition of uc-majorization on $\mathbb{R}^n$. Let $GL_n(\mathbb{R}^n)$ be the general linear group acting on $\mathbb{R}^n$ and G be a subgroup of $GL_n(\mathbb{R}^n)$. Then G induces an equivalence relation on $\mathbb{R}^n$ by $x\sim y$ if and only if $x = gy$ for some $g\in G$. The equivalence class of $x\in\mathbb{R}^n$ is $O(X) = \{gx : g\in G\}$. A vector y is said to be G-majorized by x, denoted by $y \prec_G x$, if it is in the convex hull of $O(x)$.

REMARK 1. It is common to consider a subgroup of an orthogonal group. But here we use the concept stated in [4].

Consider G be the group of permutation matrices, P_n, then the relation $\prec_{P_n}$ is the classic majorization introduced in introduction. For another example consider the relation $\prec_{E_n}$, where E_n is the group of even permutation matrices [9]. Also in [8] authors consider the group of circulant permutation matrices and the majorization induced by this group.

In the following we propose a definition of semimajorization in which we consider a monoid instead of a subgroup. The definition is similar to the above definition. Before that we mention the definition of a monoid.

DEFINITION 1. A set M with a binary operation $*: M \times M \longrightarrow M$ is called a monoid if

a) $(a*b)*c = a*(b*c), \qquad \forall a,b,c \in M$

b) There exists $e \in M$ such that for every $a \in M$, $e*a = a*e = a$, that is called the identity element.

DEFINITION 2. Let $GL_n(\mathbb{R}^n)$ be the general linear group acting on $\mathbb{R}^n$ and G be a submonoid of $GL_n(\mathbb{R}^n)$. Then G induces a relation on $\mathbb{R}^n$ by $x\mathscr{R}y$ if and only if $x = gy$ for some $g \in G$. Define the orbit $x \in \mathbb{R}^n$ by $O(X) = \{gx : g \in G\}$. A vector y is said to be G-semimajorized by x if it is in the convex hull of $O(x)$.

For continuing our main discussion we need to prove a simple lemma. We say matrix $A = (a_{ij})$ has equal column sum a if $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij+1} = a$, for $1 \leqslant j \leqslant n-1$. We have the following lemma.

LEMMA 1. *If A and B have equal column sums a and b respectively, then AB has equal column sum ab.*

Proof. Let $AB = (c_{ij})$ we have,

$$\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} \sum_{a_{it}} b_{tj} = \sum_{t=1}^{n} (b_{tj} \sum_{k=1}^{n} a_{kj}) = a \sum_{t=1}^{n} b_{tj} = ab. \quad \square$$

Now we let's get back to our main discussion. In the following lemma we prove that the set of all upper triangular matrices with exactly one entry equal to 1 in each column and 0 elsewhere, is a monoid. Consider the set

$$\mathscr{U} = \{[e_{i_1} e_{i_2} \cdots e_{i_n}] : 1 \leqslant i_k \leqslant k, k = 1, \cdots n\} \subset GL_n(\mathbb{R}^n)$$

of $n \times n$ matrices, where e_{i_j} is one of the standard basis vectors $e_1, \cdots, e_n$. Note that $i_k \leqslant k$ implies that the matrices are upper triangular. We have the following Lemma.

LEMMA 2. *The set $\mathscr{U} = \{[e_{i_1} e_{i_2} \cdots e_{i_n}] : 1 \leqslant i_k \leqslant k, k = 1, \cdots n\} \subset GL_n(\mathbb{R}^n)$ is a monoid.*

Proof. Let $u, v \in \mathscr{U}$. Since u and v are upper triangular, uv is as well. Also u and v both have equal column sum 1, hence by the above lemma uv has equal column sum 1. Since each entry of uv is an integer more than or equal to 1, we conclude that uv has exactly one 1 in each column and the other entries are 0. Hence $uv \in \mathscr{U}$. Obviously $I \in \mathscr{U}$. Hence $\mathscr{U}$ is a monoid. $\square$

Obviously the set of all upper triangular column stochastic matrices is a convex hull of $\mathscr{U}$, hence considering the above lemma and the definition of semimajorization, we define the following kind of semimajorization.

