Operators
and
Matrices
Volume 12, Number 1 (2018), 287–299 doi:10.7153/oam-2018-12-19

SOME SPECTRAL PROPERTIES OF FOURTH ORDER DIFFERENTIAL OPERATOR EQUATION

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(*Communicated by M. Zinchenko*)

Abstract. We consider boundary value problem for fourth order differential equation with unbounded discrete operator coefficient. One of the boundary conditions involves the λ parameter. The asymptotics of spectrum of corresponding selfadjoint operator is obtained. We also calculate the trace of that operator.

Introduction

In this paper we extend the spectral analysis of second order differential problems with unbounded operator coefficient in equation and eigenvalue parameter in the boundary condition to fourth order differential problems. Our main aim here is to investigate the spectrum of operator associated with that problem and derive the regularized trace formula. In scalar case boundary value problems for the second order differential equation with λ dependent boundary condition were treated in [1, 2]. In stated works considered problem is embedded into the theory of Hilbert space in which it can be considered as the eigenvalue problem of a selfadjoint operator. That approach was used for the Sturm-Liuville operator equations with unbounded operator operator-coefficient and eigenvalue dependent boundary condition in [3, 4]. Namely, by introducing the new Hilbert space they describe the selfadjoint extension of minimal operator and find asymptotics of eigenvalues.

Theory of differential-operator equations is one of the most important methods of contemporary mathematics. Many problems for partial differential equations lead to problems for differential-operator equations posed in some functional spaces. And sometimes that approach provides methods for solving problems which are impossible to treat by classical methods. For example, in asymptotic expansion of eigenvalues of boundary value problems for partial differential equations the remainder term does not form convergent series. But usually that term is known for scalar problems. That is why it is impossible to calculate regularized trace of an operator corresponding to that problem by methods existing for scalar differential operators. We may refer to monographes by M. L. Gorbachuk and V. I. Gorbachuk [5], F. S. Rofe-Beketov and A. M. Holkin [6] where the theory of differential operator equations is developed.

Mathematics subject classification (2010): 34B05, 34G20, 34L20, 34L05, 47A05, 47A10.
Keywords and phrases: Hilbert space, differential operator equation, spectrum, eigenvalues, trace class operators, regularized trace.

Paper OaM-12-19

Comprehensive list of references to works where spectral properties of differential operators with operator coefficients are studied is given in [8]. In that work authors treat in detail basic spectral theory for self adjoint Schrödinger operators with operator valued potentials including Weyl-Titchmarch theory, Green's function structure, diagonalization and a version of spectral theorem.

History and current state of the traces of linear operators are presented in survey paper [7].

In [9–14] we considered singular and regular differential operator equations of second order with λ dependent boundary conditions. For that problems we obtained the asymptotics of λ and established trace formulas. In [14] the boundary value problem for fourth order differential operator equation without spectral parameter in boundary condition is studied.

1. Problem statement

We consider in space $L_2((0,1),H)$ (where H is separable abstract Hilbert space) the boundary value problem

$$y^{IV}(x)+Ay(x)+q(x)y(x)=\lambda y(x), \tag{1}$$

$$y(0)=y''(0)=0, \tag{2}$$

$$y''(1)=0 \tag{3}$$

and

$$y'''(1)+\lambda y(1)=0. \tag{4}$$

Here A is assumed to be a selfadjoint operator in H, moreover $A>I$, I is identity operator in H, $A^{-1}\in\sigma_\infty$. Under that assumptions A^{-1} is compact operator and spectrum of A is discrete. Denote its eigen-vectors by $\varphi_1,\varphi_2,\ldots$. About $q(x)$ we suppose that it is operator-valued function in H for each $x\in[0,1]$, and has weak derivatives $q^{(l)}(x)$, where $l=1,2$. Moreover, it satisfies the next conditions:

1) $q^*(x)=q(x)$.

2) $q^{(l)}(x)\in\sigma_1$, (σ_1 is space of trace class operators),

$$\left[q^{(l)}(x)\right]^*=q^{(l)}(x), \quad l=0,1,2$$

3) $\int\limits_0^1(q(x)\varphi_j,\varphi_j)\,dx=0$.

