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COMPLEX SYMMETRIC WEIGHTED COMPOSITION LAMBERT TYPE OPERATORS ON $L^2(\Sigma)$

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Abstract. In this paper we obtain the polar decomposition and the Aluthge transform of a weighted composition Lambert type operator $M_wEM_uC_\varphi$ on $L^2(\Sigma)$. In addition, we study the complex symmetry of these types of operators induced by triple (w,u,φ).

1. Introduction and preliminaries

Let (X,Σ,μ) be a sigma finite measure space and let $\mathscr{A}$ be a sigma subalgebra of Σ such that $(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ is also sigma finite. The collection of the equivalence classes up to null sets of $\mathscr{A}$-measurable complex-valued functions on X will be denoted by $L^0(\mathscr{A})$. Take $L^0_+(\mathscr{A})=\{f\in L^0(\mathscr{A}):f\geqslant 0\}$. For $1\leqslant p\leqslant\infty$ we let $L^p(\Sigma)=L^p(X,\Sigma,\mu)$ and $L^p(\mathscr{A})=L^p(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$. Also its norm is denoted by $\|.\|_p$ on which $L^p(\mathscr{A})$ is a Banach subspace of $L^p(\Sigma)$. The support of a measurable function f is denoted by $\sigma(f)$ and defined as $\{x\in X: f(x)\neq 0\}$. A consequence of Radon-Nikodym theorem is that to each non-negative function $f\in L^0(\Sigma)$ there exists a unique function $E_{\mathscr{A}}(f)$ with the following conditions:

(i) $E_{\mathscr{A}}(f)$ is $\mathscr{A}$-measurable and integrable.

(ii) For any $\mathscr{A}$-measurable set A, for which $\int_A f d\mu$ exist, the following functional equation takes place

$$\int_A f d\mu=\int_A E_{\mathscr{A}}(f)d\mu, \qquad A\in\mathscr{A}.$$

The function $E_{\mathscr{A}}(f)$ is called the conditional expectation of f with respect to $\mathscr{A}$. This can be extend to real-valued and complex-valued functions by examining the conditional expectation of the positive and negative parts and the real and imaginary parts for real valued and complex valued functions respectively. Note that $\mathscr{D}(E_{\mathscr{A}})$, the domain of $E_{\mathscr{A}}$, contains $\cup_{p\geqslant 1}L^p(\Sigma)\cup\{f\in L^0(\Sigma): f\geqslant 0\}$. Hence a linear transformation $E_{\mathscr{A}}: L^p(\Sigma)\to L^p(\mathscr{A})$ can be defined by $f\mapsto E_{\mathscr{A}}(f)$. It is clear that $E_{\mathscr{A}}$ is idempotent, and in case of $p=2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathscr{A})$. For more

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details on the properties of $E_{\mathscr{A}}$, one can refer to [16] and [11]. Let $f,g \in \mathscr{D}(E_{\mathscr{A}})$. We list here some of properties of $E_{\mathscr{A}}$ that will be used in this article:

(1) $E_{\mathscr{A}}(fg) = fE_{\mathscr{A}}(g)$, $f \in L^0(\mathscr{A})$.

(2) If $f \geqslant 0$ then $E_{\mathscr{A}}(f) \geqslant 0$; if $f > 0$, then $E_{\mathscr{A}}(f) > 0$.

(3) (Conditional Hölder's inequality) $(E(|fg|))^2 \leqslant (E(|f|^2))(E(|g|^2))$.

(4) If $\mathscr{A}$ and $\mathscr{B}$ be two sigma subalgebra of Σ such that $(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ and $(X,\mathscr{B},\mu_{|_{\mathscr{B}}})$ are also sigma finite and $\mathscr{A} \subseteq \mathscr{B}$, then $E_{\mathscr{A}}E_{\mathscr{B}} = E_{\mathscr{B}}E_{\mathscr{A}} = E_{\mathscr{A}}$.

Suppose that φ is a measurable transformation from X into X such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ; that is, φ is non-singular. Let h denotes the Radon-Nikodym derivative $d\mu \circ \varphi^{-1}/d\mu$ and always is assumed that h is almost everywhere finite valued. Equivalently, $\varphi^{-1}(\Sigma)$ is a sub-sigma finite algebra of Σ. We shall henceforth find it convenient to write $E_{\mathscr{A}}$ simply as E. Let u and w are in $\mathscr{D}(E)$. The operator $T_\varphi : L^p(\Sigma) \to L^0(\Sigma)$ that induced by the triple (u,w,φ) is called weighted composition Lambert type operator and defined by $T_\varphi = M_wEM_uC_\varphi$, where M_w and M_u are multiplication operators and C_φ is a composition operator. This type of operator was studied for the first time in [3]. Throughout this paper, it is assumed that $u\mathscr{R}(C_\varphi) \subset \mathscr{D}(E)$, $E_\varphi = E_{\varphi^{-1}(\Sigma)}$ and φ is non-singular, where $\mathscr{R}(C_\varphi)$ denotes the range of C_φ. Let $T_\varphi = M_wEM_uC_\varphi$, $p = 2$ and $J = hE_\varphi(E(|u|^2)E(|w|^2)) \circ \varphi^{-1}$. Now suppose that $J \in L^\infty(\Sigma)$, we have

$$\begin{aligned}
\| T_\varphi f \|_2^2 &= \int_X E(|w|^2)|E(uf \circ \varphi)|^2 d\mu \\
&= \int_X |E(u\sqrt{E(|w|^2)} f \circ \varphi|^2 d\mu \\
&= \|EM_{u\sqrt{E(|w|^2)}}\, C_\varphi f\|_2^2.
\end{aligned}$$

Put $v = u\sqrt{E(|w|^2)}$. Using conditional Hölder's inequality, we have

$$\|EM_v\, C_\varphi f\|_2^2 \leqslant \int_X J|f|^2 d\mu.$$

Since $J \in L^\infty(\Sigma)$, the previous inequality implies that T_φ is bounded [3]. Note that when $\varphi = id$, the identity transformation, then $T_1 := M_wEM_u$ is bounded if and only if EM_v is bounded if and only if $E(|u|^2)E(|w|^2) \in L^\infty(\Sigma)$ (see [4, 15]).