DEFINITION 3. Let $x, y \in \mathbb{R}^n$. We say that x is uc-majorized(or simply uc-majorized) by y (written as $x \prec_{uc} y$) if there exists $D \in \mathscr{U}_n^{uc}$ such that $x = Dy$. Also we say x is lc-majorized by y ($x \prec_{lc} y$) if D is a lower triangular column stochastic matrix.

In the following we prove some properties of uc-majorization. These properties will then be used to prove our main theorem.

LEMMA 3. *Let $x = \sum_{i=1}^{n} x_i e_i \in \mathbb{R}^n$. Then $x[t] = \sum_{i=1}^{t} x_i e_1 + \sum_{i=t+1}^{n} x_i e_{i-t+1}$ is uc-majorized by x, for each $1 \leqslant t \leqslant n$.*

Proof. Consider the following column stochastic upper triangular matrices:

$$U_1 = I, U_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \cdots, U_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & 0 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

It is obvious that $x[t] = U_t x$, and hence $x[t] \prec_{uc} x$. $\square$

The following lemma gives an understanding about vectors being uc-majorized by nonnegative vectors.

LEMMA 4. *Let $x, y \in \mathbb{R}^n$:*

(i) If $x \prec_{uc} y$, then $tr(x) = tr(y)$.

(ii) Let $y \geqslant 0$. Then $x \prec_{uc} y$ if and only if $0 \leqslant \sum_{i=t}^{n} x_i \leqslant \sum_{i=t}^{n} y_i$ for every $1 \leqslant t \leqslant k$ and $tr(x) = tr(y)$.

(iii) If $y \geqslant 0$, $x \prec_{uc} y$ and k is the largest index with $y_k \neq 0$, then $x_i = 0$ for every $i \geqslant k$.

Proof. $i)$ $x \prec_{uc} y$ implies that there is $D \in \mathscr{U}_n^{uc}$ such that $x = Dy$.

$$tr(x) = (\sum_{i=1}^{n} d_{i1}) y_1 + \cdots + (\sum_{i=1}^{n} d_{in}) y_n = tr(y).$$

$ii)$ Since both y and D are nonnegative, so is x. We have

$$\begin{aligned} 0 \leqslant \sum_{i=t}^{n} x_i &= (d_{tt} y_t + \cdots + d_{tn} y_n) + \cdots + d_{nn} y_n \\ &= \sum_{i=t}^{n} d_{it} y_t + \cdots + d_{nn} y_n \\ &\leqslant \sum_{i=t}^{n} y_i \end{aligned}$$

What is more by virtue of part (i), $tr(x) = tr(y)$. Conversely, suppose x and y satisfy the conditions. Define matrix $D = (d_{ij})$, such that $d_{kk} = \frac{\sum_{i=k}^{n} x_i - \sum_{i=k+1}^{n} y_i}{y_t}$ and $d_{kk+1} = 1 - d_{k-1k-1}$ and the other entries of the matrix are 0.(Here we consider $\frac{0}{0} = 0$). It is easy to check that $D \in \mathscr{U}_n^{uc}$ and $x = Dy$.

iii) Follows from part (ii). $\square$

3. Strong linear preservers of uc(lc)-majorization

In this section we characterizes all strong linear preservers of $\prec_{uc}$.

DEFINITION 4. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a strong linear preserver of $\prec_{uc}$, if for all $x, y \in \mathbb{R}^n$

$$x \prec_{uc} y \Leftrightarrow Tx \prec_{uc} Ty.$$

In the following theorem the notation $\phi(x)$ is used for the set $\{y \in \mathbb{R}^n : y \prec_{uc} x\}$.

THEOREM 3. *A linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ strongly preserves $\prec_{uc}$ if and only if $[T]$ satisfies the following conditions:*

(i) *$[T]$ is an invertible upper triangular matrix.*

(ii) *$[T]$ has equal column sums.*

Proof. First, let T be a strong linear preserver of $\prec_{uc}$, we show that $[T]$ satisfies the conditions. To show that $[T]$ is invertible, suppose that $T(x) = 0$. Since T is a linear map, $T(x) = T(0) = 0$. Hence we have $T(x) \prec_{uc} T(0)$. T is a strong linear preserver of $\prec_{uc}$ which implies $x \prec_{uc} 0$ and hence $x = 0$.

