Operators and Matrices
Volume 12, Number 2 (2018), 321–332 doi:10.7153/oam-2018-12-21

HYPERCYCLICITY AND WEYL TYPE THEOREMS FOR OPERATOR MATRICES

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(*Communicated by I. M. Spitkovsky*)

Abstract. In this paper, we study the hypercyclicity and supercyclicity for operator matrices in the class $\mathscr{S}$ consisting 2×2 operator matrices with $(1,2)$-entries having closed range. Under some conditions, we find the necessary and sufficient conditions for 2×2 operator matrices in the class $\mathscr{S}$ for which Weyl's theorem, Browder's theorem, a-Weyl's theorem or a-Browder's theorem hold.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be complex, separable, infinite dimensional Hilbert spaces, let $\mathscr{L}(\mathscr{H},\mathscr{K})$ denote the set of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, and abbreviate $\mathscr{L}(\mathscr{H}) = \mathscr{L}(\mathscr{H},\mathscr{H})$. If $T \in \mathscr{L}(\mathscr{H},\mathscr{K})$, we write $\mathscr{N}(T)$ for the null space of T; $\mathscr{R}(T)$ for the range of T. Since every closed subspace $\mathscr{H}_1$ of $\mathscr{H}$ is complemented, $\mathscr{H}$ is decomposed as a direct sum of two closed subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$, that is, $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Thus, each bounded linear operator T on $\mathscr{H}$ can be expressed as the 2×2 operator matrix with respect to the decomposition $\mathscr{H}_1 \oplus \mathscr{H}_2$. Many authors [2, 15, 14] have studied perturbations of various spectra of such 2×2 operator matrices. In particular, upper triangular operator matrices have studied in a variety of directions (for example, invertibility and perturbation of spectra, etc) by numerous authors.

An operator $T \in \mathscr{L}(\mathscr{H})$ is *hypercyclic* if there is a vector $x \in \mathscr{H}$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in $\mathscr{H}$. Like the study of cyclic vectors in connection with the invariant subspace problem, hypercyclicity of operators has been studied by numerous authors, one of the reasons for this interest being the relation of hypercyclicity with the invariant subset problem. We refer the reader to the book [3] for the main results and problems in this area. In particular, Herrero [12] gave a spectral description of the norm-closure of the set of hypercyclic operators. Using this spectral characterization, Cao [7] has studied the hypercyclicity for 2×2 upper triangular operator matrices. He has also investigated the hypercyclicity for upper triangular operator matrix by using

Mathematics subject classification (2010): Primary 47A53, 47A55, 47A10.
Keywords and phrases: 2×2 operator matrices, Browder essential approximate point spectrum, Weyl's theorem, a-Weyl's theorem, a-Browder's theorem.
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Science and ICT (NRF-2017R1C1B1006538). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Science and ICT (NRF-2016K1A3A1A21005747).

the analytic core. In this paper, we explore the hypercyclicity and supercyclicity for 2×2 (not necessarily upper triangular) operator matrices and study a connection with the (a-)Weyl's theorem and the (a-)Browder's theorem.

Throughout this paper, we denote by $\mathscr{S}$ the collection of 2×2 operator matrices

$$M = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$$

acting on $\mathscr{H}\oplus\mathscr{K}$ where $C:\mathscr{K}\to\mathscr{H}$ has a closed range. We study a spectral characterization of 2×2 operator matrices in $\mathscr{S}$ for hypercyclicity and supercyclicity. Under some conditions, we find the equivalent conditions for 2×2 operator matrices in the class $\mathscr{S}$ for which Weyl's theorem, Browder's theorem, a-Weyl's theorem or a-Browder's theorem hold.

2. Preliminaries

For $T\in\mathscr{L}(\mathscr{H})$, the family $\{\mathscr{N}(T^k)\}$ forms an ascending sequence of subspaces. We write $\alpha(T)$ for the smallest nonnegative integer k for which $\mathscr{N}(T^k) = \mathscr{N}(T^{k+1})$ holds, and call the integer $\alpha(T)$ the *ascent* of T. If no such integer exists, we define $\alpha(T)=\infty$. In a similar way, the family $\{\mathscr{R}(T^k)\}$ forms a descending sequence. The smallest nonnegative integer k for which $\mathscr{R}(T^k)=\mathscr{R}(T^{k+1})$ is called the *descent* of T and is denoted by $\beta(T)$. If no such integer exists, we set $\beta(T)=\infty$. An operator $T\in\mathscr{L}(\mathscr{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If T is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and the *index* of T is defined by

$$\operatorname{ind}(T) := \dim\mathscr{N}(T) - \dim\mathscr{N}(T^*).$$

If both $\dim\mathscr{N}(T)$ and $\dim\mathscr{N}(T^*)$ are finite, then T is called *Fredholm*. An operator T is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent.

