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GENERALIZED LIE DERIVATIONS OF UNITAL ALGEBRAS WITH IDEMPOTENTS

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Abstract. Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e over a unital commutative ring R. We show that under suitable assumptions every generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form $F(x)=\lambda x+\Delta(x)$, where $\lambda\in Z(\mathscr{A})$ and Δ is a Lie n-derivation of $\mathscr{A}$. As an application, we give a description of generalized Lie n-derivations on classical examples of unital algebras with idempotents: triangular algebras, matrix algebras, nest algebras and algebras of all bounded linear operators.

1. Introduction

Throughout this paper, let R be a commutative ring with unity and let $\mathscr{A}$ be a unital algebra over R. Let us assume that $\mathscr{A}$ has an idempotent $e\neq 0,1$ and let f denote the idempotent $1-e$. In this case $\mathscr{A}$ can be represented in the Peirce decomposition form $\mathscr{A}=e\mathscr{A}e+e\mathscr{A}f+f\mathscr{A}e+f\mathscr{A}f$ where $e\mathscr{A}e$ and $f\mathscr{A}f$ are subalgebras with unitary elements e and f, respectively, $e\mathscr{A}f$ is an $(e\mathscr{A}e,f\mathscr{A}f)$-bimodule and $f\mathscr{A}e$ is an $(f\mathscr{A}f,e\mathscr{A}e)$-bimodule. We will assume that $\mathscr{A}$ satisfies

$$\begin{aligned} exe\cdot e\mathscr{A}f=\{0\}=f\mathscr{A}e\cdot exe \quad &\text{implies}\quad exe=0, \\ e\mathscr{A}f\cdot fxf=\{0\}=fxf\cdot f\mathscr{A}e \quad &\text{implies}\quad fxf=0, \end{aligned} \tag{1}$$

for all $x\in\mathscr{A}$. Examples of unital algebras with nontrivial idempotents having the property (1) are triangular algebras, matrix algebras, and prime (and hence in particular simple) algebras with nontrivial idempotents.

By $[x,y]=xy-yx$ we denote the commutator or the Lie product of elements $x,y\in\mathscr{A}$. Set $p_1(x)=x$ and

$$p_n(x_1,x_2,\ldots,x_n)=[p_{n-1}(x_1,x_2,\ldots,x_{n-1}),x_n] \quad \text{for all integers } n\geqslant 2.$$

Thus, $p_2(x_1,x_2)=[x_1,x_2]$, $p_3(x_1,x_2,x_3)=[[x_1,x_2],x_3]$, etc. Let $n\geqslant 2$ be an integer. A linear map $D:\mathscr{A}\to\mathscr{A}$ is called a *Lie n-derivation* if

$$D(p_n(x_1,x_2,\ldots,x_n))=\sum_{i=1}^{n} p_n(x_1,\ldots,x_{i-1},D(x_i),x_{i+1},\ldots,x_n) \tag{2}$$

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for all $x_1,x_2,\ldots,x_n \in \mathscr{A}$. In particular, a Lie 2-derivation is a Lie derivation and a Lie 3-derivation is a Lie triple derivation. A linear map $F:\mathscr{A}\to\mathscr{A}$ is said to be a *generalized Lie n-derivation* if there exists a Lie n-derivation $D:\mathscr{A}\to\mathscr{A}$ such that

$$F\left(p_n\left(x_1,x_2,\ldots,x_n\right)\right)=p_n\left(F\left(x_1\right),x_2,\ldots,x_n\right)+\sum_{i=2}^{n}p_n\left(x_1,\ldots,D\left(x_i\right),\ldots,x_n\right) \tag{3}$$

for all $x_1,x_2,\ldots,x_n \in \mathscr{A}$. In particular, a generalized Lie 2-derivation is a linear map F that satisfies

$$F([x_1,x_2])=[F(x_1),x_2]+[x_1,D(x_2)] \quad \text{for all } x_1,x_2\in\mathscr{A},$$

where D is a Lie derivation of $\mathscr{A}$. A generalized Lie 2-derivation is a generalized derivation for the Lie product. Namely, a linear map $F:\mathscr{A}\to\mathscr{A}$ is a generalized derivation with an associated derivation D of $\mathscr{A}$ if $F(x_1x_2)=F(x_1)x_2+x_1D(x_2)$ for all $x_1,x_2\in\mathscr{A}$. Let us mention that a generalized Lie 2-derivation should not be mistaken for the notion of generalized Lie derivation. Namely, a linear map $F:\mathscr{A}\to\mathscr{A}$ is a *generalized Lie derivation* if there exists a derivation $D:\mathscr{A}\to\mathscr{A}$ such that

$$F\left([x_1,x_2]\right)=F\left(x_1\right)x_2-F\left(x_2\right)x_1+x_1D\left(x_2\right)-x_2D\left(x_1\right) \quad \text{for all } x_1,x_2\in\mathscr{A}. \tag{4}$$