We know $e_i \prec_{uc} e_{i+1}$. Since T preserves $\prec_{uc}$, we have $Te_i \prec_{uc} Te_{i+1}$. So by lemma 4, $tr(Te_i) = tr(Te_{i+1})$, for every $1 \leqslant i \leqslant n$ which implies that $[T]$ has equal column sums. To complete the proof it suffices to show that $[T]$ is an upper triangular matrix.

Let $x = (x_1, \cdots, x_n)^t \in \mathbb{R}^n$. We show that k is the largest index with $x_k \neq 0$ if and only if $< \phi(x) >=< e_1, \cdots, e_k >$. First, let k be the largest index with $x_k \neq 0$. By lemma 3, we know $x[t] = \sum_{i=1}^{t} x_i e_1 + \sum_{i=t+1}^{k} x_i e_{i-t+1} \prec_{uc} x$, for each $t \leqslant k$. It is straightforward to see that $x[1], \cdots, x[k]$ are linearly independent and hence $< e_1, \cdots, e_k > \subseteq < \phi(x) >$. Also by lemma 3, if $y \in \phi(x)$, then k is the largest index that $y_k \neq 0$ hence $y \in < e_1, \cdots, e_k >$. Now suppose that $< \phi(x) >=< e_1, \cdots, e_k >$. Since $x \in \phi(x)$, $x = \sum_{i=1}^{k} a_i e_i$ and hence k is the largest index that $x_k \neq 0$.

Now since T is invertible and strongly preserves $\prec_{uc}$, we have

$$\begin{aligned} T(\langle \phi(x) \rangle) &= \langle \{Ty : y \prec_{uc} x\} \rangle \\ &= \langle \{Ty : Ty \prec_{uc} Tx\} \rangle \\ &= \langle \{z : z \prec_{uc} Tx\} \rangle \\ &= \langle \phi(Tx) \rangle \end{aligned}$$

By the above argument we know $\langle \phi(e_k) \rangle = \langle e_1, \cdots, e_k \rangle$. Since T is invertible, $dim\langle \phi(Te_k) \rangle = dimT\langle \phi(e_k) \rangle = dim\langle \phi(e_k) \rangle = k$. Repeating the above argument we conclude that $Te_k = (a_{1k}, \cdots, a_{kk}, 0, \cdots, 0)^t$ which means $[T]$ is an upper triangular matrix.

Conversely, let $[T]$ satisfy the conditions and $x \prec_{uc} y$. This means that there is a column stochastic upper triangular matrix D such that $x = Dy$. To complete the proof, it suffices to show that $D' = [T]D[T]^{-1}$ is a column stochastic upper triangular matrix, because we have:

$$\begin{aligned} & x \prec_{uc} y \\ \Leftrightarrow & x = Dy \\ \Leftrightarrow & [T]x = [T]Dy \\ \Leftrightarrow & [T]x = [T]D[T]^{-1}[T]y \end{aligned}$$

Actually we will prove even more. We will show that $D' = D$. Since $[T]$ and D are upper triangular, so is D'. Also by the above lemma D' has equal column sums 1, and therefore it is an upper triangular matrix with column sums 1. We complete proof using induction. First let's relabel the indexes of the entries of an upper triangular matrix A. Put $a_{ij} = a_{(j-i-1)n - \frac{(j-i-1)(j-i)}{2} + i}$, i.e.:

$$A = \begin{pmatrix} a_1 & a_{n+1} & & \cdots & a_n \\ & a_2 & a_{n+2} & & \vdots \\ & & \ddots & \ddots & \\ & 0 & & a_{n-1} & a_{2n-1} \\ & & & & a_n \end{pmatrix}.$$

Now we prove by induction on new indexes. Obviously $d'_1 a_{11} = a_{11} d_1$, hence $d'_1 = d_1$. Let $d'_t = d_t$ for all new indexes less than k. We prove it for $k+1$. Let $k+1$ is related to the index ij. So we must proof $d'_{ij} = d_{ij}$. Considering old indexes and the induction hypothesis, we have:

$$\begin{gathered} a_{11}d_{1j} + a_{12}d_{2j} + \cdots + a_{1j}d_{jj} = d_{11}a_{1j} + d_{12}a_{2j} + \cdots + d_{1j}a_{jj} \\ a_{22}d_{2j} + \cdots + a_{2j}d_{jj} = d_{22}a_{2j} + \cdots + d_{2j}a_{jj} \\ \vdots \\ a_{j-1j}d_{jj} = d_{j-1j}a_{jj} \end{gathered}$$

and note that the ith equality is:

$$a_{ii}d_{ij} + \cdots + a_{ij}d_{jj} = d_{ii}a_{ij} + \cdots + d_{ij-1}a_{j-1j} + d'_{ij}a_{jj}$$