Recall that $L_2((0,1),H)$ is space of vector functions with integrible square of H norms, namely class of vector functions $y(x)$ with values from H and

$$\int\limits_0^1\|y(x)\|_H^2\,dx<\infty.$$

Introduce the space $H_1 = L_2((0,1),H) \oplus H$ of vectors $Y = \{y(x), y_1\}$, where $y(x) \in L_2((0,1),H)$, $y_1 \in H$. Define in H_1 the scalar product of elements $Y, Z \in H_1$, $Y = (y(x), y_1)$, $Z = (z(x), z_1)$ by

$$(Y,Z) = \int_0^1 (y(x), z(x))_H \, dx + (y_1, z_1)_H \, .$$

Define in H_1 operators L_0, L_1 by

$$D(L_0) = \{Y \in H_1 / y'''(x) \text{ is absolutely contionous in norm } \|\cdot\|,$$

$$y^{IV}(x) + Ay(x) \in L_2((0,1),H), y_1 = y(1)\},$$

$$L_0 Y = \{y^{IV}(x) + Ay(x), -y'''(0)\},$$

$$L = L_0 + Q, \; QY = \{q(x)y(x), 0\}.$$

By the technic of [3, 4, 15] it might be shown that L and L_0 are self-adjoint operators in H_1. The main questions to be treated in that paper are to investigate the asymptotic behavior of eigenvalues of L and L_0 and derive the regularized trace formula for operator L. Under stated conditions L_0 and L are both discrete operators. Denote their eigenvalues by $\mu_1, \mu_2, \ldots$ and $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ respectively, in ascending order by counting multiplicities.

2. Asymptotics of eigenvalues

Denote eigenvalues of A by $\gamma_1 \leqslant \gamma_2 \leqslant \ldots$. Since A is a self-adjoint operator then expanding the problem (1)–(4) in its eigenvectors we get the next spectral problem in $L_2(0,1)$ for Fourier coefficients $y_k(x) = (y(x), \varphi_k)_H$:

$$y_k^{IV}(x) + \gamma_k y_k(x) = \lambda y_k(x), \tag{5}$$

$$y_k(0) = y_k''(0) = 0, \tag{6}$$

$$y_k''(1) = 0, \tag{7}$$

and

$$y_k'''(1) + \lambda y_k(1) = 0. \tag{8}$$

Solution of equation (5) satisfying conditions (6) is

$$y_k(x) = c_1 \sin \sqrt[4]{\lambda - \gamma_k} x + c_2 \operatorname{sh} \sqrt[4]{\lambda - \gamma_k} x. \tag{9}$$

Let

$$z = \sqrt[4]{\lambda - \gamma_k}.$$

Solution (9) satisfies (7) and (8) if and only if hold

$$-c_1 \sin z + c_2 \operatorname{sh} z = 0 \tag{10}$$

and

$$-c_1 z^3 \cos z + c_2 z^3 \operatorname{ch} z + c_1 \left(z^4 + \gamma_k\right) \sin z + c_2 \left(z^4 + \gamma_k\right) \operatorname{sh} z = 0. \tag{11}$$

System of equations (10), (11) in c_1, c_2 has unique solution only if

$$\begin{vmatrix} -\sin z & \operatorname{sh} z \\ -z^3 \cos z + \left(z^4 + \gamma_k\right) \sin z & z^3 \operatorname{ch} z + \left(z^4 + \gamma_k\right) \operatorname{sh} z \end{vmatrix} = 0$$

or equivalently

$$z^3 \left(\operatorname{sh} z \cos z - \operatorname{ch} z \sin z\right) - 2\left(z^4 + \gamma_k\right) \sin z \operatorname{sh} z = 0. \tag{12}$$

Writing it in form

$$tgz = \frac{z^3}{2\left(z^4 + \gamma_k\right) + z^3 \operatorname{cth} z}$$

we get that real roots are

$$\alpha_k = \pi k + O\left(\frac{1}{k}\right). \tag{13}$$

Now look for roots of (12) having form $z = iy \ (y > 0)$, if any. Taking in (12) $z = iy$ we have

$$(iy)^3 \left(\operatorname{sh} iy \cos iy - \operatorname{ch} iy \sin iy\right) - 2\left((iy)^4 + \gamma_k\right) \sin iy \operatorname{sh} iy = 0$$

which simplifies to

$$-y^3 \left(\operatorname{sh} y \cos y - \operatorname{ch} y \sin y\right) = -2\left(y^4 + \gamma_k\right) \sin y \operatorname{sh} y.$$

