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PRODUCTS OF RADIAL DERIVATIVE AND WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN–ORLICZ SPACES TO WEIGHTED–TYPE SPACES

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Abstract. Let $H(\mathbb{B}^n)$ be the space of all holomorphic functions on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, φ a holomorphic self-map of $\mathbb{B}^n$, $u \in H(\mathbb{B}^n)$, and $\Re$ the radial derivative operator on $H(\mathbb{B}^n)$. Two operators on $H(\mathbb{B}^n)$ are defined by $\Re W_{u,\varphi} f(z) = \Re(u(z)f(\varphi(z)))$ and $W_{u,\varphi}\Re f(z) = u(z)\Re f(\varphi(z))$, which are called the products of radial derivative operators and weighted composition operators. In this paper, the boundedness and compactness of the operators $\Re W_{u,\varphi}$ and $W_{u,\varphi}\Re$ from weighted Bergman-Orlicz spaces to a class of weighted-type spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$ and $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit ball of the complex vector space $\mathbb{C}^n$. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, $H(\mathbb{B}^n)$ the space of all holomorphic functions on $\mathbb{B}^n$ and $S(\mathbb{B}^n)$ the class of all holomorphic self-maps of $\mathbb{B}^n$.

Let $\varphi \in S(\mathbb{B}^n)$ and $u \in H(\mathbb{B}^n)$. The weighted composition operator $W_{u,\varphi}$ is defined on $H(\mathbb{B}^n)$ by

$$W_{u,\varphi} f(z) = u(z) f(\varphi(z)).$$

When $u(z) \equiv 1$ on $\mathbb{B}^n$, the operator $W_{u,\varphi}$ is reduced to the composition operator, usually denoted by C_φ, while if $\varphi(z) = z$, the operator $W_{u,\varphi}$ is reduced to the multiplication operator, usually denoted by M_u. It is clear that the weighted composition operator is the product of composition operator and multiplication operator. Weighted composition operators between various spaces of holomorphic functions on different domains have been studied by numerous authors (see, e.g., [4, 6, 8, 9, 11, 12, 16, 21, 23, 24, 25, 30, 31, 38, 39, 43] and the references therein).

Let D be the differentiation operator on $H(\mathbb{D})$ defined by

$$Df(z) = f'(z).$$

The products of differentiation and composition operators DC_φ and $C_\varphi D$ have been studied, for example, in [2, 7, 10, 13, 14, 17, 27, 32, 34], while some generalizations of

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these product-type operators have been studied, for example, in [15, 29, 35, 36, 37, 40]. Recently, in [22] Sharma has studied the following six operators:

$$M_uC_\varphi D, M_uDC_\varphi, C_\varphi M_uD, C_\varphi DM_u, DM_uC_\varphi, DC_\varphi M_u. \tag{1}$$

The operators in (1) have been also studied by Stević, Sharma and Bhat in a unified manner in [41] and [42]. Quite recently, the products of differentiation and weighted composition operators $DW_{u,\varphi}$ and $W_{u,\varphi}D$ have also been studied in [5]. A natural problem is to consider the products of radial derivative and weighted composition operators on some subspaces of $H(\mathbb{B}^n)$. Let $\mathfrak{R}$ be the radial derivative operator on $H(\mathbb{B}^n)$, that is

$$\mathfrak{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

The products of radial derivative and weighted composition operators on some subspaces of $H(\mathbb{B}^n)$ are defined as follows

$$\mathfrak{R}W_{u,\varphi}f(z) = \mathfrak{R}(u(z)f(\varphi(z)))$$

and

$$W_{u,\varphi}\mathfrak{R}f(z) = u(z)\mathfrak{R}f(\varphi(z)).$$

This paper is devoted to studying the operators $\mathfrak{R}W_{u,\varphi}$ and $W_{u,\varphi}\mathfrak{R}$ from weighted Bergman-Orlicz spaces to a class of weighted-type spaces. Weighted composition operators and the integral-type operators defined and studied in [26, 28, 33], from weighted Bergman-Orlicz spaces to a class of weighted-type spaces have been studied in [18]. Here it must be mentioned that Stević in [35] introduced a more general operator $\mathfrak{R}^m_{u,\varphi}$, called the weighted iterated radial composition operator. Clearly, for $m = 1$ the operator $\mathfrak{R}^m_{u,\varphi}$ becomes the operator $W_{u,\varphi}\mathfrak{R}$. This paper can be regarded as a continuation of the investigation of concrete operators between these spaces.

Let $d\nu$ be the Lebesgue measure on the unit ball $\mathbb{B}^n$, $d\sigma$ the normalized surface measure on $\mathbb{S}^n = \partial\mathbb{B}^n$ (the boundary of $\mathbb{B}^n$). Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z|^2 = \langle z, z\rangle$. For $\alpha > -1$, by $d\nu_\alpha$ we denote the normalized Lebesgue measure $c_\alpha(1-|z|^2)^\alpha d\nu(z)$ (constant c_α is chosen such that $\nu_\alpha(\mathbb{B}^n) = 1$).

We now present some facts from [18]. The function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing function from the interval $[0,\infty)$ onto itself. Clearly, these conditions, among others, imply that $\Phi(0) = 0$.

The function Φ is of positive upper type $q \geqslant 1$, if there exists $C > 0$ such that $\Phi(st) \leqslant Ct^q\Phi(s)$ for every $s > 0$ and $t \geqslant 1$. We denote by $\mathfrak{U}^q$ the set of growth functions Φ of positive upper type q (for some $q \geqslant 1$), such that the function $t \mapsto \Phi(t)/t$ is non-decreasing on $(0,\infty)$. The function Φ is of positive lower type $p > 0$, if there exists $C > 0$ such that $\Phi(st) \leqslant Ct^p\Phi(s)$ for every $s > 0$ and $0 < t \leqslant 1$. By $\mathfrak{L}_p$ we denote the set of growth functions Φ of positive lower type p (for some $0 < p \leqslant 1$), such that the function $t \mapsto \Phi(t)/t$ is non-increasing on $(0,\infty)$.

Let Φ be a growth function. The weighted Bergman-Orlicz space $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ consists of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{A^{\Phi}_{\alpha}(\mathbb{B}^n)} = \int_{\mathbb{B}^n} \Phi(|f(z)|)dv_{\alpha}(z) < \infty.$$

On $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ is defined the following quasi-norm

$$\|f\|^{lux}_{A^{\Phi}_{\alpha}(\mathbb{B}^n)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{B}^n} \Phi\Big(\frac{|f(z)|}{\lambda}\Big)dv_{\alpha}(z) \leqslant 1\Big\}.$$

If $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_p$, then the quasi-norm on $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ is finite and call the Luxembourg norm.

