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# POINTWISE–GENERALIZED–INVERSES OF LINEAR MAPS BETWEEN C$^*$–ALGEBRAS AND JB$^*$–TRIPLES

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*Abstract.* We study pointwise-generalized-inverses of linear maps between C$^*$-algebras. Let $\Phi$ and $\Psi$ be linear maps between complex Banach algebras $A$ and $B$. We say that $\Psi$ is a pointwise-generalized-inverse of $\Phi$ if $\Phi(aba) = \Phi(a)\Psi(b)\Phi(a)$, for every $a, b \in A$. The pair $(\Phi,\Psi)$ is Jordan-triple multiplicative if $\Phi$ is a pointwise-generalized-inverse of $\Psi$ and the latter is a pointwise-generalized-inverse of $\Phi$. We study the basic properties of this maps in connection with Jordan homomorphism, triple homomorphisms and strongly preservers. We also determine conditions to guarantee the automatic continuity of the pointwise-generalized-inverse of continuous operator between C$^*$-algebras. An appropriate generalization is introduced in the setting of JB$^*$-triples.

## 1. Introduction

Let $\Delta : A \to B$ be a mapping between two Banach algebras. Accordingly to the standard literature (see [22, 23, 26] and [27]) we shall say that $\Delta$ is a *Jordan triple map* (respectively, *Jordan triple product homomorphism* or a *Jordan triple multiplicative* mapping) if the identity

$$\Delta(abc+cba) = \Delta(a)\Delta(b)\Delta(c)+\Delta(c)\Delta(b)\Delta(a)$$

(respectively, $\Delta(aba) = \Delta(a)\Delta(b)\Delta(a)$) holds for every $a, b, c \in A$. For a linear map $T : A \to B$, it is easy to see that $T$ is a Jordan triple map if, and only if, it is a Jordan triple product homomorphism. In [27], L. Molnar gives a complete description of those Jordan triple multiplicative bijections $\Phi$ between the self-adjoint parts of two von Neumann algebras $M$ and $N$. F. Lu studies in [23] bijective maps from a standard operator algebra into a $\mathbb{Q}$-algebra which are generalizations of Jordan triple multiplicative maps.

In papers [22, 23, 26, 27] the mappings are not assumed to be linear, but are shown to be so. In this paper we introduce a new point of view by considering and studying pairs of linear maps which are Jordan triple multiplicative. Henceforth let $A$ and $B$ denote two complex Banach algebras.

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DEFINITION 1. Let $\Phi, \Psi : A \to B$ be linear maps. We shall say that $\Psi$ is a pointwise-generalized-inverse (pg-inverse for short) of $\Phi$ if the identity

$$\Phi(aba) = \Phi(a)\Psi(b)\Phi(a),$$

holds for all $a,b \in A$. If in addition $\Phi$ also is a pointwise-generalized-inverse of $\Psi$, we shall say that $\Psi$ is a normalized-pointwise-generalized-inverse (normalized-pg-inverse for short) of $\Phi$. In this case, we shall simply say that $(\Phi,\Psi)$ is Jordan-triple multiplicative.

Let us observe that, in the linear setting, $\Psi : A \to B$ is a pg-inverse of $\Phi$ if and only if

$$\Phi(abc + cba) = \Phi(a)\Psi(b)\Phi(c) + \Phi(c)\Psi(b)\Phi(a),$$

for all $a,b,c \in A$.

Every Jordan homomorphism (in particular, every homomorphism and every anti-homomorphism) $\pi : A \to B$ admits a pg-inverse. Actually, the couple $(\pi,\pi)$ is Jordan-triple multiplicative.

Pairs of linear maps satisfying certain properties have been previously studied in functional analysis and algebra. For example, centralizers of C$^*$-algebras [8], derivations on Banach-Jordan pairs [12], and structural transformations [25].