A combination of conditional expectation operators, multiplication and composition operators appears more often in the service of the study of other operators, such as integral operators [9], Markov and averaging operators [16].

In section 2, we investigate the polar decomposition and Aluthge transform of T_φ. In section 3, we investigate which combinations of weights u,w and self-maps φ on X give rise to complex symmetric weighted composition Lambert type operators with a special conjugation.

2. Polar decomposition and Aluthge transform of T_φ

In this section we present the polar decomposition and the Aluthge transformation of weighted composition Lambert type operators. Let $B(\mathscr{H})$ denotes the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. We write $\mathscr{N}(T)$ for the null-space of $T \in B(\mathscr{H})$. Recall that for $T \in B(\mathscr{H})$, there is a unique factorization $T = U|T|$, where $\mathscr{N}(T) = \mathscr{N}(U) = \mathscr{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T.

THEOREM 2.1. *Let $T_\varphi = M_wEM_uC_\varphi$ be a bounded weighted composition Lambert type operator on $L^2(\Sigma)$. Suppose that $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$ and $uE_\varphi(\bar{u}) \in L^0_+(\Sigma)$. Then the unique parts of the polar decomposition U, $|T_\varphi|$ for the weighted composition Lambert type operator T_φ are given by*

$$|T_\varphi|f = \left\{\frac{\chi_G[E(|w|^2)] \circ \varphi^{-1}}{[E(u(h\circ\varphi)E_\varphi(\bar{u}))] \circ \varphi^{-1})}\right\}^{\frac{1}{2}} [(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)] \circ \varphi^{-1};$$

$$Uf = \left\{\frac{\chi_{S\cap G\cap G'}}{E(u(h\circ\varphi)E_\varphi(\bar{u}))(E(|w|^2))}\right\}^{\frac{1}{2}} wE(uf\circ\varphi),$$

for all $f \in L^2(\Sigma)$, in which $G = \sigma([E(u(h\circ\varphi)E_\varphi(\bar{u}))] \circ \varphi^{-1})$, $S = \sigma(E(|w|^2))$ and $G' = \sigma(E(u(h\circ\varphi)E_\varphi(\bar{u})))$.

Proof. It is easy to check that

$$T^*_\varphi f = hE_\varphi(M_{\bar{u}}EM_{\bar{w}}(f)) \circ \varphi^{-1}$$

and

$$T^*_\varphi T_\varphi(f) = hE_\varphi(\bar{u}E(|w|^2)E(uf\circ\varphi)) \circ \varphi^{-1}.$$

Since $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$ so

$$T^*_\varphi T_\varphi(f) = [(h\circ\varphi)E(|w|^2)E_\varphi(\bar{u})E(uf\circ\varphi)] \circ \varphi^{-1}.$$

By induction for each $n \in \mathbb{N}$,

$$(T^*_\varphi T_\varphi)^n(f) = [(h\circ\varphi)E(|w|^2)^nE_\varphi(\bar{u})(E(u(h\circ\varphi)E_\varphi(\bar{u}))^{n-1}E(uf\circ\varphi)] \circ \varphi^{-1}.$$

Put

$$\Lambda = [(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)] \circ \varphi^{-1};$$
$$\gamma = [E(|w|^2)] \circ \varphi^{-1};$$
$$\lambda = [E(u(h\circ\varphi)E_\varphi(\bar{u}))] \circ \varphi^{-1}.$$

Then we have

$$(T^*_\varphi T_\varphi)^n(f) = M_{\gamma^n\lambda^{n-1}}\Lambda, \qquad n \in \mathbb{N}.$$

Let $q(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial with complex coefficients. When M_u be a bounded multiplication operator on $L^2(\Sigma)$, then $q(M_u)$ is also bounded and $q(M_u)=M_{q\circ u}$. Thus

$$\begin{aligned} q(T_\varphi^* T_\varphi)(f) &= \alpha_0 I+\sum_{i=0}^{n-1}\alpha_{i+1}(T^*T)^{i+1}(f) \\ &= \alpha_0 I+(\sum_{i=0}^{n-1}\alpha_{i+1}M_{\gamma^{i+1}\lambda^{i}})\Lambda. \end{aligned}$$

So

$$\begin{aligned} q(T_\varphi^* T_\varphi)(f) &= \alpha_0 I+(\sum_{i}^{n}\alpha_i M_{\gamma^i\lambda^{i-1}})\Lambda \\ &= \alpha_0 I+\chi_G M_{\lambda^{-1}}(\sum_{i}^{n}\alpha_i M_{\gamma^i\lambda^i})\Lambda \\ &= q(0)I+\chi_G M_{\lambda^{-1}}(M_{q\circ(\gamma\lambda)}-q(0)I)\Lambda. \end{aligned}$$

Now, let $g\in Spec(T_\varphi^* T_\varphi)$, the spectrum of $T_\varphi^* T_\varphi$. Using Weierstrass theorem and functional calculus, we have

$$g(T_\varphi^* T_\varphi)(f)=g(0)I+\chi_G M_{\lambda^{-1}}(M_{g\circ(\gamma\lambda)})-g(0)I)\Lambda.$$

Take $g(t)=t^{\frac{1}{2}}$. Then we obtain

$$\begin{aligned} |T_\varphi|f &= \frac{\chi_G}{\lambda}(\lambda\gamma)^{\frac{1}{2}}\Lambda \\ &= \left\{\frac{\chi_G[E(|w|^2)]\circ\varphi^{-1}}{[E(u(h\circ\varphi)E_\varphi(\bar{u}))]\circ\varphi^{-1}}\right\}^{\frac{1}{2}}[(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi]\circ\varphi^{-1}. \end{aligned}$$

Put $K=S\cap G\cap G'$ and define

$$Uf=\frac{\chi_K}{\sqrt{E(u(h\circ\varphi)E_\varphi(\bar{u}))E(|w|^2)}}wE(uf\circ\varphi).$$

It is easy to check that $T_\varphi=U|T_\varphi|$.