The classical weighted Bergman space $A^p_{\alpha}(\mathbb{B}^n)$, $p > 0$, $\alpha > -1$, corresponds to $\Phi(t) = t^p$ and consists of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|^p_{A^p_{\alpha}(\mathbb{B}^n)} = \int_{\mathbb{B}^n} |f(z)|^p dv_{\alpha}(z) < \infty.$$

We say that a function $\omega : (0,1] \to (0,\infty)$ belongs to class Ω_1, if ω is nonincreasing, $1/\omega$ is of some positive lower type and the function $t\omega$ is increasing. For example, the function $\omega(t) = 1/t^{\alpha}$, $0 < \alpha < 1$, belongs to class Ω_1. We say that a function $\omega : (0,1] \to (0,\infty)$ belongs to class Ω_2, if $\omega \in \mathfrak{L}_p$, and satisfies the condition:

$$\int_t^1 \frac{\omega(s)}{s^2}ds \lesssim \frac{\omega(t)}{t} \quad (0 < t < 1).$$

Let ω be a positive function defined on $(0,1]$. An $f \in H(\mathbb{B}^n)$ is said to be in $H^{\infty}_{\omega}(\mathbb{B}^n)$, if

$$\|f\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} = \sup_{z\in\mathbb{B}^n} \frac{|f(z)|}{\omega(1-|z|)} < \infty.$$

It is easy to see that $H^{\infty}_{\omega}(\mathbb{B}^n)$ is a Banach space with the norm $\|\cdot\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}$. The space $H^{\infty}_{\omega}(\mathbb{B}^n)$ with $\omega \in \Omega_1$ is not quite often used in the literature. It seems to first appear in [3] as far as we know.

An $f \in H(\mathbb{B}^n)$ belongs to $\Lambda_{\omega}(\mathbb{B}^n)$, if $\Re f \in H^{\infty}_{\omega/t}(\mathbb{B}^n)$, that is

$$b_{\Lambda_{\omega}(\mathbb{B}^n)} = \sup_{z\in\mathbb{B}^n} \frac{(1-|z|)|\Re f(z)|}{\omega(1-|z|)} < \infty.$$

$\Lambda_{\omega}(\mathbb{B}^n)$ is a Banach space under the norm

$$\|f\|_{\Lambda_{\omega}(\mathbb{B}^n)} = |f(0)| + b_{\Lambda_{\omega}(\mathbb{B}^n)}.$$

For the relations between $H^{\infty}_{\omega}(\mathbb{B}^n)$ and $\Lambda_{\omega}(\mathbb{B}^n)$, we have that, if $\omega \in \Omega_1$, then $H^{\infty}_{\omega}(\mathbb{B}^n) = \Lambda_{\omega}(\mathbb{B}^n)$; if $\omega \in \Omega_2$, then $H^{\infty}_{\omega}(\mathbb{B}^n)$ embeds continuously into $\Lambda_{\omega}(\mathbb{B}^n)$.

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively. Let $L : X \to Y$ be a linear operator. The operator $L : X \to Y$ is bounded if there exists a positive constant K such that

$$d_Y(Lf, 0) \leqslant K d_X(f, 0)$$

for all $f \in X$. The operator $L: X \to Y$ is compact if it maps bounded sets into relatively compact sets.

Throughout this paper, positive constant C may differ from one occurrence to the other. The notation $a \lesssim b$ means that $a \leqslant Cb$ for some positive constant C.

2. Preliminary results

We first have the following compactness criteria. Since the proof is similar to that of Proposition 3.11 in [1], it is omitted.

LEMMA 2.1. *Let* $\alpha > -1$, ω *a positive function defined on* $(0,1]$, *and* $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. *Let* $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$ *and* $T \in \{\Re W_{u,\varphi}, W_{u,\varphi}\Re\}$. *Then the bounded operator* $T: A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$ *is compact if and only if for every bounded sequence* $\{f_j\}$ *in* $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ *such that* $f_j \to 0$ *uniformly on any compact subset of* $\mathbb{B}^n$ *as* $j \to \infty$, *it follows that*

$$\lim_{j\to\infty} \|Tf_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} = 0.$$

We need the following estimate. See Lemma 2.16 in [19] for a complete proof.

LEMMA 2.2. *Let* $\alpha > -1$ *and* $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. *Then there exists a positive constant* C *independent of* $f \in A^{\Phi}_{\alpha}(\mathbb{B}^n)$ *and* $z \in \mathbb{B}^n$ *such that*

$$|f(z)| \leqslant C\Phi^{-1}\Big(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\Big)\|f\|^{lux}_{A^{\Phi}_{\alpha}(\mathbb{B}^n)}.$$

We also need the following estimate for derivative of functions in weighted Bergman-Orlicz spaces. See Lemma 4.8 in [20] for a complete proof.

LEMMA 2.3. *Let* $\alpha > -1$ *and* $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. *Then there exist two positive constants* $C_n = C(\alpha, n)$ *and* $D_n = D(\alpha, n)$ *independent of* $f \in A^{\Phi}_{\alpha}(\mathbb{B}^n)$ *and* $z \in \mathbb{B}^n$ *such that*

$$|\nabla f(z)| \leqslant \frac{C_n}{1-|z|^2}\Phi^{-1}\Big(\frac{D_n}{(1-|z|^2)^{n+1+\alpha}}\Big)\|f\|^{lux}_{A^{\Phi}_{\alpha}(\mathbb{B}^n)}.$$

REMARK 2.1. By the proofs of Lemma 2.16 in [19] and Lemma 4.8 in [20], we can choose the constant $C := \max\{1, D_n\}$ instead of 1 and D_n in Lemmas 2.2 and 2.3.

LEMMA 2.4. *Let* $\alpha > -1$ *and* $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. *Then for every* $t \geqslant 0$ *and* $w \in \mathbb{B}^n$, *the following function is in* $A^{\Phi}_{\alpha}(\mathbb{B}^n)$

$$f_{w,t}(z) = \Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\Big(\frac{1-|w|^2}{1-\langle z,w\rangle}\Big)^{2(n+1+\alpha)+t},$$

where C *is an arbitrary positive constant. Moreover,*

$$\sup_{w\in\mathbb{B}^n} \|f_{w,t}\|^{lux}_{A^{\Phi}_{\alpha}(\mathbb{B}^n)} \lesssim 1.$$

Proof. Let

$$g_w(z) = \Big(\frac{1-|w|^2}{1-\langle z,w\rangle}\Big)^{2(n+1+\alpha)+t}.$$

We first have

$$\int_{\mathbb{B}^n} \Phi(|f_{w,t}(z)|)d\nu_\alpha(z) = \int_{\mathbb{B}^n} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z) = I+J,$$

where

$$I = \int_{\{z\in\mathbb{B}^n:|g_w(z)|\leqslant 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z)$$

and

$$J = \int_{\{z\in\mathbb{B}^n:|g_w(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z).$$