Note, that any Lie n-derivation is an example of a generalized Lie n-derivation (set $F=D$ in (3)). On the other hand any multiplier $x\mapsto\lambda x$ where $\lambda\in Z(\mathscr{A})$ is an example of a generalized Lie n-derivation (set $F(x)=\lambda x$ for all $x\in\mathscr{A}$ and $D=0$ in (3)). These types of maps and their sums are standard examples of generalized Lie n-derivations. We expect that in several settings these maps are basically the only examples of generalized Lie n-derivations.

The main purpose of the paper is to describe generalized Lie n-derivations of unital algebras with idempotents, which satisfy (1). In the main result of the paper, Theorem 2.3, we show that under certain mild assumptions every generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form

$$F(x)=\lambda x+\Delta(x) \quad \text{for all } x\in\mathscr{A},$$

where $\lambda\in Z(\mathscr{A})$ and Δ is a Lie n-derivation of $\mathscr{A}$. As an application, we give a description of generalized Lie n-derivations on classical examples of unital algebras with idempotents: triangular algebras (Corollary 3.2), matrix algebras (Corollary 3.1), upper triangular matrix algebras (Corollary 3.3), nest algebras (Corollary 3.4) and algebras of all bounded linear operators (Corollary 3.6). We shall use some known results about the form of Lie n-derivations on unital algebras with idempotents that were obtained in papers [3, 4, 6, 13, 14]. Let us mention that in different papers [5, 8, 9, 10, 11, 12, 15] Lie derivations and Lie triple derivations on triangular algebras were studied from different perspectives. The main motivation for our study actually comes from papers [1, 2, 7]. Ashraf and Jabeen [1] recently described the form of nonlinear generalized Lie 3-derivation on triangular algebras. In papers [2, 7] generalized Lie derivations, maps satisfying (4), of triangular algebras were studied.

In this paper we present a new approach, which elegantly reduces the problem of describing a generalized Lie n-derivation to problem of describing a Lie n-derivation. It turns out, that if $F:\mathscr{A}\to\mathscr{A}$ is a generalized Lie n-derivation associated with a Lie n-derivation D, then a linear map $H=F-D$ satisfies

$$H\left(p_n\left(x_1,x_2,\ldots,x_n\right)\right)=p_n\left(H(x_1),x_2,\ldots,x_n\right) \tag{5}$$

for all $x_1,x_2,\ldots,x_n\in\mathscr{A}$. Therefore, it suffices to consider linear maps with property (5). Note that H is actually a generalized Lie n-derivation whose associated Lie n-derivation is the zero map. Under suitable assumptions on unital algebra $\mathscr{A}$ (see Proposition 2.2) any such generalized Lie n-derivation is of the form $H(x)=\lambda x+\gamma(x)$ for all $x\in\mathscr{A}$, where $\lambda\in Z(\mathscr{A})$ and $\gamma:\mathscr{A}\to Z(\mathscr{A})$ is a linear map that vanishes on $p_n(\mathscr{A},\ldots,\mathscr{A})$.

2. Preliminaries and the main theorem

Let $\mathscr{A}$ be a unital algebra with nontrivial idempotents e and $f=1-e$, which satisfies (1). For convenience we shall use the following notations $a=eae\in e\mathscr{A}e$, $m=emf\in e\mathscr{A}f$, $t=fte\in f\mathscr{A}e$ and $b=fbf\in f\mathscr{A}f$. Thus, every element $x\in\mathscr{A}$ can be represented in the form

$$x=eae+emf+fte+fbf=a+m+t+b.$$

From [3, Proposition 2.1] it follows that the center of $\mathscr{A}$ is equal to

$$Z(\mathscr{A})=\{a+b\in e\mathscr{A}e+f\mathscr{A}f\,|\,am=mb,ta=bt \text{ for all } m\in e\mathscr{A}f,t\in f\mathscr{A}e\}.$$

Furthermore, there exists a unique algebra isomorphism $\tau:Z(\mathscr{A})e\to Z(\mathscr{A})f$, such that $am=m\tau(a)$ and $ta=\tau(a)t$ for all $m\in e\mathscr{A}f,t\in f\mathscr{A}e$ and for any $a\in Z(\mathscr{A})e$. It is not difficult to see:

REMARK 2.1. Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e and $f=1-e$. For any $x\in\mathscr{A}$ and for any integer $n\geqslant 2$ we have

$$p_n(x,e,\ldots,e)=(-1)^{n-1}exf+fxe \quad \text{and}$$
$$p_n(x,f,\ldots,f)=exf+(-1)^{n-1}fxe.$$

In particular, $[x,e]=-exf+fxe$ and $[x,f]=exf-fxe$.