The last is equivalent to (12). Thus

$$y = \pi m + O\left(\frac{1}{m}\right) \tag{14}$$

and corresponding to that roots imaginary roots of (12) are eigenvalues of L_0 are

$$\beta_m = iy = i\left(\pi m + O\left(\frac{1}{m}\right)\right).$$

Eigenvalues of selfadjoint operator L_0 are real and expressible like $\mu_k = \gamma_k + z^4$, where z is root of (12). Now we will study existence of roots of (12) of form $y + iy$, since fourth degree of that numbers yields real number

$$2iy^2 (y + iy)\left[\frac{e^{y+iy} - e^{-y-iy}}{2} \cdot \frac{e^{iy-y} + e^{-iy+y}}{2} - \frac{e^{y+iy} + e^{-y-iy}}{2} \cdot \frac{e^{iy-y} + e^{-iy+y}}{2i}\right]$$

$$-2\left(-4y^4 + \gamma_k\right)\left[\frac{e^{iy-y} - e^{-iy+y}}{2i} \cdot \frac{e^{y+iy} - e^{-y-iy}}{2}\right] = 0.$$

Multiplying the terms

$$iy^2(y+iy)\left[\frac{e^{2iy}+e^{2y}-e^{-2y}-e^{-2iy}}{2}-\frac{e^{2iy}-e^{2y}+e^{-2y}-e^{-2iy}}{2i}\right]$$
$$-2\left(-4y^4+\gamma_k\right)\left[\frac{e^{2iy}-e^{-2y}-e^{2y}+e^{-2iy}}{2}\right]=0,$$

or

$$\left(iy^3-y^3\right)\left[\operatorname{sh}2y+i\sin 2y+\frac{\operatorname{sh}2y}{i}-\sin 2y\right]-\left(\gamma_k-4y^4\right)\left[\frac{\cos 2y}{i}-\frac{\operatorname{ch}2y}{i}\right]=0,$$
$$y^3(i-1)\left[\frac{i+1}{i}\operatorname{sh}2y+\sin 2y\,(i-1)\right]-\left(\gamma_k-4y^4\right)\left[\frac{\cos 2y-\operatorname{ch}2y}{i}\right]=0,$$
$$y^3\left[-2\operatorname{sh}2y+2\sin 2y\right]=\left(\gamma_k-4y^4\right)\left(\cos 2y-\operatorname{ch}2y\right).$$

Expanding the last into power series

$$2y^3\left[\sum_{k=0}^{\infty}\frac{(-1)^k(2y)^{2k+1}}{(2k+1)!}-\sum_{k=0}^{\infty}\frac{(2y)^{2k+1}}{(2k+1)!}\right]$$
$$=\left(\gamma_k-4y^4\right)\left[\sum_{k=0}^{\infty}\frac{(-1)^k(2y)^{2k}}{(2k)!}-\sum_{k=0}^{\infty}\frac{(2y)^{2k}}{(2k)!}\right].$$

After simplifications

$$-\gamma_k+\sum_{k=1}^{\infty}\frac{y^{4k}2^{4k-1}\left(4k(4k+1)(4k+2)+(4k-1)4k(4k+1)(4k+2)-4\gamma_k\right)}{(4k+2)!}=0. \tag{15}$$

Obviously starting with some k all coefficients at y^{4k} in (15) become positive. Thus by Descarte's rule of signs for each k (15) has exactly one positive root. Find asymptotics of that roots. Rewrite (15) in the form

$$\frac{2y^3(\sin 2y-\operatorname{sh}2y)}{\cos 2y-\operatorname{ch}2y}=\gamma_k-4y^4. \tag{16}$$

Since left hand side of (16) is positive, then $\gamma_k-4y^4>0$, thus $0<y<\sqrt[4]{\frac{\gamma_k}{4}}$. Expanding functions on the left of (16)

$$\frac{4y^4\sum\limits_{n=1}^{\infty}\frac{(2y)^{4n-2}}{(4n-1)!}}{\sum\limits_{n=1}^{\infty}\frac{(2y)^{4n-2}}{(4n-2)!}}=\gamma_k-4y^4, \tag{17}$$

or

$$4y^4\alpha(y)=\gamma_k-4y^4,\quad \alpha(y)=\frac{\sum\limits_{n=1}^{\infty}\frac{(2y)^{4n-2}}{(4n-1)!}}{\sum\limits_{n=1}^{\infty}\frac{(2y)^{4n-2}}{(4n-2)!}},$$

and

$$y = \sqrt[4]{\frac{\gamma_k}{4\alpha(y)+4}}.$$

Clearly, $\alpha(y) < 1$ and close to 1.