Let us start by considering the case where $\Phi\in\mathfrak{U}^q$. Since $\Phi(r)/r$ is non-decreasing on $(0,\infty)$, we have

$$\frac{\Phi(r|g_w(z)|)}{r|g_w(z)|} \leqslant \frac{\Phi(r)}{r}$$

for any $z\in\{z\in\mathbb{B}^n : |g_w(z)|\leqslant 1\}$, which shows

$$\begin{aligned} I &= \int_{\{z\in\mathbb{B}^n:|g_w(z)|\leqslant 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z) \\ &\leqslant C\int_{\mathbb{B}^n} \frac{(1-|w|^2)^{n+1+\alpha+t}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)+t}}d\nu_\alpha(z) \\ &\lesssim 1, \end{aligned}$$

where we use Theorem 1.12 in [45]. Using that Φ is of positive upper type q and $q\geqslant 1$, we obtain

$$\begin{aligned} J &= \int_{\{z\in\mathbb{B}^n:|g_w(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z) \\ &\lesssim \int_{\{z\in\mathbb{B}^n:|g_w(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\Big)\big|g_w(z)\big|^q d\nu_\alpha(z) \\ &\leqslant \frac{C}{(1-|w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \frac{(1-|w|^2)^{2q(n+1+\alpha+\frac{t}{2})}}{|1-\langle z,w\rangle|^{2q(n+1+\alpha+\frac{t}{2})}}d\nu_\alpha(z) \\ &\lesssim 1. \end{aligned}$$

We now consider the case where $\Phi\in\mathfrak{L}_p$. Using that Φ is of lower type p and Theorem 1.12 in [45], we have

$$\begin{aligned} I &= \int_{\{z\in\mathbb{B}^n:|g_w(z)|\leqslant 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big)d\nu_\alpha(z) \\ &\leqslant \frac{C}{(1-|w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} |g_w(z)|^p d\nu_\alpha(z) \end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{(1-|w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \Big(\frac{1-|w|^2}{|1-\langle z,w\rangle|}\Big)^{[2(n+1+\alpha)+t]p} dv_\alpha(z) \\
&\lesssim 1.
\end{aligned}
$$

We consider the second integral. Using the fact that $\Phi^{-1}(r)/r$ is non-decreasing on $(0,\infty)$, we obtain

$$
\frac{\Phi^{-1}(r)}{r} \leqslant \frac{\Phi^{-1}(r|g_w(z)|)}{r|g_w(z)|}
$$

for any $z \in \{z \in \mathbb{B}^n : |g_w(z)| > 1\}$, which shows

$$
|g_w(z)|\Phi^{-1}(r) \leqslant \Phi^{-1}(r|g_w(z)|)
$$

for any $z \in \{z \in \mathbb{B}^n : |g_w(z)| > 1\}$. Hence, we have

$$
\begin{aligned}
J &= \int_{\{z\in\mathbb{B}^n:|g_w(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\Big)\big|g_w(z)\big|\Big) dv_\alpha(z) \\
&\leqslant \frac{1}{(1-|w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \Big(\frac{1-|w|^2}{|1-\langle z,w\rangle|}\Big)^{2(n+1+\alpha)+t} dv_\alpha(z) \\
&\lesssim 1.
\end{aligned}
$$

This finishes the proof of the lemma. □

3. Boundedness and compactness of $\mathfrak{R}W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$

Let $\varphi = (\varphi_1,\ldots,\varphi_n) \in S(\mathbb{B}^n)$. By $(\mathfrak{R}\varphi_1,\ldots,\mathfrak{R}\varphi_n)$ we denote $\mathfrak{R}\varphi$, and by ∇f the gradient of function f. We assume that holomorphic self-map φ satisfies the following condition:

There is a $\delta \in (0,1)$ such that

$$
\big|\mathfrak{R}\varphi(z)\big| \leqslant \frac{1}{\delta}\big|\big\langle \mathfrak{R}\varphi(z), \varphi(z)\big\rangle\big| \tag{2}
$$

on $K = \{z \in \mathbb{B}^n : |\varphi(z)| \geqslant \delta\}$.

It is easily seen that if $n = 1$, all the holomorphic self-maps of $\mathbb{D}$ satisfy this condition. While if $n > 1$, we can also find some holomorphic self-maps φ to satisfy this condition. For example, if $\mathfrak{R}\varphi(z) = \varphi(z)$, then φ satisfies the condition. For such holomorphic self-maps, one can see, for example, $\varphi(z) = (z_1, z_2/2, \ldots, z_n/n)$.

Now, we characterize the boundedness of $\mathfrak{R}W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$.

THEOREM 3.1. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω be a positive function defined on $(0,1]$. Then the following statements are equivalent:*

(i) The operator $\mathfrak{R}W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded.

(ii) The functions u and φ satisfy the following conditions

$$
M_1 := \sup_{z\in\mathbb{B}^n} \frac{\big|\mathfrak{R}u(z)\big|}{\omega(1-|z|)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty \tag{3}
$$

and

$$M_2 := \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty, \tag{4}$$

where C is the positive constant in Remark 2.1.

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. First for each $h \in H(\mathbb{B}^n)$, we have

$$\frac{\partial(h(\varphi(z)))}{\partial z_j} = \sum_{i=1}^{n} \frac{\partial h}{\partial z_i}(\varphi(z))\frac{\partial \varphi_i}{\partial z_j}(z),$$

from which it follows that

$$\Re(h(\varphi(z))) = \big\langle \nabla h(\varphi(z)), \overline{\Re\varphi(z)}\big\rangle. \tag{5}$$

In Lemma 2.4, let $t=0$, C the constant in Remark 2.1, and replace w with $\varphi(w)$. Then we obtain the function $f_w(z) := f_{\varphi(w),0}(z)$. From some calculations, it follows that

$$f_w(\varphi(w)) = \Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big) \tag{6}$$

and

$$\frac{\partial f_w}{\partial z_j}(z) = c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\Big(\frac{1-|\varphi(w)|^2}{1-\langle z,\varphi(w)\rangle}\Big)^{2(n+1+\alpha)}\frac{\overline{\varphi_j(w)}}{1-\langle z,\varphi(w)\rangle}, \tag{7}$$

where $c_{\alpha,n} = 2(n+\alpha+1)$. From (7), we have

$$\frac{\partial f_w}{\partial z_j}(\varphi(w)) = c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{\overline{\varphi_j(w)}}{1-|\varphi(w)|^2}. \tag{8}$$

From (8), we obtain

$$\nabla f_w(\varphi(w)) = c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2}. \tag{9}$$

Hence, from (5) and (9) we deduce that

$$\big|\Re(f_w(\varphi(w)))\big| = c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{|\langle\Re\varphi(w),\varphi(w)\rangle|}{1-|\varphi(w)|^2}. \tag{10}$$