Direct calculations show that

$$U^*(f)=\left[(h\circ\varphi)E_\varphi(\bar{u})E\left(\frac{\bar{w}\chi_K}{\sqrt{vE(|w|^2)}}f\right)\right]\circ\varphi^{-1},$$

where $v=E(u(h\circ\varphi)E_\varphi(\bar{u}))$. Then

$$U^*U(f)=\left[(h\circ\varphi)E_\varphi(\bar{u})E\left(\frac{|w|^2\chi_K}{vE(|w|^2)}\right)E(uf\circ\varphi)\right]\circ\varphi^{-1},$$

and so

$$UU^*U(f)=\frac{\chi_K}{\sqrt{vE(|w|^2)}}wE(uf\circ\varphi).$$

Accordingly $UU^*U = U$. Hence U is a partial isometry. With respect to the definition of U it is clear that $\mathcal{N}(U) = \mathcal{N}(T_\varphi)$ and since $\mathcal{N}(|T_\varphi|) = \mathcal{N}(T_\varphi)$, therefore $\mathcal{N}(|T_\varphi|) = \mathcal{N}(U)$. Thus the decomposition is unique. □

COROLLARY 2.2. *Let $T_1 = M_wEM_u$ be a bounded Lambert type operator on $L^2(\Sigma)$. Then the parts of the polar decomposition U_1, $|T_1|$ for the Lambert type operator T_1 are given by*

$$|T_1|(f) = \left\{ \frac{\chi_{G_1} E(|w|^2)}{E(|u|^2)} \right\}^{\frac{1}{2}} \bar{u}E(uf);$$

$$U_1(f) = \left\{ \frac{\chi_{G_1 \cap S_1}}{E(|u|^2)E(|w|^2)} \right\}^{\frac{1}{2}} wE(uf),$$

where $G_1 = \sigma(E(|u|^2))$, $S_1 = \sigma(E(|w|^2))$ and $f \in L^2(\Sigma)$.

When $\varphi^{-1}(\Sigma) \subseteq \mathscr{A}$, because of $\varphi^{-1}(\Sigma)$-measurability of $f \circ \varphi$ for each Σ-measurable function f, we have $T_\varphi = M_{wE(u)}C_\varphi := W$. Put $v = wE(u)$. A result of Hoover, Lambert and Quinn [12] shows that W is bounded on $L^2(\Sigma)$ if and only if $J_W := hE_\varphi(|v|^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$ and in this case the adjoint W^* of W is given by $W^*(f) = hE_\varphi(\bar{v}f) \circ \varphi^{-1}$ for all $f \in L^2(\Sigma)$. Now, by [2, Lemma 4.1]) we have the following result.

PROPOSITION 2.3. *Let $\varphi^{-1}(\Sigma) \subseteq \mathscr{A}$. Then the unique parts of the polar decomposition U_W, $|W|$ for $W = M_wEM_uC_\varphi \in B(L^2(\Sigma))$ are given by*

$$|W|f = \left\{ hE_\varphi(|w|^2|E(u)|^2) \circ \varphi^{-1} \right\}^{\frac{1}{2}} f;$$

$$U_W f = \left\{ \frac{\chi_A}{h \circ \varphi E_\varphi(|w|^2|E(u)|^2)} \right\}^{\frac{1}{2}} wE(u)f \circ \varphi,$$

for all $f \in L^2(\Sigma)$, in which $A = \sigma(E_\varphi(|w|^2|E(u)|^2))$.

Note that, for each non-negative measurable function v, $\sigma(v) \subseteq \sigma(E(v))$ and $\sigma(E(v)) = \sigma(E(v^2))$, in which $\sigma(E(v))$ is the smallest $\mathscr{A}$-measurable set containing $\sigma(v)$. In the following theorem we determine the Aluthge transformation of T_φ in case of $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$.

THEOREM 2.4. *Let $T_\varphi = M_wEM_uC_\varphi$ be a bounded weighted composition Lambert type operator on $L^2(\Sigma)$. Suppose that $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$ and $uE_\varphi(\bar{u}) \in L_0^+(\Sigma)$. Put*

$$\gamma = [E(|w|^2)] \circ \varphi^{-1};$$

$$\lambda = [E(u(h \circ \varphi)E_\varphi(\bar{u}))] \circ \varphi^{-1};$$

$$d = [E(\chi_{\varphi^{-1}(G)} u(h \circ \varphi)E_\varphi(\bar{u}))] \circ \varphi^{-1};$$

$$c = (\lambda \chi_{\Gamma \cap G}) \circ \varphi.$$

Then the Aluthge transformation of T_φ is

$$\widetilde{T_\varphi}(f) = \chi_{\Gamma\cap G}\sqrt[4]{\frac{\gamma}{\lambda d^2}}\left\{(h\circ\varphi)E_\varphi(\bar{u})E\left[\frac{\chi_{\varphi^{-1}(K)}(wc)\circ\varphi}{\sqrt[4]{\lambda^3\gamma d^2}\circ\varphi^2}uE(uf\circ\varphi)\circ\varphi\right]\right\}\circ\varphi^{-1},$$

where $G=\sigma(\lambda)$, $S=\sigma(\gamma\circ\varphi)$, $G'=\sigma(\lambda\circ\varphi)$, $\Gamma=\sigma(d)$ *and* $K=S\cap G\cap G'$.