In this section we will prove the main result, Theorem 2.3. As we mentioned in the introduction, the problem of a description of a generalized Lie n-derivations can be reduced to a description of a map satisfying (5). Let us begin with the solution of this problem.

PROPOSITION 2.2. *Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e satisfying (1). Let us assume that $Z(e\mathscr{A}e)=Z(\mathscr{A})e$ and $Z(f\mathscr{A}f)=Z(\mathscr{A})f$. If a linear map $H:\mathscr{A}\to\mathscr{A}$ satisfies*

$$H\left(p_n\left(x_1,x_2,\ldots,x_n\right)\right)=p_n\left(H(x_1),x_2,\ldots,x_n\right) \tag{6}$$

for all $x_1,x_2,\ldots,x_n \in \mathscr{A}$*, then*

$$H(x) = \lambda x + \gamma(x) \quad \textit{for all } x \in \mathscr{A},$$

where $\lambda \in Z(\mathscr{A})$ *and* $\gamma : \mathscr{A} \to Z(\mathscr{A})$ *is a linear map such that* $\gamma(p_n(\mathscr{A},\ldots,\mathscr{A})) = 0$.

Proof. Let us prove first, that

$$H(e) = eH(e)e + fH(e)f \in e\mathscr{A}e + f\mathscr{A}f \quad \text{and} \tag{7}$$
$$H(f) = eH(f)e + fH(f)f \in e\mathscr{A}e + f\mathscr{A}f. \tag{8}$$

We can write

$$H(p_n(e,f,\ldots,f)) = H(p_{n-1}([e,f],\ldots,f)) = 0$$

and according to Remark 2.1 it holds

$$p_n(H(e),f,\ldots,f) = eH(e)f + (-1)^{n-1}fH(e)e.$$

Hence, $eH(e)f + (-1)^{n-1}fH(e)e = 0$ and so $eH(e)f = 0 = fH(e)e$. Therefore, $H(e)$ is of the form (7). We prove similarly that $H(f)$ is of the form (8).

Further, let us prove that

$$fH(e)f \in Z(\mathscr{A})f \quad \text{and} \tag{9}$$
$$eH(f)e \in Z(\mathscr{A})e. \tag{10}$$

If $n = 2$ then

$$0 = H([e,b]) = [H(e),b] = [fH(e)f,b]$$

for all $b \in f\mathscr{A}f$. Hence $fH(e)f \in Z(f\mathscr{A}f)$. With assumption $Z(f\mathscr{A}f) = Z(\mathscr{A})f$ we see that (9) holds true. Similarly, one can prove (10). Let $n \geqslant 3$ and let $b \in f\mathscr{A}f$, $m \in e\mathscr{A}f$ and $t \in f\mathscr{A}e$ be arbitrary elements. We can write

$$H(p_n(e,b,m,f,\ldots,f)) = H(p_{n-1}([e,b],m,f,\ldots,f)) = 0$$

and

$$\begin{aligned} p_n(H(e),b,m,f,\ldots,f) &= p_{n-2}([[H(e),b],m],f,\ldots,f) \\ &= p_{n-2}([[fH(e)f,b],m],f,\ldots,f) \\ &= -m[fH(e)f,b]. \end{aligned}$$

Hence, $e\mathscr{A}f \cdot [fH(e)f,b] = \{0\}$. Analogously, we can show

$$H(p_n(e,b,t,e,\ldots,e)) = H(p_{n-1}([e,b],t,e,\ldots,e)) = 0$$

and

$$p_n(H(e),b,t,e,\ldots,e) = p_{n-2}([[H(e),b],t],e,\ldots,e) = [fH(e)f,b]t.$$

Hence, $[fH(e)f,b]\cdot f\mathscr{A}e=\{0\}$. Because the algebra $\mathscr{A}$ satisfies (1), it follows $[fH(e)f,b]=0$ for all $b\in f\mathscr{A}f$. Hence, $fH(e)f\in Z(f\mathscr{A}f)=Z(\mathscr{A})f$ and (9) holds true. Similarly, one can prove (10).