From (6), (10) and the boundedness of $\Re W_{u,\varphi} : A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$, we have

$$\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{|\Re u(w)|}{\omega(1-|w|)}$$
$$- c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{|u(w)||\langle\Re\varphi(w),\varphi(w)\rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)}$$

$$
\begin{aligned}
&= \frac{1}{\omega(1-|w|)}\left|\Re u(w) f_w(\varphi(w))\right| - \frac{1}{\omega(1-|w|)}\left|u(w)\Re(f_w(\varphi(w)))\right| \\
&\leqslant \frac{1}{\omega(1-|w|)}\Big|\Re u(w) f_w(\varphi(w)) + u(w)\Re(f_w(\varphi(w)))\Big| \\
&= \frac{1}{\omega(1-|w|)}\left|\Re W_{u,\varphi} f_w(w)\right| \leqslant \|\Re W_{u,\varphi} f_w\|_{H^\infty_\omega(\mathbb{B}^n)} \leqslant C\|\Re W_{u,\varphi}\|. \qquad (11)
\end{aligned}
$$

From (11), we obtain the following inequality

$$
\begin{aligned}
&\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{\left|\Re u(w)\right|}{\omega(1-|w|)} \\
&\leqslant C\|\Re W_{u,\varphi}\| + c_{\alpha,n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{|u(w)||\langle \Re\varphi(w), \varphi(w)\rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)}. \qquad (12)
\end{aligned}
$$

On the other hand, from Lemma 2.4 it follows that the function

$$
g_w(z) := f_{\varphi(w),1}(z) - f_{\varphi(w),0}(z)
$$

is in $A^\Phi_\alpha(\mathbb{B}^n)$ and $\|g_w\|^{lux}_{A^\Phi_\alpha(\mathbb{B}^n)} \leqslant C$. From some calculations, we similarly obtain

$$
\left|\Re(g_w(\varphi(w)))\right| = \Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{\left|\langle \Re\varphi(w), \varphi(w)\rangle\right|}{1-|\varphi(w)|^2} \qquad (13)
$$

and $g_w(\varphi(w)) = 0$. From this, (13) and the boundedness of $\Re W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$, it follows that

$$
\begin{aligned}
&\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big)\frac{|u(w)||\langle \Re\varphi(w), \varphi(w)\rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)} \\
&= \frac{1}{\omega(1-|w|)}\left|u(w)\Re(g_w(\varphi(w)))\right| \\
&= \frac{1}{\omega(1-|w|)}\Big|\Re u(w) g_w(\varphi(w)) + u(w)\Re(g(\varphi(w)))\Big| \\
&= \frac{1}{\omega(1-|w|)}\left|\Re W_{u,\varphi} g_w(w)\right| \\
&\leqslant \left\|\Re W_{u,\varphi} g_w\right\|_{H^\infty_\omega(\mathbb{B}^n)}, \qquad (14)
\end{aligned}
$$

which shows

$$
\sup_{z\in\mathbb{B}^n}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\frac{|u(z)||\langle \Re\varphi(z), \varphi(z)\rangle|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \leqslant \left\|\Re W_{u,\varphi} g_w\right\|_{H^\infty_\omega(\mathbb{B}^n)}. \qquad (15)
$$

Hence, from (12) and (15), it follows that

$$
M_1 = \sup_{z\in\mathbb{B}^n}\frac{|\Re u(z)|}{\omega(1-|z|)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty.
$$

For δ and K in (2), from (15) we have

$$\begin{aligned}
&\sup_{z\in K}\frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\
&\leqslant \frac{1}{\delta}\sup_{z\in K}\frac{|u(z)||\langle\Re\varphi(z),\varphi(z)\rangle|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\
&\leqslant \frac{1}{\delta}\big\|\Re W_{u,\varphi}g_w\big\|_{H^\infty_\omega(\mathbb{B}^n)}\\
&\leqslant C\big\|\Re W_{u,\varphi}\big\|\\
&<\infty. \qquad (16)
\end{aligned}$$

Take functions $f(z)=1$ and $f(z)=z_j$, respectively. From the boundedness of $\Re W_{u,\varphi}: A^\Phi_\alpha(\mathbb{B}^n)\to H^\infty_\omega(\mathbb{B}^n)$, we get

$$\sup_{z\in\mathbb{B}^n}\frac{|\Re u(z)|}{\omega(1-|z|)}<\infty \qquad (17)$$

and

$$\sup_{z\in\mathbb{B}^n}\frac{1}{\omega(1-|z|)}\big|\varphi_j(z)\Re u(z)+u(z)\Re\varphi_j(z)\big|<\infty. \qquad (18)$$

From (17), (18) and the boundedness of $\varphi_j(z)$, we have

$$\sup_{z\in\mathbb{B}^n}\frac{|u(z)||\Re\varphi_j(z)|}{\omega(1-|z|)}<\infty. \qquad (19)$$

Since

$$|\Re\varphi(z)|\leqslant|\Re\varphi_1(z)|+|\Re\varphi_2(z)|+\cdots+|\Re\varphi_n(z)|,$$

by (19) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}^n\setminus K}\frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\
&\leqslant c_{\alpha,\delta}\sup_{z\in\mathbb{B}^n}\frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)}\\
&\leqslant c_{\alpha,\delta}\sum_{j=1}^{n}\sup_{z\in\mathbb{B}^n}\frac{|u(z)||\Re\varphi_j(z)|}{\omega(1-|z|)}\\
&<\infty, \qquad (20)
\end{aligned}$$

where

$$c_{\alpha,\delta}=\frac{1}{1-\delta^2}\Phi^{-1}\Big(\frac{C}{(1-\delta^2)^{n+1+\alpha}}\Big).$$

Consequently, from (16) and (20) we get

$$M_2=\sup_{z\in\mathbb{B}^n}\frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)<\infty.$$

$(ii) \Rightarrow (i)$. Suppose that (3) and (4) hold. Then for every $f \in A_\alpha^\Phi(\mathbb{B}^n)$, from Lemma 2.2, Lemma 2.3 and Remark 2.1 we have

$$\begin{aligned}
\|\Re W_{u,\varphi} f\|_{H_\omega^\infty(\mathbb{B}^n)} &= \sup_{z\in\mathbb{B}^n} \frac{1}{\omega(1-|z|)}\left|\Re u(z) f(\varphi(z)) + u(z)\Re(f(\varphi(z)))\right| \\
&\leqslant \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)||f(\varphi(z))|}{\omega(1-|z|)} + \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re(f(\varphi(z)))|}{\omega(1-|z|)} \\
&= \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)||f(\varphi(z))|}{\omega(1-|z|)} + \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\langle \nabla f(\varphi(z)), \overline{\Re\varphi(z)}\rangle|}{\omega(1-|z|)} \\
&\leqslant C \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)\|f\|^{lux}_{A_\alpha^\Phi(\mathbb{B}^n)} \\
&\quad + \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi(z)||\nabla f(\varphi(z))|}{\omega(1-|z|)} \\
&\leqslant C \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)\|f\|^{lux}_{A_\alpha^\Phi(\mathbb{B}^n)} \\
&\quad + C_n \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)\|f\|^{lux}_{A_\alpha^\Phi(\mathbb{B}^n)} \\
&= (CM_1 + C_n M_2)\|f\|^{lux}_{A_\alpha^\Phi(\mathbb{B}^n)},
\end{aligned}$$

which shows that the operator $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ is bounded. □

From the fact $H_\omega^\infty(\mathbb{B}^n) \hookrightarrow \Lambda_\omega(\mathbb{B}^n)$ when $\omega \in \Omega_2$, and Theorem 3.1, we can obtain the following result.