Proof. According to previous theorem we have

$$|T_\varphi|f = \chi_G\sqrt{\frac{\gamma}{\lambda}}[(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)]\circ\varphi^{-1}.$$

By induction for each $n\in\mathbb{N}$,

$$|T_\varphi|^n f = \chi_G\sqrt{\frac{\gamma}{\lambda}}\left[(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)(\sqrt{\frac{\gamma}{\lambda}}\circ\varphi)^{n-1}(d\circ\varphi)^{n-1}\right]\circ\varphi^{-1}.$$

Putting

$$\Lambda = \chi_G\sqrt{\frac{\gamma}{\lambda}}[(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)]\circ\varphi^{-1},$$

we have

$$|T_\varphi|^n f = M_{d^{n-1}t^{n-1}}\Lambda,$$

where $t=\sqrt{\frac{\gamma}{\lambda}}$. Let $q(z)$ be a polynomial with complex coefficient. With respect to the process of the proof of Theorem 2.1 we obtain

$$q(|T_\varphi|)(f) = q(0)I + \frac{\chi_\Gamma}{td}(M_{q(td)} - q(0)I)\Lambda.$$

Let $g\in C(\mathrm{Spec}(|T_\varphi|))$, using Weierstrass theorem and continuous functional calculus, we have

$$g(|T_\varphi|)(f) = g(0)I + \frac{\chi_\Gamma}{td}(M_{g(td)} - g(0)I)\Lambda.$$

Therefore

$$|T_\varphi|^{\frac{1}{2}}(f) = \frac{\chi_\Gamma}{\sqrt{td}}\Lambda = \chi_{\Gamma\cap G}\sqrt[4]{\frac{\gamma}{\lambda d^2}}[(h\circ\varphi)E_\varphi(\bar{u})E(uf\circ\varphi)]\circ\varphi^{-1}.$$

Now, the desired conclusion follows from $\widetilde{T_\varphi} = |T_\varphi|^{\frac{1}{2}}U|T_\varphi|^{\frac{1}{2}}$. □

COROLLARY 2.5. *The Aluthge transformation of $T_1 = M_wEM_u\in B(L^2(\Sigma))$ is*

$$\widetilde{T_1}(f) = \frac{\chi_{G_1}E(uw)}{E(|u|^2)}\bar{u}E(uf), \qquad f\in L^2(\Sigma),$$

where $G_1=\sigma(E(|u|^2)$.

Recall that when $\varphi^{-1}(\Sigma) \subseteq \mathscr{A}$, then $T_\varphi = M_{wE(u)}C_\varphi$ is a weighted composition operator. Thus by Proposition 2.3 we have the following result.

PROPOSITION 2.6. *Let $\varphi^{-1}(\Sigma) \subseteq \mathscr{A}$ and $T_\varphi \in B(L^2(\Sigma))$. Then the Aluthge transformation of T_φ is*

$$\widetilde{T_\varphi}(f) = wE(u)\left\{\frac{hE_\varphi(|w|^2|E(u)|^2)\circ\varphi^{-1}\chi_B}{(h\circ\varphi)E_\varphi(|w|^2|E(u)|^2)}\right\}^{\frac{1}{4}} f\circ\varphi,$$

for all $f \in L^2(\Sigma)$, in which $B = \sigma(E_\varphi(|w|^2|E(u)|^2))$.

EXAMPLE 2.7. Let $X = [0,1]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformations $\varphi, \psi : X \to X$ by $\varphi(x) = \sqrt{x}$ and

$$\psi(x) = \begin{cases} 1-2x & x \in [0,\frac{1}{2}]; \\ 2x-1 & x \in (\frac{1}{2},1]. \end{cases}$$

Put $w(x) = 2$, $u(x) = x^2$, $\mathscr{A} = \psi^{-1}(\Sigma)$ and $E^{\mathscr{A}} = E$. Then $h(x) = \frac{d\mu\circ\varphi^{-1}}{d\mu}(x) = 2x$, $E_\varphi = I$ and

$$E(f)(x) = \frac{f(x)+f(1-x)}{2}, \quad f \in L^2(\Sigma).$$

Since $J(x) = h(x)E_\varphi(E(|u|^2)E(|w|^2))\circ\varphi^{-1}(x) = 4x(x^8+(1-x^2)^4) \in L^\infty(\Sigma)$, then $T_\varphi = M_wEM_uC_\varphi$ is bounded on $L^2(\Sigma)$, where for each $f \in L^2(\Sigma)$ and $x \in [0,1]$ we have

$$T_\varphi f(x) = x^2 f(\sqrt{x}) + (1-x)^2 f(\sqrt{1-x}).$$

So a weighted composition Lambert type operator can be written as a finite sum of weighted composition operators. Here $T_\varphi f = u_1 f\circ\varphi_1 + u_2 f\circ\varphi_2$, where $u_1(x) = x^2$, $u_2(x) = (1-x)^2$, $\varphi_1(x) = \sqrt{x}$ and $\varphi_2(x) = \sqrt{1-x}$. Set $J_i = h_iE_{\varphi_i}(u_i^2)\circ\varphi_i^{-1}$. It follows that $J_1(x) = J_2(x) = 2x^9$, and so by [3, Theorem 2.1(i)] and [13, Proposition 2.3(b)] we deduce that $\|T\| = 2$.

Now by using the polar decomposition of T_φ, we can obtain from direct computations that

$$(|T_\varphi|f)(x) = \frac{2x^5}{\sqrt{x^9+(1-x^2)^{\frac{9}{2}}}}\left\{x^4 f(\sqrt{x}) + (1-x^2)^2 f(\sqrt{1-x^2})\right\};$$

$$(Uf)(x) = \frac{1}{2\sqrt{x^{\frac{9}{2}}+(1-x^2)^{\frac{9}{2}}}}\left\{x^2 f(\sqrt{x}) + (1-x)^2 f(\sqrt{1-x^2})\right\}.$$

The parts of the polar decomposition U_1, $|T_1|$ for the Lambert type operator $(T_1f)(x) = x^2f(x)+(1-x)^2f(1-x)$ are given by

$$(|T_1|f)(x) = \frac{\sqrt{2}}{\sqrt{x^4+(1-x)^4}}\left\{x^4f(x)+x^2(1-x)^2f(1-x)\right\};$$

$$(U_1f)(x) = \frac{1}{\sqrt{2x^4+2(1-x)^4}}\left\{x^2f(x)+(1-x)^2f(1-x)\right\}.$$