Let us denote $\alpha=eH(e)e-\tau^{-1}(fH(e)f)\in e\mathscr{A}e$, $\beta=fH(f)f-\tau(eH(f)e)\in f\mathscr{A}f$, and $\lambda=\alpha+\beta\in e\mathscr{A}e+f\mathscr{A}f$. Let us prove that $\lambda\in Z(\mathscr{A})$. For this purpose we describe images $H(m)$ and $H(t)$, where $m\in e\mathscr{A}f$ and $t\in f\mathscr{A}e$ arbitrary elements. The first conclusion is

$$[H(e),m]=eH(e)em-mfH(e)f=(eH(e)e-\tau^{-1}(fH(e)f))m=\alpha m,$$
$$[H(f),m]=eH(f)em-mfH(f)f=m(\tau(eH(f)e-fH(f)f)=-m\beta.$$

Considering this, we can write

$$\begin{aligned}H(m)&=H([e,m])=H\left(p_n\left(e,m,f,\dots,f\right)\right)\\&=p_n\left(H(e),m,f,\dots,f\right)=p_{n-1}\left([H(e),m],f,\dots,f\right)\\&=p_{n-1}\left(\alpha m,f,\dots,f\right)=\alpha m\end{aligned}$$

and

$$\begin{aligned}H(m)&=-H([f,m])=-H\left(p_n\left(f,m,f,\dots,f\right)\right)\\&=-p_n\left(H(f),m,f,\dots,f\right)=-p_{n-1}\left([H(f),m],f,\dots,f\right)\\&=-p_{n-1}\left(-m\beta,f,\dots,f\right)=m\beta.\end{aligned}$$

Hence

$$H(m)=\alpha m=m\beta\quad\text{for all } m\in e\mathscr{A}f. \tag{11}$$

Analogously, one can prove that

$$H(t)=\beta t=t\alpha\quad\text{for all } t\in f\mathscr{A}e. \tag{12}$$

From equalities (11) and (12) we conclude, that $\lambda=\alpha+\beta\in Z(\mathscr{A})$.

Let a linear map $\gamma\colon\mathscr{A}\to\mathscr{A}$ be defined by $\gamma(x)=H(x)-\lambda x$ for all $x\in\mathscr{A}$. Then γ also satisfies (6). Namely, it holds true that

$$\begin{aligned}\gamma\left(p_n\left(x_1,x_2,\dots,x_n\right)\right)&=H\left(p_n\left(x_1,x_2,\dots,x_n\right)\right)-\lambda p_n\left(x_1,x_2,\dots,x_n\right)\\&=p_n\left(H(x_1),x_2,\dots,x_n\right)-p_n\left(\lambda x_1,x_2,\dots,x_n\right)\\&=p_n\left(H(x_1)-\lambda x_1,x_2,\dots,x_n\right)\\&=p_n\left(\gamma(x_1),x_2,\dots,x_n\right)\end{aligned}$$

for all $x_1,x_2,\dots,x_n\in\mathscr{A}$. Next, we prove that γ maps into the center of $\mathscr{A}$.

According to (11) and (12), we see that

$$\gamma(m)=H(m)-\lambda m=\alpha m-(\alpha+\beta)m=0,$$
$$\gamma(t)=H(t)-\lambda t=\beta t-(\alpha+\beta)t=0$$

for all $m \in e\mathscr{A}f$, $t \in f\mathscr{A}e$. Let us fix $a \in e\mathscr{A}e$. Then the following equalities,

$$\gamma(p_n(a,f,\ldots,f)) = \gamma(p_{n-1}([a,f],\ldots,f)) = 0,$$
$$p_n(\gamma(a),f,\ldots,f) = e\gamma(a)f + (-1)^{n-1} f\gamma(a)e,$$

imply $e\gamma(a)f + (-1)^{n-1} f\gamma(a)e = 0$. Hence, $e\gamma(a)f = 0 = f\gamma(a)e$ and so $\gamma(a) = e\gamma(a)e + f\gamma(a)f \in e\mathscr{A}e + f\mathscr{A}f$. Let $m \in e\mathscr{A}f$ be an arbitrary element. Then $am \in e\mathscr{A}f$ and $\gamma(am) = 0$. Therefore, we may write

$$\begin{aligned} 0 = \gamma(am) &= \gamma([a,m]) = \gamma(p_n(a,m,f,\ldots,f)) \\ &= p_n(\gamma(a),m,f,\ldots,f) = p_{n-1}([\gamma(a),m],f,\ldots,f) \\ &= e\gamma(a)em - mf\gamma(a)f. \end{aligned}$$

Hence,

$$e\gamma(a)e \cdot m = m \cdot f\gamma(a)f \quad \text{for all } m \in e\mathscr{A}f. \tag{13}$$

Similarly, let $t \in f\mathscr{A}e$ be an arbitrary element. Because $ta \in f\mathscr{A}e$, we have

$$\begin{aligned} 0 = -\gamma(ta) &= \gamma([a,t]) = \gamma(p_n(a,t,e,\ldots,e)) \\ &= p_n(\gamma(a),t,e,\ldots,e) = p_{n-1}([\gamma(a),t],e,\ldots,e) \\ &= f\gamma(a)ft - te\gamma(a)e. \end{aligned}$$