PROPOSITION 3.2. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and $\omega \in \Omega_2$. If u and φ satisfy the following conditions:*

$$\sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty$$

and

$$\sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty,$$

where C is the positive constant in Remark 2.1, then the operator $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ is bounded.

Next we prove the following compactness criteria.

THEOREM 3.3. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω a positive function defined on $(0,1]$. Then the following statements are equivalent:*

(i) The operator $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ is compact.

(ii) The functions u and φ are such that $\Re u \in H_\omega^\infty(\mathbb{B}^n)$, for each $j \in \{1,2,\dots,n\}$

$$L_j := \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi_j(z)|}{\omega(1-|z|)} < \infty, \tag{21}$$

$$\lim_{|\varphi(z)|\to 1} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) = 0, \tag{22}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) = 0, \tag{23}$$

where C is the positive constant in Remark 2.1.

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Then $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ is bounded. For any $f \in A_\alpha^\Phi(\mathbb{B}^n)$, it follows that

$$\|\Re W_{u,\varphi} f\|_{H_\omega^\infty(\mathbb{B}^n)} = \sup_{z\in\mathbb{B}^n} \frac{1}{\omega(1-|z|)} \big|\Re u(z) f(\varphi(z)) + u(z)\Re(f(\varphi(z)))\big| < \infty. \tag{24}$$

Taking $f(z) = 1$ and $f(z) = z_j$ in (24), we obtain that $\Re u \in H_\omega^\infty(\mathbb{B}^n)$ and (21).

Next consider a sequence $\{\varphi(z_j)\}$ in $\mathbb{B}^n$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. If such sequence does not exist, then (22) and (23) obviously hold. Using this sequence, we define the functions $f_j(z) = f_{\varphi(z_j),0}(z)$. Then the sequence $\{f_j\}$ is uniformly bounded in $A_\alpha^\Phi(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j \to \infty$. Since f_j is defined by replacing $\varphi(w)$ by $\varphi(z_j)$ in f_w in the proof of Theorem 3.1, from (12) we have

$$\begin{aligned} &\frac{|\Re u(z_j)|}{\omega(1-|z_j|)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big) \\ &\quad - c_{\alpha,n} \frac{|u(z_j)||\langle \Re\varphi(z_j), \varphi(z_j)\rangle|}{\omega(1-|z_j|)(1-|\varphi(z_j)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big) \\ &\leqslant \|\Re W_{u,\varphi} f_j\|_{H_\omega^\infty(\mathbb{B}^n)}, \end{aligned} \tag{25}$$

We also define the function

$$g_j(z) = f_{\varphi(z_j),1}(z) - f_{\varphi(z_j),0}(z)$$

for each $j \in \mathbb{N}$. The sequence $\{g_j\}$ is uniformly bounded in $A_\alpha^\Phi(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j \to \infty$. From (15), we have

$$\frac{|u(z_j)|\big|\langle \Re\varphi(z_j), \varphi(z_j)\rangle\big|}{\omega(1-|z_j|)(1-|\varphi(z_j)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big) \leqslant \big\|\Re W_{u,\varphi} g_j\big\|_{H_\omega^\infty(\mathbb{B}^n)}. \tag{26}$$

From (25) and (26), it follows that

$$\frac{|\Re u(z_j)|}{\omega(1-|z_j|)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big) \leqslant \|\Re W_{u,\varphi} f_j\|_{H^\infty_\omega(\mathbb{B}^n)} + c_{\alpha,n}\|\Re W_{u,\varphi} g_j\|_{H^\infty_\omega(\mathbb{B}^n)}.$$

From the compactness of $\Re W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$, it follows that (22) holds.

Since $|\varphi(z_j)| \to 1$ as $j \to \infty$, there exists a $J \in \mathbb{N}$ such that $|\varphi(z_j)| > \delta$ for $j \geqslant J$, that is, $\varphi(z_j) \in K$ for $j \geqslant J$. Similar to (16), we can obtain

$$\frac{|u(z_j)||\Re\varphi(z_j)|}{\omega(1-|z_j|)(1-|\varphi(z_j)|^2)} \leqslant \frac{1}{\delta}\|\Re W_{u,\varphi} g_j\|_{H^\infty_\omega(\mathbb{B}^n)}.$$

This implies that (23) holds.

$(ii) \Rightarrow (i)$. We first check that $\Re W_{u,\varphi} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded. For this, we observe that (22) and (23) imply that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$\frac{|\Re u(z)|}{\omega(1-|z|)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \varepsilon \tag{27}$$

and

$$\frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \varepsilon, \tag{28}$$

for any $z \in K_\eta = \{z \in \mathbb{B}^n : |\varphi(z)| > \eta\}$.

Write

$$I(z) := \frac{|\Re u(z)|}{\omega(1-|z|)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)$$

and

$$J(z) := \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big).$$

Then from (21), (27) and (28) we have

$$\begin{aligned} M_1 = \sup_{z\in\mathbb{B}^n} I(z) &= \sup_{z\in\mathbb{B}^n\setminus K_\eta} I(z) + \sup_{z\in K_\eta} I(z) \\ &\leqslant \|\Re u\|_{H^\infty_\omega(\mathbb{B}^n)}\Phi^{-1}\Big(\frac{C}{(1-\eta^2)^{n+1+\alpha}}\Big) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} M_2 = \sup_{z\in\mathbb{B}^n} J(z) &= \sup_{z\in\mathbb{B}^n\setminus K_\eta} J(z) + \sup_{z\in K_\eta} J(z) \\ &\leqslant \sum_{j=1}^n L_j \Phi^{-1}\Big(\frac{C}{(1-\eta^2)^{n+1+\alpha}}\Big) + \varepsilon. \end{aligned}$$

From Theorem 3.1, it follows that $\Re W_{u,\varphi}: A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$ is bounded.