Moreover, $(Wf)(x) = (2x^4 - 2x^2 + 1)f(\sqrt{x})$ and

$$J_W(x) = 2x(2x^4 - 2x^2 + 1)^2;$$
$$(|W|f)(x) = \sqrt{2x}(2x^4 - 2x^2 + 1)f(x);$$
$$(U_W f)(x) = \frac{2x^4 - 2x^2 + 1}{\sqrt[4]{4x}(2x^2 - 2x + 1)}\chi_{(0,1]}(x)f(\sqrt{x}).$$

Also, by Theorem 2.4 we have

$$(|T_\varphi|^{\frac{1}{2}} f)(x) = \frac{\sqrt{2}x^5}{\{x^9 + (1-x^2)^{\frac{9}{2}}\}^{\frac{3}{4}}}\left\{x^4 f(x) + (1-x^2)^2 f(\sqrt{1-x^2})\right\}.$$

It follows that

$$\widetilde{T_\varphi}(f) = 2x^5 \sqrt[4]{\frac{4}{[x^9 + (1-x^2)^{\frac{9}{2}}]^3}}\ [\Theta(f) + \beta\Delta(f)],$$

where

$$\Theta(f) = \frac{\frac{2[x^{\frac{9}{2}} + (1-x)^{\frac{9}{2}}]}{\sqrt[4]{4[x^{\frac{9}{2}} + (1-x)^{\frac{9}{4}}]^5}}[x^6 f(\sqrt{x}) + x^4(1-x^2)f(\sqrt{1-x})]}{4},$$

$$\beta = \frac{2[(1-x^2)^{\frac{9}{4}} + (1-\sqrt{1-x^2})^{\frac{9}{2}}]}{\sqrt[4]{4[(1-x^2)^{\frac{9}{4}} + (1-\sqrt{1-x^2})^{\frac{9}{4}}]^5}},$$

and

$$\Delta(f) = \frac{[(1-x^2)^3 f(\sqrt[4]{1-x^2}) + (1-\sqrt{1-x^2})^2(1-x^2)^2 f(\sqrt{1-\sqrt{1-x^2}})]}{4}.$$

EXAMPLE 2.8. Let $\mathbb{N}$ denotes the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\{e_n\}_{n\in\mathbb{N}_0}$ be an orthornormal basis for $\ell^2(\mathbb{N}_0)$ and let $u \in \ell^2(\mathbb{N}_0)$ with $u(0) = u_0 = 0$ and $u(n) = u_n \geqslant 0$ for all $n \in \mathbb{N}$. Define $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ as

$$\varphi(n) = \begin{cases} 0 & n = 0, 1, \\ n-1 & n \geqslant 2. \end{cases}$$

Then the matrix representation of the forward weighted shift $M_u C_\varphi$ can now be written as:

$$M_u C_\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ u_1 & 0 & 0 & 0 & \ldots \\ 0 & u_2 & 0 & 0 & \ldots \\ 0 & 0 & u_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For fix $r,s\in\mathbb{N}$, define a non-singular measurable transformation ψ on $\mathbb{N}_0$ such that $\psi^{-1}(\{0\})=\{0,1\}$ and

$$\psi^{-1}(\{2k\})=\{(k-1)(r+s)+r+i+1:1\leqslant i\leqslant s\},\quad k=1,2,3,\cdots$$
$$\psi^{-1}(\{2k-1\})=\{(k-1)(r+s)+i+1:1\leqslant i\leqslant r\},\qquad k=1,2,3,\cdots$$

Put $\mathscr{A}_{r,s}=\psi^{-1}(2^{\mathbb{N}_0})=\{\{0,1\},\{2,\cdots,r+1\},\{r+2,\cdots,r+s+1\},\{r+s+2,\cdots,2r+s+1\},\{2r+s+2,\cdots,2r+2s+1\},\cdots\}$. Then by [14, Example 4.1] we obtain

$$E^{\mathscr{A}_{r,s}}(e_i)(k)=\frac{\sum_{j\in\psi^{-1}(\psi(k))}e_i(j)}{\sum_{j\in\psi^{-1}(\psi(k))}1}.$$

Then the matrix representation of $E^{\mathscr{A}_{r,s}}$ is diagonal. More precisely, $E^{\mathscr{A}_{r,s}}=\operatorname{diag}(A_{1,1},A_{2,2},\cdots,A_{n,n})$, where $A_{i,j}=0$ for $i\neq j$,

$$A_{1,1}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

and for $k=1,2,3,\ldots$, we have

$$A_{2k,2k}=\begin{pmatrix}\frac{1}{r}&\cdots&\frac{1}{r}\\ \vdots&\vdots&\vdots\\ \frac{1}{r}&\cdots&\frac{1}{r}\end{pmatrix},\qquad A_{2k+1,2k+1}=\begin{pmatrix}\frac{1}{s}&\cdots&\frac{1}{s}\\ \vdots&\vdots&\vdots\\ \frac{1}{s}&\cdots&\frac{1}{s}\end{pmatrix}.$$

To avoid tedious calculations, take $r=s=1$. In this case $\varphi=\psi$, therefore $\mathscr{A}_{1,1}=\varphi^{-1}(\Sigma)$ hence the operator $T_\varphi=M_wEM_uC_\varphi$ reduces to weighted composition operator M_vC_φ, where $v=wE(u)$. In situations like this, it seems that using the matrix representation of the operator facilitates the process of computing factors of polar decomposition of the operator (U and $|T|$) and its Aluthge transform. In the following we use of the matrix representation of the mentioned operator and the general method that we described in Theorem 2.1 to obtain the polar decomposition and the Aluthge transform. So one can compare them and choose the way that is simpler.