Eigenvalues corresponding to that roots are

$$\mu_k = \gamma_k - 4y^4 = \frac{4\alpha(y)}{4\alpha(y)+4}\gamma_k, \;\; k = 1,2,\dots .$$

Analogously putting in (12) $z = y - iy$ we again get equation (16). Resuming all stated above we conclude that for each k equation (12) has countable real roots and exactly one root of forms $y \pm iy$. Thus we have proved the next theorem.

THEOREM 1. *Eigenvalues of operator L_0 are repeated with multiplicities 2 and they form two series*

$$\lambda_{k,m} = \gamma_k + \left(\pi m + O\left(\frac{1}{m}\right)\right)^4, \tag{18}$$

$$\lambda_k \sim c(y)\gamma_k, \tag{19}$$

where $c(y) < \frac{1}{2}$ and close to $\frac{1}{2}$.

Assume that $\gamma_k \sim a \cdot k^{\alpha}$, $a > 0$, $\alpha > 0$. Then in virtue of (18), (19) also asymptotics (13), (14) and since Q is bounded operator in H_1, by the way of Theorem 1 [15] we get the next lemma.

LEMMA 1. *If eigenvalues of operator A for big k values satisfy $\gamma_k \sim a \cdot k^{\alpha}$ $(a>0, \alpha>0)$, then for eigenvalues of L_0 and L the next asymptotic relation is true:*

$$\mu_n \sim \lambda_n \sim c n^{\frac{4\alpha}{4+\alpha}}.$$

Now we turn to deriving trace formula for L.

From that lemma it follows that the resolvent of L_0 and L is trace class operator (from σ_1) if $\alpha > \frac{4}{3}$. So under that condition by the way of Theorem 1 and Theorem 2 from [16] we get that

$$\sum_{n=1}^{\infty}{}' (\lambda_n - \mu_n) \equiv \lim_{m\to\infty} \sum_{n=1}^{n_m} (\lambda_n - \mu_n) = \sum_{n=1}^{\infty} (Q\psi_n, \psi_n), \tag{20}$$

where $\{\psi_n\}_{n=1}^{\infty}$ are orthonormal eigenvectors of L_0, n_m some subsequence of natural numbers. Sum $\sum_{n=1}^{\infty}{}' (\lambda_n - \mu_n)$ in accordance with [16] we call regularized trace. Thus for evaluating sum of series in (20) we have to find orthonormal eigenvectors of L_0. Eigenvectors of L_0 are

$$Y = \{[c_1 \sin(\alpha_m x) + c_2 \mathrm{sh}\,(\alpha_m x)]\,\varphi_k, -[c_1 \sin\alpha_m + c_2 \mathrm{sh}\,\alpha_m]\,\varphi_k\},\; m = 0, \infty,\; (\alpha_0 = y + iy)$$

$$Y = \{[c_1 \sin(\beta_m x) + c_2 \mathrm{sh}\,(\beta_m x)]\,\varphi_k, -[c_1 \sin\beta_m + c_2 \mathrm{sh}\,\beta_m]\,\varphi_k\},\; m = 0, \infty,\; (\beta_0 = y - iy).$$

From (8) $c_2 = c_1 \dfrac{\sin\alpha_m}{\operatorname{sh}\alpha_m}$.

Norm of Y is

$$
\begin{aligned}
\|Y\|_{H_1}^2 &= c_1^2 \int_0^1 \left[\sin^2(\alpha_m x) + \frac{2\sin\alpha_m}{\operatorname{sh}\alpha_m}\operatorname{sh}(\alpha_m x)\sin(\alpha_m x) + \frac{\sin^2\alpha_m}{\operatorname{sh}^2\alpha_m}\operatorname{sh}^2(\alpha_m x)\right] dx \\
&\quad + 4\sin^2\alpha_m \\
&= \frac{1}{2} - \frac{\sin 2\alpha_m}{4\alpha_m} + \frac{\sin\alpha_m}{\operatorname{sh}\alpha_m}\cdot\frac{1}{\alpha_m}\left(\operatorname{ch}\alpha_m\sin\alpha_m - \operatorname{sh}\alpha_m\cos\alpha_m\right) \\
&\quad + \frac{\sin^2\alpha_m}{\operatorname{sh}^2\alpha_m}\frac{\sin 2\alpha_m}{4\alpha_m} - \frac{1}{2}\frac{\sin^2\alpha_m}{\operatorname{sh}^2\alpha_m} + 4\sin^2\alpha_m .
\end{aligned}
$$