If $T\in\mathscr{L}(\mathscr{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $\sigma_s(T)$ for the spectrum, the point spectrum, the approximate point spectrum, and the surjective spectrum of T, respectively. The left essential spectrum $\sigma_{le}(T)$, the right essential spectrum $\sigma_{re}(T)$, the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T\in\mathscr{L}(\mathscr{H})$ are defined by

$$\begin{aligned}
\sigma_{le}(T) &= \{\lambda\in\mathbb{C} : T-\lambda \text{ is not upper semi-Fredholm}\};\\
\sigma_{re}(T) &= \{\lambda\in\mathbb{C} : T-\lambda \text{ is not lower semi-Fredholm}\};\\
\sigma_{e}(T) &= \{\lambda\in\mathbb{C} : T-\lambda \text{ is not Fredholm}\};\\
\sigma_{w}(T) &= \{\lambda\in\mathbb{C} : T-\lambda \text{ is not Weyl}\};\\
\sigma_{b}(T) &= \{\lambda\in\mathbb{C} : T-\lambda \text{ is not Browder}\}.
\end{aligned}$$

Evidently, we have the following inclusions

$$\sigma_{le}(T)\cup\sigma_{re}(T) = \sigma_e(T)\subseteq\sigma_w(T)\subseteq\sigma_b(T) = \sigma_e(T)\cup\operatorname{acc}\sigma(T),$$

where we write acc Δ for the accumulation points of $\Delta \subseteq \mathbb{C}$.

Let $T \in \mathscr{L}(\mathscr{H})$. If we write iso $\Delta = \Delta \setminus \operatorname{acc} \Delta$ for the isolated points of Δ, we write

$$\pi_{00}(T) := \pi_{0f}(T) \cap \operatorname{iso} \sigma(T) \quad \text{and} \quad p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$$

where $\pi_{0f}(T) = \{\lambda \in \mathbb{C} : 0 < \dim \mathscr{N}(T-\lambda) < \infty\}$. We say that *Weyl's theorem holds* for T if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem holds* for T if $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$. Hermann Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to several classes of operators by many authors [4, 8, 9, 10, 14].

We recall the Weyl essential approximate point spectrum $\sigma_{ea}(T)$ and the Browder essential approximate point spectrum $\sigma_{ab}(T)$ given by

$$\sigma_{ea}(T) = \bigcap \{\sigma_a(T+K) : K \in \mathscr{K}(\mathscr{H})\},$$
$$\sigma_{ab}(T) = \bigcap \{\sigma_a(T+K) : TK = KT \text{ and } K \in \mathscr{K}(\mathscr{H})\}$$

where $\mathscr{K}(\mathscr{H})$ is the set of all compact operators on $\mathscr{H}$. We put

$$\pi_{00}^a(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \dim \mathscr{N}(T-\lambda) < \infty\},$$
$$p_{00}^a(T) = \sigma_a(T) \setminus \sigma_{ab}(T).$$

We say that *a-Browder's theorem holds* for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T)$, and that *$a$-Weyl's theorem holds* for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$. It is well-known that

a-Weyl's theorem $\Longrightarrow$ a-Browder's theorem $\Longrightarrow$ Browder's theorem,

a-Weyl's theorem $\Longrightarrow$ Weyl's theorem $\Longrightarrow$ Browder's theorem.

Recall that $T \in \mathscr{L}(\mathscr{H})$ is *hypercyclic* if there is a vector $x \in \mathscr{H}$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in $\mathscr{H}$. In such a case, x is called a *hypercyclic vector* for T. A vector x is called *supercyclic* for T if its projective orbit, $\{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in $\mathscr{H}$. In this case, T is called a *supercyclic operator*. We denote by $\overline{HC(\mathscr{H})}$ (resp. $\overline{SC(\mathscr{H})}$) the norm-closure of all hypercyclic (resp. supercyclic) operators in $\mathscr{L}(\mathscr{H})$. The spectral characterizations of $\overline{HC(\mathscr{H})}$ and $\overline{SC(\mathscr{H})}$ have been studied by several authors [6, 7, 12, 11].

An operator $T \in \mathscr{L}(\mathscr{H})$ has the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0, the only analytic function $f : U \to \mathscr{H}$ which satisfies the equation $(T-\lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$ on U. The operator T has the *single-valued extension property* if T has the single-valued extension property at every $\lambda_0 \in \mathbb{C}$. The basic role of the single-valued extension property arises in local spectral theory. Trivially, an operator $T \in \mathscr{L}(\mathscr{H})$ has the single-valued extension property at every point of the resolvent $\rho(T)$. By the identity theorem for analytic function, we see that $T \in \mathscr{L}(\mathscr{H})$ has the single-valued extension property at every point of the boundary $\partial\sigma(T)$ of the spectrum. We need this property in the proof of Theorem 3.4. We refer the book [13] for a detailed information.

3. Main results

In this section, we explore the hypercyclicity of general 2×2 operator matrices and study the relation between hypercyclic operator matrices and the operator matrices which satisfy Weyl type theorems under some conditions. The hypercyclicity (or supercyclicity) for 2×2 diagonal or upper triangular operator matrices have been studied in [6, 7].