Put $E^{\mathscr{A}_{1,1}}=E$. Then we have

$$E=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0&0&\ldots\\ \frac{1}{2}&\frac{1}{2}&0&0&\ldots\\ 0&0&1&0&\ldots\\ 0&0&0&1&\ldots\\ \vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Let $w=\{w_n\}_{n=0}^{\infty}\in l^\infty(\mathbb{N}_0)$ be a sequence of nonzero real numbers. Hence the matrix representation of $T_\varphi=U_\varphi|T_\varphi|$ can be represented by

$$T_\varphi=M_wEM_uC_\varphi=\begin{pmatrix}\frac{1}{2}u_1w_0&0&0&\ldots\\ \frac{1}{2}u_1w_1&0&0&\ldots\\ 0&u_2w_2&0&\ldots\\ 0&0&u_3w_3&\ldots\\ \vdots&\vdots&\vdots&\ddots\end{pmatrix}\in B(l^2(\mathbb{N}_0)),$$

in which

$$|T_\varphi| = \begin{pmatrix} \frac{1}{2}\sqrt{(u_1w_0)^2+(u_1w_1)^2} & 0 & 0 & \dots \\ 0 & u_2|w_2| & 0 & \dots \\ 0 & 0 & u_3|w_3| & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With respect to the assumptions of the example we have

$$h(k) = \begin{cases} 2 & k=0, \\ 1 & k \geqslant 1, \end{cases}$$

and

$$E(u)(k) = \begin{cases} \frac{u_0+u_1}{2} & k=0,1, \\ u_k & k \geqslant 2. \end{cases}$$

According to Proposition 2.3, $|T_\varphi| = M_{\sqrt{J_W}}$ where $J_W = hE_\varphi(|w|^2|E(u)|^2)\circ\varphi^{-1}$. Hence

$$J_W(k) = \begin{cases} (w_0E(u)(0))^2 + (w_1E(u)(1))^2 & k=0, \\ (w_{k+1}u_{k+1})^2 & k \geqslant 1. \end{cases}$$

So we have

$$M_{\sqrt{J_W}}(e_0)(k) = \begin{cases} \frac{1}{2}\sqrt{(w_0u_1)^2+(w_1u_1)^2} & k=0, \\ 0 & k \geqslant 1, \end{cases}$$

and

$$M_{\sqrt{J_W}}(e_i)(k) = \begin{cases} u_{i+1}|w_{i+1}| & k=i, \\ 0 & k \neq i. \end{cases}$$

Again, according to the matrix form of T_φ, we have

$$U_\varphi = \begin{pmatrix} \frac{u_1w_0}{\sqrt{(u_1w_0)^2+(u_1w_1)^2}} & 0 & 0 & \cdots \\ \frac{u_1w_1}{\sqrt{(u_1w_0)^2+(u_1w_1)^2}} & 0 & 0 & \cdots \\ 0 & \frac{w_2}{|w_2|} & 0 & \dots \\ 0 & 0 & \frac{w_3}{|w_3|} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since in this case, $U_\varphi(f) = \{\frac{\chi_A}{h\circ\varphi E_\varphi(|w|^2|E(u)|^2)}\}^{\frac{1}{2}} wE(u)f\circ\varphi$, so we have

$$\begin{aligned} U_\varphi(e_0)(0) &= \frac{wE(u)(e_0\circ\varphi)(0)}{\sqrt{h\circ\varphi(E(u))^2(E_\varphi(|w|^2))(0)}} \\ &= \frac{\frac{u_1w_0}{2}}{\sqrt{2(\frac{w_0{}^2+w_1{}^2}{2})(\frac{u_1{}^2}{4})}} \\ &= \frac{u_1w_0}{\sqrt{(w_0{}^2+w_1{}^2)(u_1{}^2)}}. \end{aligned}$$

By same calculations,

$$U_\varphi(e_0)(1) = \frac{u_1 w_1}{\sqrt{({w_0}^2 + {w_1}^2)({u_1}^2)}},$$

and for $i \geqslant 1$,

$$U_\varphi(e_i)(k) = \begin{cases} \frac{w_k}{|w_k|} & k = i+1, \\ 0 & k \neq i+1. \end{cases}$$

Thus, the two methods give the same result. Moreover, the matrix form of the Aluthge transformation of T_φ is

$$\widetilde{T_\varphi} = \begin{pmatrix} \frac{1}{2}(u_1|w_0|) & 0 & 0 & \dots \\ \frac{(u_1 w_1)\sqrt{u_2 w_2}}{\sqrt{2}((u_1 w_0)^2 + (u_1 w_1)^2)^{\frac{1}{4}}} & 0 & 0 & \dots \\ 0 & \sqrt{(u_2|w_2|)(u_3|w_3|)} & 0 & \dots \\ 0 & 0 & \sqrt{(u_3|w_3|)(u_4|w_4|)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

3. Complex symmetry

We recall that a conjugation on a complex Hilbert space $\mathscr{H}$ is a function $C : \mathscr{H} \to \mathscr{H}$ that is conjugate linear, involutive and isometric. By involutive and isometric we mean that $C^2 = I$ and $\langle f, g\rangle = \langle Cg, Cf\rangle$ for all f, g belonging to $\mathscr{H}$, respectively. Let C be a conjugation on $\mathscr{H}$. A bounded linear operator on $\mathscr{H}$ is called C-symmetric if $T = CT^*C$. If there exists a C with respect to which T be C-symmetric, then T is called complex symmetric operator. The class of complex symmetric operators includes all normal and binormal operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators, see [6, 7, 8]. The problem of describing all complex symmetric weighted composition operators on various analytic function spaces is very active recently (see [1, 10, 17]). The problem in this section is: Which (weighted composition) Lambert type operators are complex symmetric?

Let $u \in L^0(\Sigma)$ for which $uf \in \mathscr{D}(E)$ for all $f \in L^2(\Sigma)$. Thus, the operator $R_u := EM_u$ is defined on all $L^2(\Sigma)$. Recall that R_u is bounded on $L^2(\Sigma)$ if and only if $E(|u|^2) \in L^\infty(\Sigma)$ and in this case $\|R_u\| = \sqrt{\|E(|u|^2)\|_\infty}$ and $R_u^* = M_{\bar{u}}E$. For more details, see [11]. One can easily check that $M_u : L^2(\Sigma) \to L^2(\Sigma)$ is C-symmetric, where $Cf = \overline{f}$. The class of complex symmetric operators contains the class of normal operators (see [8]). It is known that the bounded operator R_u is normal if and only if u is an $\mathscr{A}$-measurable function [11]. Hence when u is an $\mathscr{A}$-measurable function, R_u is complex symmetric.