Taking into consideration from (12) that

$$
\operatorname{ch}\alpha_m\sin\alpha_m - \operatorname{sh}\alpha_m\cos\alpha_m = \frac{-2\left(\alpha_m^4 + \gamma_k\right)\sin\alpha_m \operatorname{sh}\alpha_m}{\alpha_m^3}
$$

after simplifications we get

$$
\begin{aligned}
\|Y\| = {} & \frac{2\alpha_m\operatorname{sh}^2\alpha_m - \sin 2\alpha_m\operatorname{sh}^2\alpha_m + 8\alpha_m\operatorname{sh}^2\alpha_m\sin^2\alpha_m}{4\alpha_m\operatorname{sh}^2\alpha_m} \\
& + \frac{\sin^2\alpha_m\operatorname{sh}2\alpha_m - 2\alpha_m\sin^2\alpha_m - \frac{8\gamma_k\sin^2\alpha_m\operatorname{sh}^2\alpha_m}{\alpha_m^3}}{4\alpha_m\operatorname{sh}^2\alpha_m} .
\end{aligned}
$$

The same result we get writing β_m in place of α_m. Thus taking $c_1 = \sqrt{\frac{4\alpha_m\operatorname{sh}^2\alpha_m}{H_{k,m}}}$, where

$$
\begin{aligned}
H_{k,m} = {} & \alpha_m\operatorname{sh}^2\alpha_m - \sin 2\alpha_m\operatorname{sh}^2\alpha_m + 8\alpha_m\operatorname{sh}^2\alpha_m\sin^2\alpha_m \\
& + \sin^2\alpha_m\operatorname{sh}2\alpha_m - 2\alpha_m\sin^2\alpha_m - \frac{8\gamma_k\sin^2\alpha_m\operatorname{sh}^2\alpha_m}{\alpha_m^3}
\end{aligned}
$$

(also expression $H_{k,m}$ with β_m instead of α_m which we shall denote $H'_{k,m}$) we obtain orthonormal eigen-vectors of operator L_0.

Now we will prove absolute convergence of series $\sum\limits_{n=1}^{\infty}(Q\psi_n, \psi_n)$.

LEMMA 2. *Under conditions 1) – 3)*

$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left|\frac{4\alpha_m \mathrm{sh}^2\alpha_m}{H_{k,m}}\left(\int_0^1 q_k(x)\sin^2(\alpha_m x)\,dx+\int_0^1 2q_k(x)\frac{\sin\alpha_m}{\mathrm{sh}\,\alpha_m}\mathrm{sh}\,(\alpha_m x)\sin(\alpha_m x)\,dx\right.\right.$$
$$\left.+\int_0^1 q_k(x)\frac{\sin^2\alpha_m}{\mathrm{sh}^2\alpha_m}\mathrm{sh}^2(\alpha_m x)\sin^2(\alpha_m x)\,dx\right)$$
$$+\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left|\frac{4\beta_m \mathrm{sh}^2\beta_m}{H'_{k,m}}\left(\int_0^1 q_k(x)\sin^2(\beta_m x)\,dx+\int_0^1 2q_k(x)\frac{\sin\beta_m}{\mathrm{sh}\,\beta_m}\mathrm{sh}\,(\beta_m x)\sin(\beta_m x)\,dx\right.\right.$$
$$\left.\left.+\int_0^1 q_k(x)\frac{\sin^2\beta_m}{\mathrm{sh}^2\beta_m}\mathrm{sh}^2(\beta_m x)\sin^2(\beta_m x)\,dx\right)\right|<\infty. \tag{21}$$

Proof. For the first of series in left of (21)

$$\frac{4\alpha_m \mathrm{sh}^2\alpha_m}{H_{k,m}}\int_0^1 q_k(x)\sin^2(\alpha_m x)\,dx=\frac{-2\alpha_m \mathrm{sh}^2\alpha_m}{H_{k,m}}\int_0^1 q_k(x)\cos(\alpha_m x)\,dx. \tag{22}$$