To prove that $\Re W_{u,\varphi}: A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_j\}$ is a sequence in $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ such that $\|f_j\|_{A^{\Phi}_{\alpha}(\mathbb{B}^n)} \leqslant M$ and $\{f_j\}$ uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j \to \infty$, then

$$\lim_{j\to\infty} \|\Re W_{u,\varphi} f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} = 0.$$

For any $\varepsilon > 0$ and the associated η in (27) and (28), by using again $\Re u \in H^{\infty}_{\omega}(\mathbb{B}^n)$, (27), (28), Lemmas 2.2, 2.3 and Remark 2.1, we have

$$\begin{aligned}
&\|\Re W_{u,\varphi} f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}\\
&= \sup_{z\in\mathbb{B}^n} \frac{1}{\omega(1-|z|)}\Big|\Re(u(z)f_j(\varphi(z)))\Big|\\
&= \sup_{z\in\mathbb{B}^n} \frac{1}{\omega(1-|z|)}\Big|\Re u(z)f_j(\varphi(z)) + u(z)\Re(f_j(\varphi(z)))\Big|\\
&= \sup_{z\in\mathbb{B}^n} \frac{1}{\omega(1-|z|)}\Big|\Re u(z)f_j(\varphi(z)) + u(z)\big\langle \nabla f_j(\varphi(z)), \overline{\Re\varphi(z)}\big\rangle\Big|\\
&\leqslant \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)}\big|f_j(\varphi(z))\big| + \sup_{z\in\mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)}\Big|\big\langle \nabla f_j(\varphi(z)), \overline{\Re\varphi(z)}\big\rangle\Big|\\
&\leqslant \sup_{z\in\mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)}\big|f_j(\varphi(z))\big| + \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)}\big|\nabla f_j(\varphi(z))\big|\\
&\leqslant \sup_{z\in\mathbb{B}^n\setminus K_\eta} \frac{|\Re u(z)|}{\omega(1-|z|)}\big|f_j(\varphi(z))\big| + \sup_{z\in K_\eta} \frac{|\Re u(z)|}{\omega(1-|z|)}\big|f_j(\varphi(z))\big|\\
&\quad + \sup_{z\in\mathbb{B}^n\setminus K_\eta} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)}\big|\nabla f_j(\varphi(z))\big| + \sup_{z\in K_\eta} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)}\big|\nabla f_j(\varphi(z))\big|\\
&\leqslant \|\Re u\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} \sup_{\{z:|z|\leqslant\eta\}} \big|f_j(z)\big| + CM \sup_{z\in K_\eta} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\
&\quad + \sum_{j=1}^{n} L_j \sup_{\{z:|z|\leqslant\eta\}} \big|\nabla f_j(z)\big| + C_n M \sup_{z\in K_\eta} \frac{|u(z)||\Re\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\
&\leqslant \|\Re u\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} \sup_{\{z:|z|\leqslant\eta\}} \big|f_j(z)\big| + \sum_{j=1}^{n} L_j \sup_{\{z:|z|\leqslant\eta\}} \big|\nabla f_j(z)\big| + (C+C_n)M\varepsilon.
\end{aligned} \tag{29}$$

It is easy to see that, if $\{f_j\}$ uniformly converges to zero on any compact subset of $\mathbb{B}^n$, then $\{\frac{\partial f_j}{\partial z_i}\}$ also does as $j \to \infty$ for each $i = 1, 2, \ldots, n$. This shows that $\{|\nabla f_j|\}$ uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j \to \infty$. Since $\{z \in \mathbb{B}^n : |z| \leqslant \eta\}$ is compact subset of $\mathbb{B}^n$, by letting $j \to \infty$ in (29) we have

$$\lim_{j\to\infty} \|\Re W_{u,\varphi} f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} = 0.$$

This shows that $\Re W_{u,\varphi}: A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$ is compact. $\square$

4. Boundedness and compactness of $W_{u,\varphi}\mathfrak{R} : A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$

THEOREM 4.1. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω a positive function defined on $(0,1]$. Then the following statements are equivalent:*
(i) The operator $W_{u,\varphi}\mathfrak{R} : A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$ is bounded.
(ii) The functions u and φ satisfy the following condition

$$M_3 := \sup_{z\in\mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty, \tag{30}$$

where C is the positive constant in Remark 2.1.

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Then there exists a positive constant C such that for every $f \in A^{\Phi}_{\alpha}(\mathbb{B}^n)$,

$$\|W_{u,\varphi}\mathfrak{R} f\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} \leqslant C\|f\|^{lux}_{A^{\Phi}_{\alpha}(\mathbb{B}^n)}.$$

Considering the function $f_w(z) = f_{\varphi(w),0}(z)$, we have

$$\big|\mathfrak{R} f_w(\varphi(w))\big| = c_{\alpha,n}\frac{|\varphi(w)|^2}{1-|\varphi(w)|^2}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big).$$

From this and the boundedness of $W_{u,\varphi}\mathfrak{R} : A^{\Phi}_{\alpha}(\mathbb{B}^n) \to H^{\infty}_{\omega}(\mathbb{B}^n)$, we obtain

$$\begin{aligned}\frac{1}{\omega(1-|w|)}\big|W_{u,\varphi}\mathfrak{R} f_w(w)\big| &= \frac{1}{\omega(1-|w|)}\big|u(w)\mathfrak{R} f_w(\varphi(w))\big| \\ &= c_{\alpha,n}\frac{|u(w)||\varphi(w)|^2}{\omega(1-|w|)(1-|\varphi(w)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\Big) \\ &\leqslant \|W_{u,\varphi}\mathfrak{R} f_w\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} \leqslant C\big\|W_{u,\varphi}\mathfrak{R}\big\|,\end{aligned}$$

which shows

$$I := \sup_{z\in\mathbb{B}^n} \frac{|u(z)||\varphi(z)|^2}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty.$$

Then for a fixed $\delta \in (0,1)$, we have

$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) \leqslant \frac{I}{\delta^2} < \infty. \tag{31}$$

Also taking the function $f(z) = z_j$, we have

$$\|W_{u,\varphi}\mathfrak{R} f\|_{H^{\infty}_{\omega}(\mathbb{B}^n)} = \sup_{z\in\mathbb{B}^n}\frac{|u(z)||\varphi_j(z)|}{\omega(1-|z|)} < \infty.$$

From this and the boundedness of φ_j, we get

$$J := \sup_{z\in\mathbb{B}^n}\frac{|u(z)|}{\omega(1-|z|)} < \infty. \tag{32}$$

Hence,

$$\sup_{\{z:|\varphi(z)|\leqslant\delta\}} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) \leqslant d_\delta J < \infty, \tag{33}$$

where

$$d_\delta = \frac{1}{1-\delta^2}\Phi^{-1}\Big(\frac{C}{(1-\delta^2)^{n+1+\alpha}}\Big).$$

Consequently, (31) and (33) show that $M_3 < \infty$.

$(ii) \Rightarrow (i)$. Suppose that (30) holds. Since

$$|\mathfrak{R}f(z)| = \Big|\sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}\Big| \leqslant \sum_{j=1}^{n}|z_j|\Big|\frac{\partial f(z)}{\partial z_j}\Big| \leqslant \sum_{j=1}^{n}\Big|\frac{\partial f(z)}{\partial z_j}\Big| = \sqrt{n}|\nabla f(z)|$$

for every $f \in A^\Phi_\alpha(\mathbb{B}^n)$, by Lemma 2.3 we have

$$\begin{aligned}\|W_{u,\varphi}\mathfrak{R}f\|_{H^\infty_\omega(\mathbb{B}^n)} &= \sup_{z\in\mathbb{B}^n}\frac{1}{\omega(1-|z|)}\big|u(z)\mathfrak{R}f(\varphi(z))\big| \\ &\leqslant \sqrt{n}C_n \sup_{z\in\mathbb{B}^n}\frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\|f\|^{lux}_{A^\Phi_\alpha(\mathbb{B}^n)} \\ &= \sqrt{n}C_n M_3\|f\|^{lux}_{A^\Phi_\alpha(\mathbb{B}^n)},\end{aligned}$$

which shows that the operator $W_{u,\varphi}\mathfrak{R} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded. □

The following result is proved similarly.