Throughout this section, we denote by $\mathscr{S}$ the collection of all 2×2 operator matrices such that the operator at $(1,2)$-entry has closed range, i.e.,

$$\mathscr{S}=\left\{\begin{pmatrix} A & C\\ Z & B\end{pmatrix} : \mathscr{H}\oplus\mathscr{K}\to\mathscr{H}\oplus\mathscr{K} \mid R(C) \text{ is closed}\right\}.$$

Let $M=\begin{pmatrix} A & C\\ Z & B\end{pmatrix}$ be an operator matrix in the class $\mathscr{S}$. Since $\mathscr{R}(C)$ is closed, M has the following matrix representation (cf. [2]);

$$M=\begin{pmatrix} A_1 & 0 & 0\\ A_2 & 0 & C_1\\ Z & B_1 & B_2\end{pmatrix}, \tag{1}$$

which maps from $\mathscr{H}\oplus\mathscr{K}=\mathscr{H}\oplus\mathscr{N}(C)\oplus\mathscr{N}(C)^{\perp}$ into $\mathscr{H}\oplus\mathscr{K}=\mathscr{R}(C)^{\perp}\oplus\mathscr{R}(C)\oplus\mathscr{K}$ where $C_1=C|_{\mathscr{N}(C)^{\perp}}$, $A_1=P_{\mathscr{R}(C)^{\perp}}A$, $A_2=P_{\mathscr{R}(C)}A$, $B_1=B|_{\mathscr{N}(C)}$ and $B_2=B|_{\mathscr{N}(C)^{\perp}}$. Here, $P_{\mathscr{N}(C)}$ (resp. $P_{\mathscr{N}(C)^{\perp}}$) denotes the projection of $\mathscr{K}$ onto $\mathscr{N}(C)$ (resp. $\mathscr{N}(C)^{\perp}$). When we consider an operator matrix M in the class $\mathscr{S}$, we use the matrix representation (1).

We first study various spectra for the operator matrices in the class $\mathscr{S}$.

Let $M=\begin{pmatrix} A & C\\ Z & B\end{pmatrix}$ be in $\mathscr{S}$. Since $R(C)$ is closed, C_1 is invertible. Let $\lambda\in\mathbb{C}$ be given. Using the representation (1), we write $M-\lambda$ as follows;

$$\begin{aligned} M-\lambda &= \begin{pmatrix} A_1-\lambda & 0 & 0\\ A_2-\lambda & 0 & C_1\\ Z & B_1-\lambda & B_2-\lambda\end{pmatrix}\\ &= \begin{pmatrix} 0 & I & 0\\ 0 & 0 & I\\ I & 0 & (B_2-\lambda)C_1^{-1}\end{pmatrix}\begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0\\ 0 & A_1-\lambda & 0\\ 0 & 0 & C_1\end{pmatrix}\begin{pmatrix} 0 & I & 0\\ I & 0 & 0\\ C_1^{-1}(A_2-\lambda) & 0 & I\end{pmatrix} \end{aligned} \tag{2}$$

where $A_1-\lambda=P_{\mathscr{R}(C)^{\perp}}(A-\lambda)|_{\mathscr{H}}$, $A_2-\lambda=P_{\mathscr{R}(C)}(A-\lambda)|_{\mathscr{H}}$, $B_1-\lambda=(B-\lambda)|_{\mathscr{N}(C)}$, $B_2-\lambda=(B-\lambda)|_{\mathscr{N}(C)^{\perp}}$ and $\triangle_\lambda=Z-(B_2-\lambda)C_1^{-1}(A_2-\lambda)$. From now on, we use the above matrix representation (2) for $M-\lambda$.

LEMMA 3.1. *Let M be an operator matrix in $\mathscr{S}$ with the representation (1). Suppose that $\sigma(B_1)=\sigma_a(B_1)$ and $\pi_{0f}(B_1)\setminus acc\,\sigma_{ea}(B_1)=\emptyset$. If $B_1\oplus A_1\in\overline{HC(\mathscr{K}\oplus\mathscr{H})}$, then $\sigma_w(M)\cup\partial\mathbb{D}$ is connected for any $Z\in\mathscr{L}(\mathscr{H},\mathscr{K})$.*

Proof. To prove connectedness of the set $\sigma_w(M)\cup\partial\mathbb{D}$ for any $Z\in\mathcal{L}(\mathcal{H},\mathcal{K})$, we first show that $\sigma_w(B_1\oplus A_1)=\sigma_w(M)$ for every $Z\in\mathcal{L}(\mathcal{H},\mathcal{K})$.

If $\lambda\notin\sigma_w(B_1\oplus A_1)$, then $B_1-\lambda$ and $A_1-\lambda$ are Fredholm. We have that $\operatorname{ind}(B_1-\lambda)+\operatorname{ind}(A_1-\lambda)=0$. Since we have the decomposition

$$\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}=\begin{pmatrix} I & 0 \\ 0 & A_1-\lambda \end{pmatrix}\begin{pmatrix} I & \triangle_\lambda \\ 0 & I \end{pmatrix}\begin{pmatrix} B_1-\lambda & 0 \\ 0 & I \end{pmatrix}, \tag{3}$$

the operator $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is also Fredholm. On the other hand, the operator matrices

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2-\lambda)C_1^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix} \tag{4}$$

are invertible, so that $M-\lambda$ is Fredholm. Moreover, we have that

$$\operatorname{ind}(M-\lambda)=\operatorname{ind}\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}=\operatorname{ind}(B_1-\lambda)+\operatorname{ind}(A_1-\lambda)=0.$$

Hence $M-\lambda$ is Weyl, which implies that $\sigma_w(B_1\oplus A_1)\supseteq\sigma_w(M)$.