LEMMA 3.1. *If C is an isometric antilinear involution on the Hilbert space $\mathscr{H}$, then there exist an orthogonal basis e_n such that $e_n = Ce_n$ for all $n \in \mathbb{N}$.*

Proof. See [6]. $\square$

PROPOSITION 3.2. *Let $0 \neq R_u \in B(L^2(\Sigma))$. Then R_u is complex symmetric if and only if $u \in L^0(\mathscr{A})$.*

Proof. According to the comment before Lemma 3.1, sufficiency is clear. Conversely, suppose that R_u is complex symmetric. With respect to Lemma 3.1 there exist an orthogonal basis $\{e_n\}$ for $L^2(\mathscr{A})$ such that $Ce_n = e_n$ for all $n \in \mathbb{N}$. Since $0 \neq R_u$ is complex symmetric, so $\bar{u}Ee_n = CEue_n$ and $Ee_{n_0} \neq 0$ for some $n_0 \in \mathbb{N}$. Since $Eue_{n_0} \in L^2(\mathscr{A})$, thus $Eue_{n_0} = \sum_{i=0}^{\infty} c_i e_i$ for some $\{c_i\} \subset \mathbb{C}$ and hence $C(Eue_{n_0}) = \sum_{i=0}^{\infty} \bar{c_i} e_i \in L^2(\mathscr{A})$. Thus, $\bar{u} = \frac{C(Eue_{n_0})}{Ee_{n_0}}$, and so $u \in L^0(\mathscr{A})$. $\square$

PROPOSITION 3.3. *Let $\sigma(w) = X$ and $\frac{u}{|w|} \in L^0(\mathscr{A})$. Then the bounded operator $T_1 = M_w E M_u : L^2(\Sigma) \to L^2(\Sigma)$ is complex symmetric.*

Proof. Define $C : L^2(\Sigma) \to L^2(\Sigma)$ as $C(f) = \frac{w}{|w|}\bar{f}$. It is easy to verify that C is conjugate linear, involutive and isometric. Since $\frac{u}{|w|} \in L^0(\mathscr{A})$, we obtain that $CT^*C = T$. Therefore, the proof is completed. $\square$

LEMMA 3.4. *If T is a C-symmetric operator, then $T = U|T|$ where U is a C-symmetric unitary operator.*

Proof. See [7]. $\square$

PROPOSITION 3.5. *If the bounded operator $T_1 = M_w E M_u : L^2(\Sigma) \to L^2(\Sigma)$ is complex symmetric, then u and w are $\mathscr{A}$-measurable functions.*

Proof. Let $T_1 = U_1|T_1|$ be the polar decomposition of T_1. Then by Corollary 2.2, $U_1 = M_v E M_u$ where

$$v = \left\{ \frac{\chi_{S_1 \cap G_1}}{E(|u|^2)E(|w|^2)} \right\}^{\frac{1}{2}} w, \quad G_1 = \sigma(E(|u|^2)), \text{and } S_1 = \sigma(E(|w|^2)).$$

By Lemma 3.4, $U_1^* U_1 = U_1 U_1^* = I$ where I denotes the identity operator. It follows that $M_{vE(|u|^2)} E M_{\bar{v}} = M_{\bar{u}E(|v|^2)} E M_u = I$. Therefore $\bar{u}E(|v|^2)E(u) = vE(|u|^2)E(\bar{v}) = 1$, which implies that u and w are $\mathscr{A}$-measurable functions. $\square$

PROPOSITION 3.6. *Let $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$, $\varphi^2 = id$ and $\sigma(h) = \sigma(w) = X$. If $\sqrt{h(h \circ \varphi)} = \frac{w}{|w|} \frac{\bar{w} \circ \varphi}{|w| \circ \varphi}$ and $\frac{E_\varphi(u)}{|w|} \sqrt{h \circ \varphi} \in L^0(\mathscr{A})$, then the bounded Lambert type operator $T_\varphi : L^2(\Sigma) \to L^2(\Sigma)$ is complex symmetric.*

Proof. Define $C : L^2(\Sigma) \to L^2(\Sigma)$ whose acting is giving by

$$C(f) = \frac{w}{|w|} \frac{\bar{f} \circ \varphi}{\sqrt{h \circ \varphi}}.$$

Then C is conjugate linear, $C^2 = I$ and

$$\langle Cg, Cf\rangle = \int_X \frac{(f\circ\varphi)(\bar{g}\circ\varphi)}{h\circ\varphi} d\mu = \int_X \frac{h}{h} f\bar{g} d\mu = \langle f, g\rangle.$$

Now, since $\mathscr{A} \subseteq \varphi^{-1}(\Sigma)$ and $\frac{E_\varphi(u)\sqrt{h\circ\varphi}}{|w|} \in L^0(\mathscr{A})$ we get that

$$CT_\varphi^* C(f) = \frac{w\sqrt{h\circ\varphi}}{|w|} E_\varphi\left(uE(|w|\frac{f\circ\varphi}{\sqrt{h\circ\varphi}})\right)$$

$$= \frac{E_\varphi(u)\sqrt{h\circ\varphi}}{|w|} wE\left(|w|\frac{f\circ\varphi}{\sqrt{h\circ\varphi}}\right)$$

$$= wE(E_\varphi(uf\circ\varphi)).$$

Using $E = EE_\varphi$, we obtain $CT_\varphi^* C = T_\varphi$. $\square$

EXAMPLE 3.7. Suppose that $1 < a < \infty$. Let $X = [\frac{1}{a}, a]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformations $\varphi, \psi : X \to X$ by $\varphi(x) = \frac{1}{x}$ and

$$\psi(x) = \begin{cases} (a+1)(\frac{1}{a} - x) + a & x \in [\frac{1}{a}, 1]; \\ (\frac{a+1}{a})(x - a) + a & x \in (1, a]. \end{cases}$$