Thus,

$$f\gamma(a)f \cdot t = t \cdot e\gamma(a)e \quad \text{for all } t \in f\mathscr{A}e. \tag{14}$$

From (13) and (14) we conclude that $\gamma(a) = e\gamma(a)e + f\gamma(a)f \in Z(\mathscr{A})$ for all $a \in e\mathscr{A}e$. Similarly one can prove, that $\gamma(b) = e\gamma(b)e + f\gamma(b)f \in Z(\mathscr{A})$ for all $b \in f\mathscr{A}f$. Therefore γ maps into the center of $\mathscr{A}$. From $\gamma(\mathscr{A}) \subseteq Z(\mathscr{A})$, it follows

$$\gamma(p_n(x_1,x_2,\ldots,x_n)) = p_n(\gamma(x_1),x_2,\ldots,x_n) = p_{n-1}([\gamma(x_1),x_2],\ldots,x_n) = 0$$

for all $x_1,x_2,\ldots,x_n \in \mathscr{A}$. Therefore $\gamma(p_n(\mathscr{A},\ldots,\mathscr{A})) = 0$ and the proposition is proved. $\square$

The main result of the paper states:

THEOREM 2.3. *Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e satisfying (1). Let us assume that*

(i) $Z(e\mathscr{A}e) = Z(\mathscr{A})e$,

(ii) $Z(f\mathscr{A}f) = Z(\mathscr{A})f$.

Then any generalized Lie n-derivation $F : \mathscr{A} \to \mathscr{A}$ is of the form $F(x) = \lambda x + \Delta(x)$ for all $x \in \mathscr{A}$, where $\lambda \in Z(\mathscr{A})$ and $\Delta : \mathscr{A} \to \mathscr{A}$ is a Lie n-derivation.

Proof. Let $F : \mathscr{A} \to \mathscr{A}$ be a generalized Lie n-derivation with an associated Lie n-derivation D. According to the definition

$$F(p_n(x_1,x_2,\ldots,x_n)) = p_n(F(x_1),x_2,\ldots,x_n) + \sum_{i=2}^{n} p_n(x_1,\ldots,D(x_i),\ldots,x_n),$$

$$D(p_n(x_1,x_2,\ldots,x_n)) = p_n(D(x_1),x_2,\ldots,x_n) + \sum_{i=2}^{n} p_n(x_1,\ldots,D(x_i),\ldots,x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. Let us denote $H = F - D$. If we subtract upper equalities we see that a linear map H satisfies

$$H(p_n(x_1, x_2, \ldots, x_n)) = p_n(H(x_1), x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. Since all assumptions from Proposition 2.2 are fulfilled there exists $\lambda \in Z(\mathscr{A})$ and a linear map $\gamma\colon \mathscr{A} \to Z(\mathscr{A})$ that vanishes on $p_n(\mathscr{A}, \ldots, \mathscr{A})$ such that $H(x) = \lambda x + \gamma(x)$ for all $x \in \mathscr{A}$. Then $F(x) = \lambda x + D(x) + \gamma(x)$ for all $x \in \mathscr{A}$. Note that γ is a Lie n-derivation and hence $\Delta = D + \gamma$ is also a Lie n-derivation. Thus, we get the desired result $F(x) = \lambda x + \Delta(x)$ for $x \in \mathscr{A}$. $\square$

In general, there exist generalized Lie n-derivations, which are not the sums of a multiplier $x \mapsto \lambda x$, $\lambda \in Z(\mathscr{A})$, and a Lie n-derivation. We will construct such an example on a triangular algebra, which does not satisfy assumption (ii) of Theorem 2.3. Thus, this example justifies the assumptions of our main theorem.

EXAMPLE. Let $R[X]$ be the algebra of all polynomials with coefficients from a commutative ring R with unity. Let $A = R[X]/(X^2)$. We construct a triangular algebra

$$\mathscr{A} = \begin{pmatrix} R & A \\ & A \end{pmatrix} = \left\{ \begin{pmatrix} r_0 & t_0 + t_1 X \\ & s_0 + s_1 X \end{pmatrix} ;\ r, t_0, t_1, s_0, s_1 \in R \right\}.$$

Let

$$e = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix}.$$

Note that $Z(\mathscr{A}) = R\mathbf{1}$. Since A is commutative it follows that $Z(f\mathscr{A}f) = f\mathscr{A}f \neq Rf = Z(\mathscr{A})f$. Hence $\mathscr{A}$ does not satisfy assumption (ii) of Theorem 2.3.