Summing up the sides of the equations and considering that D and D' both have column sums 1, we have:

$$\begin{aligned}
ad_{1j}+\cdots+ad_{j-1j}+(a-a_{jj})d_{jj} = a_{1j}+\cdots+a_{j-1j}+ \\
(d_{1j}+\cdots+d_{i-1j}+d'_{ij}+d_{i+1j}+\cdots+d_{j-1j})a_{jj}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow a-a_{jj}d_{jj} = a-a_{jj}+(1-d_{ij}+d'_{ij}-d_{jj})a_{jj}\\
&\Rightarrow a_{jj}(1-d_{jj}) = (1-d_{ij}+d'_{ij}-d_{jj})a_{jj}\\
&\Rightarrow d_{ij} = d'_{ij}
\end{aligned}$$

Hence $D=D'$, and the proof is completed. $\square$

The definition of a strong linear preserver of $\prec_{lc}$ is similar to definition 4. In what follows we use the above theorem to characterize all linear maps that strongly preserve lc-majorization. We use the notation P for backward identity matrix.

LEMMA 5. *A linear map $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ strongly preserves $\prec_{lc}$ if and only if $P[T]P$ strongly preserves $\prec_{uc}$.*

Proof. Let $x,y\in\mathbb{R}^n$. It is easy to see that $x\prec_{uc} y$ if and only if $Px\prec_{lc} Py$. Let T strongly preserves $\prec_{lc}$. We have:

$$\begin{aligned}
x\prec_{uc} y &\Leftrightarrow Px\prec_{lc} Py\\
&\Leftrightarrow [T]Px\prec_{lc} [T]Py\\
&\Leftrightarrow P[T]Px\prec_{uc} P[T]Py
\end{aligned}$$

Hence if T strongly preserves $\prec_{lc}$, then $P[T]P$ strongly preserves $\prec_{uc}$. Proof of the converse is similar. $\square$

COROLLARY 1. *A linear map $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ strongly preserves $\prec_{lc}$ if and only if $[T]$ satisfies the following conditions:*

(i) *$[T]$ is an invertible lower triangular matrix.*

(ii) *$[T]$ has equal column sums.*

Proof. The proof follows from theorem 3 and lemma 5. $\square$

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REFERENCES

[1] T. ANDO, *Majorization, Doubly stochastic matrices, and comparision of eigenvalues*, Linear Algebra and its Applications **118** (1989), 163–248.

[2] A. ARMANDNEJAD AND A. ILKHANIZADEH MANESH, *GUT-majorization and its linear preservers*, Electronic Journal of Linear Algebra **23** (2012) 646–654.

[3] A. ARMANDNEJAD AND S. MOHTASHAMI AND M. JAMSHIDI, *On linear preservers of g-tridiagonal majorization on* $\mathbb{R}^n$, Linear Algebra and its Applications **459** (2014) 145–153.

[4] A. GIOVAGNOLI AND H.P. WYNN, *G-majorization with applications to matrix orderings*, Linear Algebra and its Applications **67** (1985) 111–135.

[5] A. M. HASANI AND M. RADJABALIPOUR, *The structure of linear operators strongly preserving majorizations of matrices*, Electronic Journal of Linear Algebra **15** (2006) 260–268.

[6] F. KHALOOEI AND A. SALEMI, *The Structure of linear preservers of left matrix majorization on* $\mathbb{R}^p$, Electronic Journal of Linear Algebra **18** (2009) 88–97.

[7] A. W. MARSHALL, I. OLKIN AND B. C. ARNOLD, *Inequalities: Theory of Majorization and Its Applications*, Springer, 2011.

[8] M. SOLEYMANI AND A. ARMANDNEJAD, *Linear preservers of circulant majorization on* $\mathbb{R}^n$, Linear Algebra and its Applications **440** (2014) 286–292.

[9] M. SOLEYMANI AND A. ARMANDNEJAD, *Linear preserver of even majorization on* $M_{n,m}$, Linear Multilinear Algebra **62** (2014) 1437–1449.

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