To prove the converse inclusion, we let $\lambda\notin\sigma_w(M)$. Then we have that $M-\lambda$ is Fredholm and $\operatorname{ind}(M-\lambda)=0$. Since C_1 is invertible, it follows from (2) and (4) that the operator

$$\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$$

is also Fredholm. Thus, $B_1-\lambda$ is upper semi-Fredholm and $A_1-\lambda$ is lower semi-Fredholm by [15, Lemma 4]. We now show that $B_1-\lambda$ is Fredholm. If $B_1-\lambda$ is injective, then $\lambda\notin\sigma_a(B_1)$. Since $\sigma(B_1)=\sigma_a(B_1)$, $B_1-\lambda$ is invertible, which implies that $B_1-\lambda$ is Fredholm. If $B_1-\lambda$ is not injective, then it follows from $\pi_{0f}(B_1)\setminus \operatorname{acc}\sigma_{ea}(B_1)=\emptyset$ that $\lambda\in\operatorname{acc}\sigma_{ea}(B_1)$. However, $B_1-\lambda$ is upper semi-Fredholm, hence $\operatorname{ind}(B_1-\lambda)>0$. This means that $\dim\mathcal{N}(B_1^*-\overline{\lambda})<\infty$. Hence $B_1-\lambda$ is Fredholm, so that $A_1-\lambda$ is also Fredholm. Thus, $(B_1\oplus A_1)-\lambda$ is Fredholm. Moreover, we have that $\operatorname{ind}(B_1\oplus A_1-\lambda)=0$ and $\lambda\notin\sigma_w(B_1\oplus A_1)$. Therefore we obtain that $\sigma_w(B_1\oplus A_1)=\sigma_w(M)$, which implies that $\sigma_w(M)\cup\partial\mathbb{D}=\sigma_w(B_1\oplus A_1)\cup\partial\mathbb{D}$ is connected. □

In [7], Cao discussed the hypercyclicity for the upper triangular operator matrices and the diagonal operator matrices using a variant of the Weyl essential approximate point spectrum. In the following theorem, we explore the hypercyclicity for the full operator matrices in the class $\mathcal{S}$ using variants of the (Weyl essential) approximate point spectra and the hypercyclicity of the diagonal operator matrices. Note that, in general, $\sigma(B_1\oplus A_1)=\sigma_b(B_1\oplus A_1)$ does not imply $\sigma(M)=\sigma_b(M)$ where M has the representation (1).

THEOREM 3.2. *Let M be an operator matrix in $\mathcal{S}$ with the representation* (1). *Suppose that $\sigma(B_1)=\sigma_a(B_1)$ and $\pi_{0f}(B_1)\setminus acc\,\sigma_{ea}(B_1)=\emptyset$.*

(i) If $B_1 \oplus A_1 \in \overline{HC(\mathcal{K} \oplus \mathcal{H})}$*, then* $M \in \overline{HC(\mathcal{H} \oplus \mathcal{K})}$*.*

(ii) If $B_1 \oplus A_1 \in \overline{SC(\mathcal{K} \oplus \mathcal{H})}$*, then* $M \in \overline{SC(\mathcal{H} \oplus \mathcal{K})}$*.*

Proof. We first show that $\sigma_b(M) = \sigma(M)$. To show this, it suffices to show $\sigma_b(M) \supseteq \sigma(M)$. Suppose that $\lambda \notin \sigma_b(M)$. Then $M-\lambda$ is Browder. Then it follows from (2) and (4) that

$$\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$$

is Fredholm with finite ascent and descent. In fact, it is known that for each positive integer k,

$$\dim \mathcal{N}(M-\lambda)^k = \dim \mathcal{N}\left(\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}^k\right).$$

However, $\dim \mathcal{N}(M-\lambda) < \infty$ and $M-\lambda$ has finite ascent, hence $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ also has finite ascent. By the index product theorem, we get that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is Weyl, so it follows from [1, Theorem 3.4] that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ has finite descent. On the other hand, $B_1-\lambda$ is upper semi-Fredholm with finite ascent. Since $\pi_{0f}(B_1) \setminus \operatorname{acc} \sigma_{ea}(B_1) = \emptyset$, we have that $\dim \mathcal{N}(B_1^* - \overline{\lambda}) < \infty$. This means that $B_1-\lambda$ is Browder, so that $A_1-\lambda$ is also Browder. Thus, the direct sum $(B_1 \oplus A_1)-\lambda$ is Browder. Since $B_1 \oplus A_1 \in \overline{HC(\mathcal{K} \oplus \mathcal{H})}$, we have that $\sigma_b(B_1 \oplus A_1) = \sigma(B_1 \oplus A_1)$. Thus, $(B_1 \oplus A_1)-\lambda$ is invertible, so that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is invertible. Since C_1 is invertible, it follows from (2) that $M-\lambda$ is invertible, i.e., $\lambda \notin \sigma(M)$. Therefore, $\sigma_b(M) = \sigma(M)$.