Here condition 3) is used. In virtue of asymptotics α_m (13) coefficient at integral in (22) for big m values is equivalent to 2. Integrating twise by parts

$$\begin{aligned}\int_0^1 q_k(x)\cos(\alpha_m x)\,dx &= \frac{1}{2\alpha_m}\sin(2\alpha_m x)\,q_k(x)\Big|_0^1-\int_0^1\frac{1}{2\alpha_m}\sin(2\alpha_m x)\,q_k'(x)\,dx\\ &=\frac{1}{2\alpha_m}\sin 2\alpha_m q_k(1)+\frac{1}{4\alpha_m^2}\cos 2\alpha_m q_k'(1)\\ &\quad-\frac{1}{4\alpha_m^2}q_k'(0)-\frac{1}{4\alpha_m^2}\int_0^1\cos(2\alpha_m x)\,q_k''(x)\,dx.\end{aligned}$$

Now convergence of series with term (22) follow from asymptotics α_m and condition 2)

$$\left(\left\|q^{(l)}(x)\right\|_{\sigma_1}\leqslant const,\ l=0,1,2\right).$$

Now consider series with terms

$$\int_0^1 q_k(x)\frac{\sin\alpha_m}{\mathrm{sh}\,\alpha_m}\mathrm{sh}\,(\alpha_m x)\sin(\alpha_m x)\,dx. \tag{23}$$

From asymptotics α_m it follows that

$$\sin\alpha_m=O\left(\frac{1}{\alpha_m}\right),$$

$$\frac{\operatorname{sh}(\alpha_m x)}{\operatorname{sh}\alpha_m} \sim e^{\alpha_m (x-1)}. \tag{24}$$

By condition 2) it follows that $\left|\sum_{k=1}^{\infty} q_k(x)\right| < const$. Therefore from (24) and (2)

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_0^1 q_k(x)\frac{\sin\alpha_m}{\operatorname{sh}\alpha_m}\operatorname{sh}(\alpha_m x)\sin(\alpha_m x)\,dx\right| \\
&\sim \sum_{m=1}^{\infty}\int_0^1\left|\frac{e^{\alpha_m(x-1)}}{\alpha_m}\sum_{k=1}^{\infty} q_k(x)\,dx\right| < \sum_{m=1}^{\infty} const\int_0^1 \frac{e^{\alpha_m(x-1)}}{\alpha_m}dx \\
&= const\sum_{m=1}^{\infty}\frac{1}{\alpha_m^2}\left(1-\frac{1}{e^{\alpha_m}}\right) < const.
\end{aligned}$$

For $m = 0$ we have series

$$\sum_{k=1}^{\infty}\int_0^1 q_k(x)\frac{\sin\alpha_0}{\operatorname{sh}\alpha_0}\operatorname{sh}(\alpha_0 x)\sin(\alpha_0 x)\,dx. \tag{25}$$

For big k values root of (21) y is also quite big and reciprocal of norm is equivalent to

$$\frac{1}{2\sin^2\alpha_0} \sim \frac{1}{2e^{2y}} \quad (\alpha_0 = y + iy).$$

The product in (25) with exception $q_k(x)$ is equivalent to $e^{2y(x-1)}$.

Now convergence of (25) follow from condition $\|q(x)\|_{\sigma_1} < const$. Consider series with terms

$$\int_0^1 q_k(x)\frac{\sin^2\alpha_m}{\operatorname{sh}^2\alpha_m}\operatorname{sh}^2(\alpha_m x)\,dx. \tag{26}$$

Again from asymptotics of α_m (26) for big m values behave like

$$O\left(\frac{1}{\alpha_m^2}\right)\int_0^1 q_k(x)\frac{\operatorname{sh}^2\alpha_m x}{\operatorname{sh}^2\alpha_m}dx \sim O\left(\frac{1}{\alpha_m^2}\right),$$

$$\int_0^1 q_k(x)\,e^{\alpha_m(x-1)}dx < O\left(\frac{1}{\alpha_m^2}\right)\int_0^1 q_k(x)\,dx. \tag{27}$$

Convergence follow from (27). Using similar arguments we could justify also convergence of series with terms β_m instead of α_m. Lemma 2 is proved. $\square$