PROPOSITION 4.2. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and $\omega \in \Omega_2$. If u and φ satisfy the condition*

$$\sup_{z\in\mathbb{B}^n}\frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \infty,$$

where C is the positive constant in Remark 2.1, then the operator $W_{u,\varphi}\mathfrak{R} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded.

Now we prove the compactness criteria.

THEOREM 4.3. *Let $p \geqslant 1$, $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω be a positive function defined on $(0,1]$. Then the following statements are equivalent:*

(i) The operator $W_{u,\varphi}\mathfrak{R} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is compact.

(ii) The functions u and φ are such that $u \in H^\infty_\omega(\mathbb{B}^n)$ and

$$\lim_{|\varphi(z)|\to 1}\frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) = 0, \tag{34}$$

where C is the positive constant in Remark 2.1.

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Then $W_{u,\varphi}\mathfrak{R} : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded, so for any $f \in A^\Phi_\alpha(\mathbb{B}^n)$,

$$\|W_{u,\varphi}\mathfrak{R}f\|_{H^\infty_\omega(\mathbb{B}^n)} = \sup_{z\in\mathbb{B}^n}\frac{\big|u(z)\mathfrak{R}f(\varphi(z))\big|}{\omega(1-|z|)} < \infty. \tag{35}$$

Taking $f(z) = z_j$ in (35), we obtain

$$\sup_{z\in\mathbb{B}^n} \frac{|u(z)||\varphi_j(z)|}{\omega(1-|z|)} < \infty. \tag{36}$$

From the boundedness of $\varphi_j(z)$ and (36), it follows that

$$\sup_{z\in\mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)} < \infty,$$

which means that $u \in H^\infty_\omega(\mathbb{B}^n)$.

Next consider a sequence $\{\varphi(z_j)\}$ in $\mathbb{B}^n$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. If such sequence does not exist, then (34) obviously holds. Using this sequence, we define the functions $f_j(z) = f_{\varphi(z_j),0}(z)$. Then $\{f_j\}$ is uniformly bounded in $A^\Phi_\alpha(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j \to \infty$. From some calculations, we obtain

$$\left|\Re f_j(\varphi(z_j))\right| = c_{\alpha,n} \frac{|\varphi(z_j)|^2}{1-|\varphi(z_j)|^2} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big).$$

Hence, the compactness of $W_{u,\varphi}\Re : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ implies that

$$\lim_{j\to\infty} \|W_{u,\varphi}\Re f_j\|_{H^\infty_\omega(\mathbb{B}^n)} = 0.$$

From this, we have

$$\lim_{j\to\infty} \frac{|u(z_j)|}{\omega(1-|z_j|)(1-|\varphi(z_j)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}}\Big) = 0.$$

This shows that (34) holds.

$(ii) \Rightarrow (i)$. We first prove that $W_{u,\varphi}\Re : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded. For this we observe that (34) implies that for every $\varepsilon > 0$, there exists an $\eta \in (0,1)$ such that

$$\frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big) < \varepsilon, \tag{37}$$

for any $z \in K_\eta = \{z \in \mathbb{B}^n : |\varphi(z)| > \eta\}$.

Let

$$K(z) := \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big).$$

Then we have

$$M_3 = \sup_{z\in\mathbb{B}^n\setminus K_\eta} K(z) + \sup_{z\in K_\eta} K(z) \leqslant \|u\|_{H^\infty_\omega(\mathbb{B}^n)} \frac{1}{1-\eta^2} \Phi^{-1}\Big(\frac{C}{(1-\eta^2)^{n+1+\alpha}}\Big) + \varepsilon, \tag{38}$$

which shows that $W_{u,\varphi}\Re : A^\Phi_\alpha(\mathbb{B}^n) \to H^\infty_\omega(\mathbb{B}^n)$ is bounded.

To prove that $W_{u,\varphi}\mathfrak{R}: A^{\Phi}_{\alpha}(\mathbb{B}^n)\to H^{\infty}_{\omega}(\mathbb{B}^n)$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_j\}$ is a sequence in $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ such that $\|f_j\|_{A^{\Phi}_{\alpha}(\mathbb{B}^n)}\leqslant M$ and $\{f_j\}$ uniformly converges to zero on any compact subset of $\mathbb{B}^n$ as $j\to\infty$, then

$$\lim_{j\to\infty}\|W_{u,\varphi}\mathfrak{R}f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}=0.$$

For any $\varepsilon>0$ and the associated η in (37), we have, by using again $u\in H^{\infty}_{\omega}(\mathbb{B}^n)$, (37) and Lemma 2.3,

$$\begin{aligned}\|W_{u,\varphi}\mathfrak{R}f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}&=\sup_{z\in\mathbb{B}^n}\frac{1}{\omega(1-|z|)}\big|W_{u,\varphi}\mathfrak{R}f_j(z)\big|=\sup_{z\in\mathbb{B}^n}\frac{1}{\omega(1-|z|)}\Big|u(z)\mathfrak{R}f_j(\varphi(z))\Big|\\&\leqslant\sup_{z\in\mathbb{B}^n\setminus K_\eta}\frac{|u(z)|}{\omega(1-|z|)}\big|\mathfrak{R}f_j(\varphi(z))\big|+\sup_{z\in K_\eta}\frac{|u(z)|}{\omega(1-|z|)}\big|\mathfrak{R}f_j(\varphi(z))\big|\\&\leqslant\|u\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}\sup_{\{z\in\mathbb{B}^n:|z|\leqslant\eta\}}\big|\mathfrak{R}f_j(z)\big|\\&\quad+C_nM\sup_{z\in K_\eta}\frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)}\Phi^{-1}\Big(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\Big)\\&\leqslant\|u\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}\sup_{\{z\in\mathbb{B}^n:|z|\leqslant\eta\}}\big|\mathfrak{R}f_j(z)\big|+C_nM\varepsilon.\end{aligned}\tag{39}$$

It is easy to see that, if $\{f_j\}$ uniformly converges to zero on any compact subset of $\mathbb{B}^n$, then $\{\mathfrak{R}f_j\}$ also does as $j\to\infty$. From this, and since $\{z\in\mathbb{B}^n:|z|\leqslant\eta\}$ is compact subset of $\mathbb{B}^n$, letting $j\to\infty$ in (39) gives

$$\lim_{j\to\infty}\|W_{u,\varphi}\mathfrak{R}f_j\|_{H^{\infty}_{\omega}(\mathbb{B}^n)}=0.$$

This shows that the operator $W_{u,\varphi}\mathfrak{R}: A^{\Phi}_{\alpha}(\mathbb{B}^n)\to H^{\infty}_{\omega}(\mathbb{B}^n)$ is compact. □

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REFERENCES

[1] C. C. COWEN, B. D. MACCLUER, *Composition operators on spaces of analytic functions*, CRC Press, 1995.