We claim that $\operatorname{ind}(M-\lambda) \geqslant 0$ for any $\lambda \in \rho_e(M)$ and each $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Indeed, assume the contrary, i.e., there exists $\lambda \in \rho_e(M)$ with $\operatorname{ind}(M-\lambda) < 0$. Then it follows from (2) that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is semi-Fredholm with the negative index. This implies that $B_1-\lambda$ is upper semi-Fredholm, so that it is Fredholm by the preceding argument. By [5, Theorem 2.1], $A_1-\lambda$ is upper semi-Fredholm, so that $(B_1 \oplus A_1)-\lambda$ is upper semi-Fredholm with

$$\operatorname{ind}(B_1-\lambda) + \operatorname{ind}(A_1-\lambda) = \operatorname{ind}\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix} = \operatorname{ind}(M-\lambda) < 0.$$

This is a contradiction because of $B_1 \oplus A_1 \in \overline{HC(\mathcal{K} \oplus \mathcal{H})}$.

By [12, Theorem 2.1], it follows from Lemma 3.1 and above arguments that $M \in \overline{HC(\mathcal{H} \oplus \mathcal{K})}$ for any $Z \in \mathcal{L}(\mathcal{H}, \mathcal{K})$.

(ii) If $B_1 \oplus A_1 \in \overline{SC(\mathcal{K} \oplus \mathcal{H})}$, then the similar proof as that of (i) shows that $M \in \overline{SC(\mathcal{H} \oplus \mathcal{K})}$. □

COROLLARY 3.3. *Let M be in $\mathscr{S}$. Suppose that $\sigma(B_1) = \sigma_a(B_1)$, $\pi_{0f}(B_1) \setminus acc\ \sigma_{ea}(B_1) = \emptyset$, $\sigma(A_1) = \sigma_a(A_1)$ and $\pi_{0f}(A_1) \setminus acc\ \sigma_{ea}(A_1) = \emptyset$.*

(i) $M \in \overline{HC(\mathscr{H} \oplus \mathscr{K})}$ if and only if $\sigma(B_1 \oplus A_1) \cap \partial\mathbb{D}$ is connected.

(ii) $M \in \overline{SC(\mathscr{H} \oplus \mathscr{K})}$ if and only if $\sigma(B_1 \oplus A_1) \cap \partial(r\mathbb{D})$ is connected for some $r \geqslant 0$.

Proof. By hypothesis, we have that $\sigma(B_1 \oplus A_1) = \sigma_w(B_1 \oplus A_1)$ and $\pi_{00}(B_1 \oplus A_1) = \emptyset$. This implies that $B_1 \oplus A_1$ satisfies Weyl's theorem. By [7, Corollary 1.2], we get that $B_1 \oplus A_1 \in \overline{HC(\mathscr{K} \oplus \mathscr{H})}$ if and only if $\sigma(B_1 \oplus A_1) \cap \partial\mathbb{D}$ is connected, and that $B_1 \oplus A_1 \in \overline{SC(\mathscr{K} \oplus \mathscr{H})}$ if and only if $\sigma(B_1 \oplus A_1) \cap \partial(r\mathbb{D})$ is connected for some $r \geqslant 0$. By Theorem 3.2, these complete the proof. □

The following theorem gives the relations between hypercyclicity and Weyl type theorems for 2×2 (not necessarily upper triangular) operator matrices, which is regarded as the full matrix version of [7, Corollary 1.7 and 1.8]. That is, under some condition different from that of Theorem 3.2, we discuss several necessary and sufficient conditions for 2×2 operator matrices in $\mathscr{S}$ to hold a-Weyl's Theorem.

THEOREM 3.4. *Let $M \in \mathscr{S}$ have the matrix representation (1) such that*

$$\pi_{0f}(A_1) \setminus acc\ \sigma_{ea}(A_1) = \emptyset. \tag{5}$$

If $B_1 = B|_{\mathscr{N}(C)} \in \overline{HC(\mathscr{K})}$ and B_1 is isoloid and satisfies Weyl's theorem, then $\pi_{00}^a(M) = \emptyset$ and the followings are equivalent.

(i) $\pi_{0f}(B_1 \oplus A_1) \setminus acc\ \sigma_{ea}(B_1 \oplus A_1) = \emptyset$;

(ii) a-Weyl's theorem holds for $B_1 \oplus A_1$;

(iii) a-Weyl's theorem holds for M;

(iv) a-Browder's theorem holds for $B_1 \oplus A_1$;

(v) a-Browder's theorem holds for M;

(vi) $\pi_{0f}(M) \setminus acc\ \sigma_{ea}(M) = \emptyset$.

Proof. We first claim that $\pi_{00}^a(M) = \emptyset$. Indeed, assume the contrary, that is, $\lambda \in \pi_{00}^a(M)$. Then there exists $\varepsilon > 0$ such that $M - \mu$ is bounded below if $0 < |\lambda - \mu| < \varepsilon$, so that

$$\begin{pmatrix} B_1 - \mu & \triangle_\mu \\ 0 & A_1 - \mu \end{pmatrix} \oplus C_1$$

is bounded below. Since C_1 is invertible, $B_1 - \mu$ is bounded below. Since $B_1 \in \overline{HC(\mathscr{K})}$ and B_1 satisfies Weyl's theorem, it follows from [7, Corollary 1.2] that $\sigma_a(B_1) = \sigma(B_1)$, so that $B_1 - \mu$ is invertible. This implies that $A_1 - \mu$ is bounded below.