Let us define a linear map $F : \mathscr{A} \to \mathscr{A}$ as

$$F : \begin{pmatrix} r_0 & t_0 + t_1 X \\ & s_0 + s_1 X \end{pmatrix} \mapsto \begin{pmatrix} 0 & t_0 X \\ & (s_0 - r_0) X \end{pmatrix}$$

for all $r, t_0, t_1, s_0, s_1 \in R$. Let

$$x = \begin{pmatrix} r_0 & t_0 + t_1 X \\ & s_0 + s_1 X \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} r_0' & t_0' + t_1' X \\ & s_0' + s_1' X \end{pmatrix}$$

be an arbitrary elements of $\mathscr{A}$. By a straightforward computation one can prove that

$$F([x,y]) = [F(x), y] = \begin{pmatrix} 0 & (r_0 t_0' + t_0 s_0' - r_0' t_0 - t_0' s_0) X \\ & 0 \end{pmatrix}.$$

Hence, F is a generalized Lie 2-derivation with the associated map $D = 0$. Now, assume that there exists $\lambda \in Z(\mathscr{A}) = R\mathbf{1}$ and a Lie derivation $\Delta : \mathscr{A} \to \mathscr{A}$ such that $F(x) = \lambda x + \Delta(x)$ for all $x \in \mathscr{A}$. Set

$$m = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}.$$

Then

$$\Delta(e)=F(e)-\lambda e=\begin{pmatrix}-\lambda & 0\\ & -X\end{pmatrix},\quad \Delta(m)=F(m)-\lambda m=\begin{pmatrix}0 & -\lambda+X\\ & 0\end{pmatrix}.$$

Since Δ is a Lie derivation, we get

$$\Delta([e,m])-[\Delta(e),m]-[e,\Delta(m)]=\begin{pmatrix}0 & \lambda-X\\ & 0\end{pmatrix}\neq 0,$$

a contradiction. Thus, F is not the sum of a multiplier and a Lie derivation. Note, that F is actually a generalized Lie n-derivation for all $n\geqslant 2$.

3. Applications

In this section we apply Theorem 2.3 to the classical examples of unital algebras: triangular algebras (upper triangular matrix algebras, nest algebras), matrix algebras and algebras of bounded linear operators. Our main result reduces the description of a generalized Lie n-derivation to the description of a Lie n-derivation. Lie n-derivations of triangular rings were considered in [4] and on unital algebras with nontrivial idempotents in [13, 14].

Matrix algebras

Let $\mathscr{A}=M_s(A)$, $s\geqslant 2$, be a matrix algebra, where A is a unital algebra. Let $\{e_{ij}|i,j=1,2,\ldots,s\}$ be the system of matrix units of $\mathscr{A}$ and let 1 be the identity of $\mathscr{A}$. Let us denote the idempotent $e=e_{11}$ and $f=1-e$. In this case

$$\mathscr{A}=\begin{pmatrix}A & M_{1\times(s-1)}(A)\\ M_{(s-1)\times 1}(A) & M_{s-1}(A)\end{pmatrix}$$

satisfies (1). Note that the subalgebra $e\mathscr{A}e$ is isomorphic to A and $f\mathscr{A}f$ is isomorphic to the matrix algebra $M_{n-1}(A)$. Clearly, $(e\mathscr{A}e,f\mathscr{A}f)$-bimodule $e\mathscr{A}f\cong M_{1\times(s-1)}(A)$ is faithful as a left $e\mathscr{A}e$-module and as a right $f\mathscr{A}f$-module. Since $Z(M_s(A))=Z(A)\mathbf{1}$, it holds $Z(\mathscr{A})e=Z(A)e=Z(e\mathscr{A}e)$ and $Z(\mathscr{A})f=Z(A)f=Z(f\mathscr{A}f)$. Theorem 2.3 implies that any generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form $F(x)=\lambda x+\Delta(x)$ for all $x\in\mathscr{A}$, where $\lambda\in Z(A)$ and Δ is a Lie n-derivation. Wang [13, Corollary 3.1] proved that every Lie n-derivation Δ of $M_s(A)$, $s\geqslant 3$, has a standard form $\Delta=d+\gamma$, where $d:\mathscr{A}\to\mathscr{A}$ is a derivation and $\gamma:\mathscr{A}\to Z(A)\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{A},\ldots,\mathscr{A})$. We get:

COROLLARY 3.1. *Let $\mathscr{A}=M_s(A)$, $s\geqslant 3$, where A is a unital $(n-1)$-torsion free algebra. Then every generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form $F(x)=\lambda x+d(x)+\gamma(x)$, where $\lambda\in Z(A)$, $d:\mathscr{A}\to\mathscr{A}$ is a derivation and $\gamma:\mathscr{A}\to Z(A)\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{A},\ldots,\mathscr{A})$.*

Let us mention, that on the matrix algebra $M_2(A)$ the problem of description of Lie n-derivations remains open. From [3, Corollary 5.7] it follows that any Lie 2-derivation of $M_2(A)$ is of a standard form. However, Corollary 3.1 holds for $M_2(A)$, where A is a prime algebra since in this case any Lie n-derivation of $M_2(A)$ has a standard form (see [13, Corollary 3.2]).