An element $a$ in an associative ring $\mathscr{R}$ is called *regular* or *von Neumann regular* if it admits a *generalized inverse* $b$ in $\mathscr{R}$ satisfying $aba = a$. The element $b$ is not, in general, unique. Under these hypothesis $ab$ and $ba$ are idempotents with $(ab)a = a(ba) = a$. If the identities $aba = a$ and $bab = b$ hold we say that $b$ is a *normalized generalized inverse* of $a$. An element $a$ may admit many different normalized generalized inverses. However, every regular element $a$ in a C$^*$-algebra $A$ admits a unique *Moore-Penrose inverse* that is, a normalized generalized inverse $b$ such that $ab$ and $ba$ are projections (i.e. self-adjoint idempotents) in $A$ (see [15, Theorems 5 and 6]). The unique Moore-Penrose inverse of a regular element $a$ will be denoted by $a^\dagger$.

A linear map between C$^*$-algebras admitting a pg-inverse is a *weak preserver*, that is, maps regular elements to regular elements (see Lemma 1). However, we shall show later the existence of linear maps between C$^*$-algebras preserving regular elements but not admitting a pg-inverse (see Example 1). Being a linear weak preserver between C$^*$-algebras is not a completely determining condition, actually, for an infinite-dimensional complex separable Hilbert space $H$, a bijective continuous unital linear map preserving generalized invertibility in both directions $\Phi : B(H) \to B(H)$ leaves invariant the ideal of all compact operators, and the induced linear map on the Calkin algebra is either an automorphism or an antiautomorphism (see [24]).

In Proposition 2 we show that a linear map $\Phi : A \to B$ between complex Banach algebras with $A$ unital, admits a normalized-pg-inverse if and only if one of the following statements holds:

$(b)$ There exists a Jordan homomorphism $T : A \to B$ such that $\Phi = R_{\Phi(1)} \circ T$ and $\Phi(1)B = T(1)B$;

$(c)$ There exists a Jordan homomorphism $S : A \to B$ such that $\Phi = L_{\Phi(1)} \circ S$ and $B\Phi(1) = BS(1)$.

A similar conclusion remains true for a pair of bounded linear maps between general $C^*$-algebras which are Jordan-triple multiplicative (see Corollary 1).

A linear map $\Phi$ between $C^*$-algebras satisfying that $\Phi(a^\dagger) = \Phi(a)^\dagger$ for every regular element $a$ in the domain is called a strongly preserver. Strongly preservers between $C^*$-algebras and subsequent generalizations to $JB^*$-triples have been studied in [5, 6, 7]. Following the conclusions of the above paragraph we can easily find a bounded linear map between $C^*$-algebras admitting a normalized-pg-inverse which is not a strongly preserver. In this setting, we shall show in Theorem 1 that for each pair of linear maps between $C^*$-algebras $\Phi,\Psi : A \to B$ such that $(\Phi,\Psi)$ is Jordan-triple multiplicative, the following statements are equivalent:

$(a)$ $\Phi$ and $\Psi$ are contractive;

$(b)$ $\Psi(a) = \Phi(a^*)^*$, for every $a \in A$;

$(c)$ $\Phi$ and $\Psi$ are triple homomorphisms.

When $A$ is unital the above conditions are equivalent to the following:

$(d)$ $\Phi$ and $\Psi$ are strongly preservers,

(see [6, Theorem 3.5]). As a consequence, we prove that every contractive Jordan homomorphism between $C^*$-algebras or between $JB^*$-algebras is a Jordan $^*$-homomorphism (cf. Corollaries 2 and 4).

Let $\Phi,\Psi : A \to B$ be linear maps between complex Banach algebras. If $A$ is unital and $(\Phi,\Psi)$ is Jordan-triple multiplicative, then $\Phi$ is norm continuous if and only if $\Psi$ is norm continuous (cf. Lemma 1). In the non-unital setting this conclusion becomes a difficult question. In section 3, we explore this problem by showing that if $\Phi,\Psi : c_0 \to c_0$ are linear maps such that $\Phi$ is continuous and $(\Phi,\Psi)$ is Jordan-triple multiplicative, then $\Psi$ is continuous (see Proposition 3). In the non-commutative setting, we prove that if $\Phi,\Psi : K(H_1) \to K(H_2)$ are linear maps such that $\Phi$ is continuous and $(\Phi,\Psi)$ is Jordan-triple multiplicative, then $\Phi$ admits a continuous normalized-pg-inverse (see Theorem 2).