From lemma 2 it directly follows that

$$\sum_{n=1}^{\infty}{}' (\lambda_n - \mu_n) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{H_{k,m}} \left[-2\alpha_m \operatorname{sh}^2\alpha_m \int_0^1 q_k(x)\cos(\alpha_m x)\,dx + \right. \tag{28}$$

$$+ \int_0^1 2q_k(x)\frac{\sin\alpha_m}{\operatorname{sh}\alpha_m}\operatorname{sh}(\alpha_m x)\sin(\alpha_m x)\,dx +$$

$$\left. + \int_0^1 q_k(x)\frac{\sin^2\alpha_m}{\operatorname{sh}^2\alpha_m}\operatorname{sh}^2(\alpha_m x)\sin^2(\alpha_m x)\,dx \right] +$$

$$+ \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{H'_{k,m}}\left[-2\beta_m \operatorname{sh}^2\beta_m \int_0^1 \cos(2\beta_m x)\, q_k(x)\,dx + \right.$$

$$+ \int_0^1 2q_k(x)\frac{\sin\beta_m}{\operatorname{sh}\beta_m}\operatorname{sh}(\beta_m x)\sin(\beta_m x)\,dx +$$

$$\left. + \int_0^1 q_k(x)\frac{\sin^2\beta_m}{\operatorname{sh}^2\beta_m}\operatorname{sh}^2(\beta_m x)\sin^2(\beta_m x)\,dx \right]. \tag{29}$$

To evaluate the sum of that series we shall select a function of complex variable z having as poles the roots of equation (12) and residues of which are terms of series (28). Then by selecting appropriate extending closed contours involving zeros of that function and investigating behavior of that function along that contours, by applying Cauchy theorem we derive the trace formula.

Consider

$$S_N(x) = \sum_{m=0}^{\infty}\left[\frac{-2\alpha_m \operatorname{sh}^2\alpha_m}{H_{k,m}}\cos(\alpha_m x)\, q_k(x) + \frac{-2\beta_m \operatorname{sh}^2\beta_m}{H'_{k,m}}\cos(2\beta_m x)\, q_k(x) \right].$$

To evaluate $\lim\limits_{N\to\infty} S_N(x)$ we take the function

$$f_1(z) = \frac{z\cos 2zx}{\sin^2 z\left(-z\operatorname{cth} z + z\operatorname{ctg} z - 2z^2 - 2\frac{\gamma_k}{z^2}\right)}.$$

Obviously it has poles at points α_m, β_m, $y \pm iy$ and πm.

$$\operatorname*{res}_{z=\alpha_m} f_1(z) = \frac{\alpha_m\cos(2\alpha_m x)}{\sin^2\alpha_m\left(-z\operatorname{cth} z + z\operatorname{ctg} z - 2z^2 - 2\frac{\gamma_k}{z^2}\right)'_{z=\alpha_m}}$$

$$= \frac{\alpha_m\cos(2\alpha_m x)}{\sin^2\alpha_m\left(-\operatorname{cth} z + \frac{z}{\operatorname{sh}^2 z} + \operatorname{ctg} z - \frac{z}{\sin^2 z} - 4z + \frac{4\gamma_k}{z^3}\right)_{z=\alpha_m}}$$

$$= \frac{-2\alpha_m \operatorname{sh}^2\alpha_m\cos(2\alpha_m x)}{H_{k,m}}.$$

Then

$$\operatorname*{res}_{z=\pi m} f_1(z) = \frac{\pi m \cos(2\pi m x)}{\cos(2\pi m x)\left(-\pi m \operatorname{cth}(\pi m)\sin(\pi m) + \pi m\cos(\pi m) - 2(\pi m)^2 \sin(\pi m) - \frac{2\gamma_k \sin(\pi m)}{(\pi m)^2}\right)}$$
$$= \frac{\pi m \cos(2\pi m x)}{\pi m} = \cos(2\pi m x).$$

Consider also functions

$$f_2(z) = \frac{-2z\sin z \operatorname{sh}(zx)\sin(zx)}{\operatorname{sh} z \sin^2 z\left(-z\operatorname{cth} z + z\operatorname{ctg} z - 2z^2 - 2\frac{\gamma_k}{z^2}\right)}$$

and

$$f_3(z) = \frac{\sin^2 z \operatorname{ch}(2zx)}{\operatorname{sh}^2 z \sin^2 z\left(-z\operatorname{cth} z + z\operatorname{ctg} z - 2z^2 - 2\frac{\gamma_k}{z^2}\right)}.$$

It might be shown by direct calculation that residues of that functions at pales are second and third terms of series (28). $f_2(z)$ and $f_3(z)$ have no other poles with exception α_m and β_m.