Triangular algebras

In case $\mathscr{A}$ is a unital algebra with a nontrivial idempotent e such that $f\mathscr{A}e=\{0\}$, and that the bimodule $e\mathscr{A}f$ is faithful as a left $e\mathscr{A}e$-module and also as a right $f\mathscr{A}f$-module, the algebra $\mathscr{A}$ is a triangular algebra. When studying Lie n-derivations on triangular algebras the following assumption is very useful

$$x\in\mathscr{A},\quad [x,\mathscr{A}]\in Z(\mathscr{A})\Longrightarrow x\in Z(\mathscr{A}). \tag{15}$$

Note that (15) is equivalent to the requirement that $[[x,\mathscr{A}],\mathscr{A}]=0$ implies $[x,\mathscr{A}]=0$ and it is equivalent to the condition that there do not exist nonzero central inner derivations of $\mathscr{A}$. Some examples of algebras satisfying (15) are (see [3, page 143]): any commutative algebra, any prime algebra (the algebra of bounded linear operators $B(X)$ over a complex Banach space X), any triangular algebra and any unital algebra $\mathscr{A}$ with a nontrivial idempotent e and property (1).

Using Theorem 2.3 and [4, Theorem 5.9] we get:

COROLLARY 3.2. *Let $\mathscr{A}=e\mathscr{A}e+e\mathscr{A}f+f\mathscr{A}f$ be a $(n-1)$-torsion free triangular algebra such that:*

(i) $Z(e\mathscr{A}e)=Z(\mathscr{A})e$,

(ii) $Z(f\mathscr{A}f)=Z(\mathscr{A})f$,

(iii) $e\mathscr{A}e$ or $f\mathscr{A}f$ satisfies (15).

Then every generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form $F(x)=\lambda x+d(x)+\gamma(x)$ for all $x\in\mathscr{A}$, where $\lambda\in Z(\mathscr{A})$, $d:\mathscr{A}\to\mathscr{A}$ is a derivation and $\gamma:\mathscr{A}\to Z(\mathscr{A})$ is a linear map that vanishes on $p_n(\mathscr{A},\dots,\mathscr{A})$.

Let $\mathscr{A}=T_s(A)$, $s\geqslant 2$, be an upper triangular matrix algebra over a unital algebra A. If we choose the idempotent $e=e_{11}$ then $\mathscr{A}$ can be represented as a triangular algebra of the form

$$\mathscr{A}=\begin{pmatrix} A & M_{1\times(s-1)}(A)\\ & T_{s-1}(A)\end{pmatrix}.$$

Since $Z(T_s(A))=Z(A)\mathbf{1}$, we have $Z(e\mathscr{A}e)=Z(A)e$, $Z(f\mathscr{A}f)=Z(A)f$ and assumptions $(i),(ii)$ of Corollary 3.2 hold true. If $s\geqslant 3$, then $f\mathscr{A}f\cong T_{s-1}(A)$ is a triangular algebra and satisfies (15) and so assumption (iii) also holds true. Thus, Corollary 3.2 implies:

COROLLARY 3.3. *Let $\mathscr{A}=T_s(A)$, $s\geqslant 3$, where A is a unital $(n-1)$-torsion free algebra. Then every generalized Lie n-derivation $F:\mathscr{A}\to\mathscr{A}$ is of the form $F(x)=\lambda x+d(x)+\gamma(x)$ for all $x\in\mathscr{A}$, where $\lambda\in Z(A)$, $d:\mathscr{A}\to\mathscr{A}$ is a derivation and $\gamma:\mathscr{A}\to Z(A)\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{A},\dots,\mathscr{A})$.*

It should be mentioned that Corollary 3.3 does not hold if $s=2$. However, Corollary 3.3 holds for $s=2$ if we assume that A satisfies (15).