Hence, $\lambda \in \text{iso}\ \sigma_a(B_1 \oplus A_1)$ and $\lambda \in \text{iso}\ \sigma(B_1) \cup \rho(B_1)$. Since $\dim \mathcal{N}(B_1 - \lambda) \leqslant \dim \mathcal{N}(M) < \infty$ and B_1 is isoloid, we have that $\lambda \in \pi_{00}(B_1) \cup \rho(B_1)$.

Moreover, Weyl's theorem holds for B_1 and $\sigma(B_1) = \sigma_b(B_1)$, so that $B_1 - \lambda$ is invertible. It follows from [14, Theorem 2.4] that $\dim \mathcal{N}(A_1 - \lambda) < \infty$. Since $\pi_{0f}(A_1) \setminus \text{acc}\ \sigma_{ea}(A_1) = \emptyset$, we have that $\dim \mathcal{N}(A_1 - \lambda) = 0$. By (2) and (4), this implies that $M - \lambda$ is injective. This contradicts to the fact that $0 < \dim \mathcal{N}(M - \lambda) < \infty$.

(i) $\Rightarrow$ (ii) By hypothesis, we have that

$$\pi_{0f}(B_1 \oplus A_1) \setminus \text{acc}\ \sigma_{ea}(B_1 \oplus A_1) = \emptyset \subseteq p_{00}^a(B_1 \oplus A_1).$$

So it follows from [7, Theorem 1.4] that a-Weyl's theorem holds for $B_1 \oplus A_1$.

(ii) $\Rightarrow$ (i) Assume that there exists $\lambda_0 \in \pi_{0f}(B_1 \oplus A_1) \setminus \text{acc}\ \sigma_{ea}(B_1 \oplus A_1)$. Then we have that $0 < \dim \mathcal{N}(B_1 \oplus A_1 - \lambda_0) < \infty$ and that there exists $\varepsilon > 0$ such that $B_1 \oplus A_1 - \lambda$ is upper semi-Fredholm and $\text{ind}(B_1 \oplus A_1 - \lambda) \leqslant 0$ for $0 < |\lambda - \lambda_0| < \varepsilon$. This means that

$$\lambda \in \sigma_a(B_1 \oplus A_1) \setminus \sigma_{ea}(B_1 \oplus A_1).$$

However, a-Weyl's theorem holds for $B_1 \oplus A_1$, so that $\lambda \in \text{iso}\ \sigma_a(B_1 \oplus A_1)$. Thus, $B_1 \oplus A_1$ has the single valued extension property at λ. Since $B_1 \oplus A_1 - \lambda$ is upper semi-Fredholm, $B_1 \oplus A_1 - \lambda$ has finite ascent. This implies that both $B_1 - \lambda$ and $A_1 - \lambda$ have finite ascent. Since $B_1 \in \overline{HC(\mathcal{K})}$, λ is a pole of the resolvent $\rho(B_1)$ of B_1, so that $B_1 - \lambda_0$ is invertible. Furthermore, $A_1 - \lambda_0$ is bounded below since $\pi_{0f}(A_1) \setminus \text{acc}\ \sigma_{ea}(A_1) = \emptyset$. Thus, $\mathcal{N}(B_1 \oplus A_1) = \{0\}$, which is a contradiction.

(ii) $\Rightarrow$ (iii) We first show that $\sigma_a(M) = \sigma_{ea}(M)$. For any $\lambda \notin \sigma_{ea}(M)$, $M - \lambda$ is upper semi-Fredholm and $\text{ind}(M - \lambda) \leqslant 0$. It follows from (2) and (4) that

$$\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} \oplus C_1$$

is upper semi-Fredholm and $\text{ind}(B_1 - \lambda) + \text{ind}(A_1 - \lambda) \leqslant 0$. Since C_1 is invertible, $B_1 \oplus A_1 - \lambda$ is also upper semi-Fredholm and $\text{ind}(B_1 \oplus A_1 - \lambda) \leqslant 0$, i.e.,

$$\lambda \in \sigma_a(B_1 \oplus A_1) \setminus \sigma_{ea}(B_1 \oplus A_1).$$

Since a-Weyl's theorem holds for $B_1 \oplus A_1$, we have that $\lambda \in \pi_{00}^a(B_1 \oplus A_1)$. Then $B_1 \oplus A_1$ has the single valued extension property at λ, equivalently, $B_1 \oplus A_1 - \lambda$ has finite ascent because $B_1 \oplus A_1 - \lambda$ is upper semi-Fredholm. This implies that $B_1 - \lambda$ and $A_1 - \lambda$ have finite ascent, so that $\text{ind}(B_1 - \lambda) \leqslant 0$. On the other hand, $B_1 \in \overline{HC(\mathcal{K})}$, so that $\text{ind}(B_1 - \lambda) \geqslant 0$. Moreover, $B_1 - \lambda$ is invertible, so that $\text{ind}(A_1 - \lambda) \leqslant 0$. Since $\pi_{0f}(A_1) \setminus \text{acc}\ \sigma_{ea}(A_1) = \emptyset$, $A_1 - \lambda$ is bounded below. However,

$$\mathcal{N}\left(\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}\right) \subseteq (B_1 - \lambda)^{-1}[\triangle_\lambda \mathcal{N}(A_1 - \lambda)] \oplus \mathcal{N}(A_1 - \lambda),$$

which implies that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is bounded below. By (2) and (4), we have that $M - \lambda$ is also bounded below, so that $\sigma_a(M) = \sigma_{ea}(M)$ for any $Z \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Therefore we have that

$$\sigma_a(M) \setminus \sigma_{ea}(M) = \pi_{00}^a(M) = \emptyset,$$

which means that a-Weyl's theorem holds for M.