[2] R. A. HIBSCHWEILER, N. PORTNOY, *Composition followed by differentiation between Bergman and Hardy spaces*, Rocky Mountain J. Math. **35** (3) (2005), 843–855.

[3] S. JANSON, *Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation*, Duke Math. J. **47** (4) (1980), 959–982.

[4] Z. J. JIANG, S. STEVIĆ, *Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces*, Appl. Math. Comput. **217** (7) (2010), 3522–3530.

[5] Z. J. JIANG, *On a class of operators from weighted Bergman spaces to some spaces of analytic functions*, Taiwan. J. Math. Soc. **15** (5) (2011), 2095–2121.

[6] Z. J. JIANG, *Weighted composition operators from weighted Bergman spaces to some spaces of analytic functions on the upper half plane*, Util. Math. **93** (2014), 205–212.

[7] S. LI, S. STEVIĆ, *Composition followed by differentiation between Bloch type spaces*, J. Comput. Anal. Appl. **9** (2) (2007), 195–206.

[8] S. LI, S. STEVIĆ, *Weighted composition operators from Bergman-type spaces into Bloch spaces*, Proc. Indian Acad. Sci. Math. Sci. **117** (3) (2007), 371–385.

[9] S. LI, S. STEVIĆ, *Weighted composition operators from α-Bloch space to H^∞ on the polydisk*, Numer. Funct. Anal. Optimization. **28** (7) (2007), 911–925.

[10] S. LI, S. STEVIĆ, *Composition followed by differentiation from mixed norm spaces to α-Bloch spaces*, Sb. Math. **199** (12) (2008), 1847–1857.

[11] S. LI, S. STEVIĆ, *Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball*, Taiwan. J. Math. Soc. **12** (2008), 1625–1639.

[12] S. LI, S. STEVIĆ, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Appl. Math. Comput. **206** (2) (2008), 825–831.

[13] S. LI, S. STEVIĆ, *Composition followed by differentiation between H^∞ and α-Bloch spaces*, Houston J. Math. **35** (1) (2009), 327–340.

[14] S. LI, S. STEVIĆ, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput. **217** (2010), 3144–3154.

[15] S. LI, S. STEVIĆ, *Weighted differentiation composition operators from the logarithmic Bloch space to the weighted-type space*, Analele Stiint. Univ. Ovidius C. **24** (3) (2016), 223–240.

[16] S. OHNO, *Weighted composition operators between H^∞ and the Bloch space*, Taiwan. J. Math. Soc. **5** (3) (2001), 555–563.

[17] S. OHNO, *Products of composition and differentiation on Bloch spaces*, Bull. Korean Math. Soc. **46** (6) (2009), 1135–1140.

[18] B. SEHBA, S. STEVIĆ, *On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces*, Appl. Math. Comput. **233** (2014), 565–581.

[19] B. SEHBA, *Duality for large Bergman-Orlicz spaces and boundedness of Hankel operators*, `https://arxiv.org/pdf/1501.03416v1`.

[20] B. SEHBA, *Derivatives characterization of Bergman-Orlicz spaces and applications*, `https://arxiv.org/abs/1610.01954`.

[21] A. K. SHARMA, Z. ABBAS, *Weighted composition operators between weighted Bergman-Nevanlinna and Bloch-type spaces*, Appl. Math. Sci. **41** (4) (2010), 2039–2048.

[22] A. K. SHARMA, *Products of composition multiplication and differentiation between Bergman and Bloch type spaces*, Turkish. J. Math. **35** (2011), 275–291.

[23] S. STEVIĆ, *Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball*, Abstr. Appl. Anal. **2008** (2008), Article ID 279691, 11 pages.

[24] S. STEVIĆ, *Norms of some operators from Bergman spaces to weighted and Bloch-type space*, Util. Math. **76** (2008), 59–64.

[25] S. STEVIĆ, *Norm of weighted composition operators from Bloch space to H^∞ on the unit ball*, Ars Combin. **88** (2008), 125–127.

[26] S. STEVIĆ, *On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball*, Discrete Dyn. Nat. Soc. **2008** (2008), Article ID 154263, 14 pages.

[27] S. STEVIĆ, *Products of composition and differentiation operators on the weighted Bergman space*, Bull. Belg. Math. Soc. **16** (2009), 623–635.

[28] S. STEVIĆ, *On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces*, Nonlinear Anal. TMA **71** (2009), 6323–6342.

[29] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces*, Appl. Math. Comput. **211** (2009), 222–233.

[30] S. STEVIĆ, *Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput. **212** (2009), 499–504.

[31] S. STEVIĆ, *Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces*, Appl. Math. Comput. **215** (2009), 818–820.

[32] S. STEVIĆ, *Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞*, Appl. Math. Comput. **207** (2009), 225–229.

[33] S. STEVIĆ, *On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball*, Appl. Math. Comput. **215** (2010), 4248–4255.

[34] S. STEVIĆ, *Composition followed by differentiation from H^∞ and the Bloch space to n-th weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3450–3458.

[35] S. STEVIĆ, *Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball*, Abstr. Appl. Anal. **2010** (2010), Article ID 801264, 14 pages.

[36] S. STEVIĆ, *Weighted differentiation composition operators from H^∞ and Bloch spaces to n-th weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3634–3641.

[37] S. STEVIĆ, *Weighted differentiation composition operators from the mixed-norm space to the n-th weigthed-type space on the unit disk*, Abstr. Appl. Anal. **2010** (2010), Article ID 246287, 15 pages.

[38] S. STEVIĆ, R. P. AGARWAL, *Weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces*, J. Inequal. Appl. **2009** (2009), Article ID 964814, 21 pages.

[39] S. STEVIĆ, Z. J. JIANG, *Differences of weighted composition operators on the unit polydisk*, Siberian Math. J. **52** (2) (2011), 454–468.

[40] S. STEVIĆ, A. K. SHARMA, *Iterated differentiation followed by composition from Bloch-type spaces to weighted BMOA spaces*, Appl. Math. Comput. **218** (2011), 3574–3580.

[41] S. STEVIĆ, A. K. SHARMA, A. BHAT, *Products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput. **217** (2011), 8115–8125.

[42] S. STEVIĆ, A. K. SHARMA, A. BHAT, *Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput. **218** (2011), 2386–2397.

[43] S. STEVIĆ, S. I. UEKI, *Weighted composition operators and integral-type operators between weighted Hardy spaces on the unit ball*, Discrete Dyn. Nat. Soc. **2009** (2009), Article ID 952831, 20 pages.

[44] W. YANG, W. YAN, *Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces*, Bull. Korean Math. Soc. **48** (6) (2011), 1195–1205.

[45] K. ZHU, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2005.

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