A nest is a chain $\mathscr{N}$ of closed subspaces of a complex Hilbert space H containing $\{0\}$ and H which is closed under arbitrary intersections and closed linear span. The nest algebra associated to $\mathscr{N}$ is the algebra

$$\mathscr{T}(\mathscr{N})=\{T\in\mathscr{B}(H)\,|\,T(N)\subseteq N \text{ for all } N\in\mathscr{N}\}.$$

Let us assume that $\mathscr{N}$ is a nontrivial nest and $N\in\mathscr{N}\setminus\{0,H\}$. Let $e\in T(\mathscr{N})$ denote the orthogonal projection on the subspace N. Then $T(\mathscr{N})$ can be represented as a triangular algebra $T(\mathscr{N})=eT(\mathscr{N})e+eT(\mathscr{N})f+fT(\mathscr{N})f$, where $eT(\mathscr{N})e$ and $fT(\mathscr{N})f$ are both nest algebras. Since the center of each nest algebra coincides with $\mathbb{C}\mathbf{1}$, it follows that assumptions $(i),(ii)$ of Corollary 3.2 hold true. Further, assumption (iii) holds true as well and hence, Corollary 3.2 yields:

COROLLARY 3.4. *Let $\mathscr{N}$ be a nontrivial nest of a complex Hilbert space H,* $\dim H\geqslant 2$. *Then every generalized Lie n-derivation of a nest algebra $\mathscr{T}(\mathscr{N})$ is of the form $F(x)=\lambda x+d(x)+\gamma(x)$ for all $x\in\mathscr{T}(\mathscr{N})$, where $\lambda\in\mathbb{C}$, $d:\mathscr{T}(\mathscr{N})\to\mathscr{T}(\mathscr{N})$ is a derivation and $\gamma:\mathscr{T}(\mathscr{N})\to\mathbb{C}\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{T}(\mathscr{N}),\ldots,\mathscr{T}(\mathscr{N}))$.*

Let us mention, that Corollary 3.4 holds true also if $\mathscr{N}=\{0,H\}$ is a trivial nest. In this case $T(\mathscr{N})=B(H)$ is the algebra of all bounded linear operators on H (see Corollary 3.6).

Algebras of all bounded linear operators

Let X be a Banach space over $\mathbb{C}$ of dimension greater than 1. By $\mathscr{B}=B(X)$ we denote the algebra of all bounded linear operators on X. $\mathscr{B}$ contains a nontrivial idempotent e and hence can be presented in the form $\mathscr{B}=e\mathscr{B}e+e\mathscr{B}f+f\mathscr{B}e+f\mathscr{B}f$. Since $\mathscr{B}$ is a prime algebra, $\mathscr{B}$ satisfies (1). Note that $e\mathscr{B}e$, $f\mathscr{B}f$ are algebras of all bounded linear operators and all $\mathscr{B}$, $e\mathscr{B}e$, $f\mathscr{B}f$ are central algebras over $\mathbb{C}$. Therefore, $eZ(\mathscr{B})e=Z(e\mathscr{B}e)=\mathbb{C}e$ and $fZ(\mathscr{B})f=Z(f\mathscr{B}f)=\mathbb{C}f$. Hence $\mathscr{B}$ meets assumptions of Theorem 2.3 and we have:

COROLLARY 3.5. *Let X be a complex Banach space over $\mathbb{C}$,* $\dim X\geqslant 2$. *Then every generalized Lie n-derivation F of $\mathscr{B}(X)$ is of the form $F(x)=\lambda x+\Delta(x)$ for all $x\in\mathscr{B}(X)$, where $\lambda\in\mathbb{C}$ and $\Delta:\mathscr{B}(X)\to\mathscr{B}(X)$ is a Lie n-derivation.*

What can be said about Lie n-derivations of the algebra $\mathscr{B}$? We know that $\mathscr{B}$ is a prime algebra and $\mathscr{B}$, $e\mathscr{B}e$, $f\mathscr{B}f$ are central algebras over $\mathbb{C}$. Thus, the assumptions of the Wang's result [13, Theorem 3.1] are fulfilled. Hence any Lie n-derivation $\Delta:\mathscr{B}\to\mathscr{B}$ has the standard form $\Delta=d+\gamma$, where d is a derivation of $\mathscr{B}$ and $\gamma:\mathscr{B}\to\mathbb{C}\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{B},\ldots,\mathscr{B})$. This fact and Corollary 3.5 yields:

COROLLARY 3.6. *Let X be a complex Banach space over $\mathbb{C}$,* $\dim X\geqslant 2$. *Then every generalized Lie n-derivation $F:\mathscr{B}(X)\to\mathscr{B}(X)$ is of the form $F(x)=\lambda x+$*

$d(x)+\gamma(x)$ for all $x \in \mathscr{B}(X)$, where $\lambda \in \mathbb{C}$, d is a derivation of $\mathscr{B}(X)$ and $\gamma : \mathscr{B}(X) \to \mathbb{C}\mathbf{1}$ is a linear map that vanishes on $p_n(\mathscr{B}(X), \ldots, \mathscr{B}(X))$.

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