Take as contour of integration a regular contour l with vertices at $\pm B_M$, $B_M \pm iA_N$, where $B_M = \pi M + \frac{\pi}{2}$, $A_N = \pi N + \frac{\pi}{2}$. For such choice of A_N inside considered countour lie N real and M imaginary zeros of function $f_1(z)$. To include also zeros of form $y+iy$ we shall take $A_N > \sqrt[4]{\frac{\gamma_k}{4}}$, $B_M > \sqrt[4]{\frac{\gamma_k}{4}}$. Since $f_1(z)$ is odd function points $-i\alpha_m$ are also zeros of that function. That is why contour l should pass by imaginary zero $iy, y>0$ along small semicircle centered at iy on the left and $-iy$ on the right.

Let $z = u+iv$. The order of $f_1(z)$ for large v values and for $u \geqslant 0$ is $O\left(\frac{1}{|v|e^{2|v|(1-x)}}\right)$ and integrals along upper and lower sides of contour vanish when $M \to \infty$.

Thus we have

$$S_N(x) + \sum_{m=0}^{N} \cos(2\pi m x) = \frac{1}{2\pi i}\int_{B_M - iA_N}^{B_M + iA_N} \frac{z\cos(2zx)\,dz}{\sin^2 z\left(-z\operatorname{cth} z + z\operatorname{ctg} z - 2z^2 - 2\frac{\gamma_k}{z^2}\right)}. \tag{30}$$

$f_1(z)$ for big z values is of order $O\left(\dfrac{\cos 2zx}{z\cos 2z}\right)$.

When $N \to \infty$ we have

$$\lim_{M\to\infty} \frac{1}{2\pi i}\int_{A_N - iB_M}^{A_N + iB_M} f_1(z)\,dz \sim \frac{1}{\pi i}\int_{A_N - i\infty}^{A_N + i\infty} \frac{\cos 2zx}{z\cos 2zx}dz \tag{31}$$
$$\sim \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos \pi x(2N+1)\cos 2ivx - \sin \pi x(2N+1)\sin 2ivx}{-(A_N+iv)\cos 2iv}dv.$$

Absolute value of the last is less than

$$\frac{2}{A_N}\int_0^\infty\left[\frac{\cos 2vx}{\operatorname{ch} 2v}+\frac{\sin 2vx}{\operatorname{sh} 2v}\right]dx<\frac{2+2\sin\frac{\pi x}{2}}{A_N\cos\frac{\pi x}{2}}.$$

Therefore under condition

$$\int_0^1 \frac{q_k(x)}{\cos\frac{\pi x}{2}}dx<\infty \tag{32}$$

it follows that

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{2\pi i}\int_0^1\int_{A_N-iB_M}^{A_N+iB_M} f_1(z)\,dz q_k(x)\,dx=0. \tag{33}$$

Analogously it can be shown that

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{2\pi i}\int_0^1\int_{A_N-iB_M}^{A_N+iB_M}(f_2(z)+f_3(z))\,dz q_k(x)\,dx=0. \tag{34}$$

Taking under consideration (29), (31), (32),

$$\sum_{m=0}^\infty \frac{-2\alpha_m \operatorname{sh}^2\alpha_m}{H_{k,m}}\int_0^1\cos(2\alpha_m x)\,q_k(x)\,dx=\lim_{N\to\infty}\int_0^1 S_N(x)\,q_k(x)\,dx$$

$$=-\lim_{N\to\infty}\int_0^1\sum_{m=0}^N\cos(2\pi_m x)\,q_k(x)\,dx=-\frac{q_k(0)+q_k(1)}{4}. \tag{35}$$

From (28), (32) and (33) we obtain

$$\sum_{n=1}^\infty(\lambda_n-\mu_n)=-\sum_{k=1}^\infty\frac{q_k(0)+q_k(1)}{4}=-\frac{trq(0)+trq(1)}{4}.$$

Thus we have proved the next theorem.

THEOREM 2. *Let be satisfied the conditions 1) – 3). If eigenvalues of A satisfy*

$$\gamma_k\sim ak^\alpha,\ \ \alpha>\frac{4}{3}$$

and (31) is fulfilled then for the regularized trace the formulae

$$\sum_{n=1}^{\infty}{}'(\lambda_n-\mu_n)=-\frac{trq(0)+trq(1)}{4}$$

is valid.

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(Received April 28, 2017)

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