(iii) $\Rightarrow$ (v) Since a-Weyl's theorem implies a-Browder's theorem, this is clear.

(v) $\Rightarrow$ (iv) If $\lambda \notin \sigma_{ab}(B_1 \oplus A_1)$, then $(B_1 \oplus A_1) - \lambda I$ is upper semi-Fredholm and has finite ascent. Thus, both $B_1 - \lambda$ and $A_1 - \lambda$ are upper semi-Fredholm with finite ascent. By [6, Lemma 2.2], the operator matrix $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ has finite ascent. Since C_1 is invertible, it follows from the decomposition (2) that $M - \lambda$ has finite ascent. Further, since $\begin{pmatrix} I & \triangle_\lambda \\ 0 & I \end{pmatrix}$ is invertible, we see from the decomposition (3) that $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is upper semi-Fredholm and so is M. Therefore, we obtain that $\lambda \notin \sigma_{ab}(M)$. Since M satisfies a-Browder's theorem, $\lambda \notin \sigma_{ea}(M)$. Hence, we have that

$$\text{ind}(B_1 - \lambda) + \text{ind}(A_1 - \lambda) \leqslant 0,$$

which implies that $\lambda \notin \sigma_{ea}(B_1 \oplus A_1)$. This shows the inclusion $\sigma_{ab}(B_1 \oplus A_1) \supseteq \sigma_{ea}(B_1 \oplus A_1)$. Since the reverse inclusion is obvious, a-Weyl's theorem holds for $B_1 \oplus A_1$.

(iv) $\Rightarrow$ (ii) It is well known that a-Browder's theorem holds for $B_1 \oplus A_1$ and $\pi_{00}(B_1 \oplus A_1) = p^a_{00}(B_1 \oplus A_1)$ if and only if a-Weyl's theorem holds for $B_1 \oplus A_1$. To show that a-Weyl's theorem holds for $B_1 \oplus A_1$, it suffices to prove that

$$\pi^a_{00}(B_1 \oplus A_1) \subseteq p^a_{00}(B_1 \oplus A_1).$$

If $\lambda \in \pi^a_{00}(B_1 \oplus A_1)$, then we have that $\lambda \in \text{iso}\, \sigma_a(B_1 \oplus A_1)$ and $0 < \dim \mathcal{N}(B_1 \oplus A_1) < \infty$. Thus, there exists $\varepsilon > 0$ such that $(B_1 \oplus A_1) - \lambda_0$ is bounded below for $0 < |\lambda - \lambda_0| < \varepsilon$. Hence $B_1 - \lambda$ is bounded below for $0 < |\lambda - \lambda_0| < \varepsilon$.

Since B_1 satisfies Weyl's theorem and $B_1 \in \overline{HC(\mathscr{K})}$, it follows from [7, Corollary 1.2] that $\sigma_a(B_1) = \sigma(B_1)$, so that the operator $B_1 - \lambda$ is also Weyl. Since B_1 has the single valued extension property at λ, $B_1 - \lambda$ has finite ascent. We also have that $\dim \mathcal{N}(A_1 - \lambda) < \infty$. But, $\pi_{0f}(A_1) \setminus \text{acc}\, \sigma_{ea}(A_1) = \emptyset$, so that $\dim \mathcal{N}(A_1 - \lambda) = 0$. So, $A_1 - \lambda$ has finite ascent, which implies that $\lambda \in p^a_{00}(B_1 \oplus A_1)$.

It remains to prove the equivalence of (i) and (vi). To do this, we prove that (i) implies (vi) and that (vi) implies that (iii) since (i) and (iii) are equivalent. But, the proof of (vi) $\Rightarrow$ (iii) is same as that of (i) $\Rightarrow$ (ii) by [7, Theorem 1.4]. Thus, we only show that (i) implies (vi).

(i) $\Rightarrow$ (vi) Assume that (vi) is not true, i.e., there exists $\lambda_0 \in \pi_{0f}(M) \setminus \text{acc}\, \sigma_{ea}(M)$. Then we see that $\dim \mathcal{N}(M - \lambda_0) > 0$ and there exists $\varepsilon > 0$ such that $M - \lambda$ is upper semi-Fredholm and $\text{ind}(M - \lambda) \leqslant 0$ for $0 < |\lambda - \lambda_0| < \varepsilon$. By the decomposition (2), $B_1 \oplus A_1 - \lambda$ is also upper semi-Fredholm and $\text{ind}(B_1 \oplus A_1 - \lambda) \leqslant 0$. Since $\pi_{0f}(B_1 \oplus A_1) \setminus \text{acc}\, \sigma_{ea}(B_1 \oplus A_1) = \emptyset$, $B_1 \oplus A_1$ is bounded below. Thus, $\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}$ is bounded below and so is $M - \lambda$ by the decomposition (2). Hence $\lambda_0 \in \pi^a_{00}(M)$, which is a contradiction from $\pi^a_{00}(M) = \emptyset$. $\square$