Put $u(x) = x^2$, $w(x) = x^3$, $\mathscr{A} = \psi^{-1}(\Sigma)$ and $E^{\mathscr{A}} = E$. Then $h(x) = \frac{1}{x^2}$, $E_\varphi = I$ and

$$E(f)(x) = \frac{af(x) + f(a(1-x)+1)}{(a+1)}, \quad f \in L^2(\Sigma).$$

Since $\varphi^{-1}(\Sigma) = \Sigma$, for $b \in (\frac{1}{a}, 1)$, it is clear that $(\frac{1}{a}, b) \in \varphi^{-1}(\Sigma)$ but $(\frac{1}{a}, b) \notin \psi^{-1}(\Sigma)$, hence $\mathscr{A} \subset \varphi^{-1}(\Sigma)$. Calculations show that

$$J(x) = h(x)E_\varphi(E(|u|^2)E(|w|^2)) \circ \varphi^{-1}(x) \leqslant [\frac{a^6 + a^7}{a+1}]^2.$$

Thus, $J \in L^\infty(\Sigma)$, and so $T_\varphi = M_w E M_u C_\varphi$ is bounded on $L^2(\Sigma)$, where for each $f \in L^2(\Sigma)$ and $x \in [\frac{1}{a}, a]$ we have

$$T_\varphi f(x) = [\frac{ax^5}{a+1}]f(\frac{1}{x}) + [\frac{x^3(a(1-x)+1)^2}{a+1}]f(\frac{1}{a(1-x)+1}).$$

Simple computations show that $\varphi^2 = id$, $\sigma(h) = \sigma(w) = X$, $\sqrt{h(h\circ\varphi)} = \frac{w}{|w|}\frac{\bar{w}\circ\varphi}{|w|\circ\varphi} = 1$ and $\frac{E_\varphi(u)}{|w|}\sqrt{h\circ\varphi} = 1 \in L^0(\mathscr{A})$, so T_φ is complex symmetric. Direct computations confirm correctness of this. Define $C(f)=\frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. It is clear that C is conjugate linear, $C^2(f)=\frac{f\circ\varphi^2}{\sqrt{hh\circ\varphi}}=f$ and $\langle Cg,Cf\rangle=\langle f,g\rangle$. Now, by a direct computation we get that

$$
\begin{aligned}
CT_\varphi^* C(f) &= \frac{x^3}{x^3\sqrt{h\circ\varphi}} h\circ\varphi E_\varphi(M_{x^2}EM_{x^3}\frac{f\circ\varphi}{\sqrt{h\circ\varphi}}) \\
&= \frac{x^2\sqrt{h\circ\varphi}}{x^3} x^3 E(x^3\frac{f\circ\varphi}{\sqrt{h\circ\varphi}}).
\end{aligned}
$$

Since $\frac{x^2\sqrt{h\circ\varphi}}{x^3}=1\in L^0(\mathscr{A})$, hence $CT_\varphi^* C(f)=x^3E(x^2f\circ\varphi)=T_\varphi(f)$. Thus $T_\varphi: L^2(\Sigma)\to L^2(\Sigma)$ is complex symmetric.

COROLLARY 3.8. *Let $\varphi^{-1}(\Sigma)\subseteq\mathscr{A}$, $\varphi^2=id$ and $\sigma(h)=X$. If $h(h\circ\varphi)=1$ and $wE(u)\in L^0(\mathscr{A})$, then the bounded Lambert type operator T_φ on $L^2(\Sigma)$ is complex symmetric.*

Proof. By Proposition 2.3, $T_\varphi=M_vC_\varphi$ where $v=wE(u)$. Define

$$C: L^2(\Sigma)\to L^2(\Sigma)$$

$$C(f)=\frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}.$$

According to assumptions it is clear that C is a conjugation and $CT^*C=T$. $\square$

COROLLARY 3.9. *Let $\sigma(h)=X$ and $\varphi^2=id$. If $h(h\circ\varphi)=1$ and $u\in L^0(\mathscr{A})$, where $\mathscr{A}=\varphi^{-1}(\Sigma)$, then the bounded weighted composition operator $M_uC_\varphi: L^2(\Sigma)\to L^2(\Sigma)$ is complex symmetric.*

It is well-known fact that if $T:\mathscr{H}\to\mathscr{H}$ is bounded C-symmetric operator, then $T=CJ|T|$ where J is a conjugation that commutes with $|T|$ (see [5]).

EXAMPLE 3.10. Again suppose that $1<a<\infty$. Let $X=[\frac{1}{a},a]$, $d\mu=dx$ and Σ be the Lebesgue sets. Define the non-singular transformation $\varphi: X\to X$ by $\varphi(x)=\frac{1}{x}$. Put $w=1$, $u(x)=x^2$ and $\mathscr{A}=\varphi^{-1}(\Sigma)$. Then $h(x)=\frac{1}{x^2}$ and $E=I$. Then $T_\varphi f=u.f\circ\varphi$ is a bounded C-symmetric operator on $L^2(\Sigma)$, where $C(f)=\frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. Computations show that $|T|f(x)=\frac{1}{x^3}f(x)$ and $Uf(x)=\frac{1}{x}f(\frac{1}{x})$. Since for each $f\in L^2(\Sigma)$, $x^3f\in L^2(\Sigma)$ thus $|T|$ is onto. Put $Jf=CUf=U^*Cf=\bar{f}$. It is clear that J is a conjugation on $L^2(\Sigma)$ and $T=CJ|T|$.

REMARK 3.11. Let $T_1\in B(L^2(\Sigma))$. By Corollary 2.5, $\widetilde{T_1}=M_{v\bar{u}}EM_u$ in which $v=\chi_{G_1}\frac{E(uw)}{E(|u|^2)}\in L^0(\mathscr{A})$. It is easy to check that $(\widetilde{T_1})^*=\widetilde{(T_1^*)}$ and $\widetilde{T_1}\widetilde{T_1}^*=\widetilde{T_1}^*\widetilde{T_1}$. Thus, the Aluthge transformation of the weighted Lambert type operators are always normal and so $\widetilde{T_1}$ is complex symmetric.

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