REMARK 3.5. If $M \in \mathscr{S}$, $B_1 = B|_{\mathcal{N}(C)} \in \overline{HC(\mathscr{K})}$, and B_1 satisfies Weyl's theorem, then it follows from [7, Corollary 1.2] that $\sigma(B_1) = \sigma_a(B_1)$ and $\pi_{0f}(B_1) \setminus$

acc $\sigma_{ea}(B_1) = \emptyset$. Then it immediately follows from Theorem 3.2 that the following arguments are true;

(a) If $B_1 \oplus A_1 \in \overline{HC(\mathscr{K} \oplus \mathscr{H})}$, then $M \in \overline{HC(\mathscr{H} \oplus \mathscr{K})}$.

(b) If $B_1 \oplus A_1 \in \overline{SC(\mathscr{K} \oplus \mathscr{H})}$, then $M \in \overline{SC(\mathscr{H} \oplus \mathscr{K})}$.

In this case, the condition (5) in Theorem 3.4 is not necessary for which (a) and (b) hold.

EXAMPLE 3.6.

(1) If T is the unilateral shift on $l^2(\mathbb{N})$ or a one-to-one quasinilpotent operator, then $\pi_{0f}(T) = \emptyset$, so that the condition (5) in Theorem 3.4 trivially holds.

(2) Let an operator A on $l^2(\mathbb{N})$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_2, 0, x_4, 0, x_6, 0, \cdots).$$

We see that $\sigma(A) = \sigma_w(A) = \{0, 1\}$, and $\pi_{0f}(A) = \emptyset$. Hence, the condition (5) in Theorem 3.4 holds for A.

(3) We now construct a simple example satisfying the equation (5) and $\pi_{0f}(T) \neq \emptyset$. We are indebted to Professor W.Y. Lee for providing the following example. To do this, let $(\mathscr{H}_n)_{n \geqslant 0}$ be a sequence of Hilbert spaces where $\dim \mathscr{H}_0 < \infty$ and $\dim \mathscr{H}_n = \infty$ for each $n \geqslant 1$. Set $\mathscr{H} = \oplus_{n \geqslant 0} \mathscr{H}_n$. We define a bounded linear operator T on $\mathscr{H}$ by

$$T = I_{\mathscr{H}_0} \oplus \left(\bigoplus_{n \geqslant 1} \frac{n}{n+1} I_{\mathscr{H}_n} \right)$$

where $I_{\mathscr{H}_n}$ is the identity operator on $\mathscr{H}_n$ $(n \geqslant 1)$. It is not hard to see that $\pi_{0f}(T) = \{1\}$ and $\sigma_{ea}(T) = \{\frac{n}{n+1} : n = 1, 2, \ldots\}$ since $(T - \frac{n}{n+1} I)|_{\mathscr{H}_n} = 0|_{\mathscr{H}_n}$ for $n \geqslant 1$. It is well known that every compact operator on an infinite dimensional space is not bounded below. Thus, we have that

$$\left(T - \frac{n}{n+1} I + K \right) \Big|_{\mathscr{H}_n}$$

is not bounded below for any compact operator K. Hence, $\pi_{0f}(T) = \{1\} = \operatorname{acc} \sigma_{ea}(T)$, which implies that (5) holds.

EXAMPLE 3.7. In this example, we give an operator matrix M satisfying the assumptions in Theorem 3.4, so that the a-Weyl's theorem holds for this M.

Let U be the unilateral shift on $l^2(\mathbb{N})$ and let $A \in \mathscr{L}(l^2(\mathbb{N}))$ (cf. [6, Remark 3.2]) be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, 0, \cdots).$$

We denote an operator M on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ as follows;

$$M := \begin{pmatrix} A & 0 \\ Z & U^* \end{pmatrix} \quad \text{for any } Z \in \mathscr{L}(l^2(\mathbb{N})).$$

Since the zero operator has closed range, M belongs to $\mathscr{S}$. Since $B_1 = U^*|_{\mathscr{N}(0)} = U^*$, we have that $\sigma(B_1) = \sigma_w(B_1) = \sigma_a(B_1) = \overline{\mathbb{D}}$, so that iso $\sigma(B_1) = \emptyset$. Thus, $B_1 \in \overline{HC(l^2(\mathbb{N}))}$ is isoloid and satisfies Weyl's theorem. Since $A_1 = P_{\mathscr{R}(0)^\perp} A = A$, it follows that

$$\pi_{0f}(A_1) \setminus \text{acc}\, \sigma_{ea}(A_1) = \emptyset.$$

Since $\pi_{0f}(B_1 \oplus A_1) \setminus \text{acc}\, \sigma_{ea}(B_1 \oplus A_1) = \emptyset$, it follows from Theorem 3.4 that M satisfies a-Weyl's theorem. Moreover, we see that $B_1 \oplus A_1 \in \overline{HC(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))}$. By Remark 3.5, M is also in $\overline{HC(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))}$.

Acknowledgement. The authors would like to thank to Professor W. Y. Lee for his fruitful discussion and comments.

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(Received March 30, 2017)

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