In the last section we extend the notion of being pg-invertible to the setting of $JB^*$-triples.

### 1.1. Preliminaries and background

We gather some basic facts, definitions, and references in this subsection. We recall that a *JB*-triple* is a complex Jordan triple system $(E,\{.,.,.\})$ which is also a Banach space satisfying the following axioms:

$(a)$ The map $L(x,x)$ is an hermitian operator with non-negative spectrum for all $x \in E$.

$(b)$ $\|\{x,x,x\}\| = \|x\|^3$ for all $x \in E$.

where $L(x,y)(z) := \{x,y,z\}$, for all $x,y,z$ in $E$ (see [19] and [9]).

The attractive of this definition relies, among other holomorphic properties, on the fact that every C$^*$-algebra is a JB$^*$-triple with respect to

$$\{x,y,z\} := 2^{-1}(xy^*z + zy^*x).$$

The Banach space $B(H,K)$ of all bounded linear operators between two complex Hilbert spaces $H,K$ is also an example of a JB$^*$-triple with respect to the triple product given above, and every JB$^*$-algebra is a JB$^*$-triple with triple product

$$\{a,b,c\} := (a\circ b^*)\circ c + (c\circ b^*)\circ a - (a\circ c)\circ b^*.$$

In a clear analogy with von Neumann algebras, a JB$^*$-triple which is also a dual Banach space is called a *JBW$^*$-triple*. Every JBW$^*$-triple admits a (unique) isometric predual and its triple product is separately weak$^*$ continuous [2]. The second dual of a JB$^*$-triple $E$ is a JBW$^*$-triple with a product extending the product of $E$ [10].

Projections are frequently applied to produce approximation and spectral resolutions of hermitian elements in von Neumann algebras. In the wider setting of JBW$^*$-triple this role is played by tripotents. We recall that an element $e$ in a JB$^*$-triple $E$ is called a *tripotent* if $\{e,e,e\} = e$. Each tripotent $e$ in $E$ produces a *Peirce decomposition* of $E$ in the form

$$E = E_2(e)\oplus E_1(e)\oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$ (compare [9, §4.2.2]). The projection of $E$ onto $E_i(e)$ is denoted by $P_i(e)$.

It is known that the Peirce space $E_2(e)$ is a JB$^*$-algebra with product $x\circ_e y := \{x,e,y\}$ and involution $x^{\sharp_e} := \{e,x,e\}$.

For additional details on JB$^*$-algebras and JB$^*$-triples the reader is referred to the encyclopedic monograph [9].

For the purposes of this paper, we also consider von Neumann regular elements in the wider setting of JB$^*$-triples (see subsection 1.1 for the concrete definitions). Let $a$ be an element in a JB$^*$-triple $E$. Following the standard notation in [11], [20] and [4] we shall say that $a$ is *von Neumann regular* if $a\in Q(a)(E) = \{a,E,a\}$. It is known that $a$ is von Neumann regular if, and only if, $a$ is *strongly von Neumann regular* (i.e. $a\in Q(a)^2(E)$) if, and only if, there exists (a unique) $b\in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a),Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$ if, and only if, $Q(a)(E)$ is norm-closed (compare [11, Theorem 1], [20, Lemma 3.2, Corollary 3.4, Proposition 3.5, Lemma 4.1], [4, Theorem 2.3, Corollary 2.4]). The unique element $b$ given above is denoted by $a^\wedge$. The set of all von Neumann regular elements in $E$ is denoted by $E^\wedge$.

Let us recall that an element $a$ in a unital Jordan Banach algebra $J$ is called invertible whenever there exists $b\in J$ satisfying $a\circ b = 1$ and $a^2\circ b = a$. Under the above circumstances, the element $b$ is unique and will be denoted by $a^{-1}$. The symbol $J^{-1} = \mathrm{inv}(J)$ will denote the set of all invertible elements in $J$. It is well known in Jordan theory that $a$ is invertible if, and only if, the mapping $x\mapsto U_a(x) := 2(a\circ x)\circ a - a^2\circ x$ is invertible in $L(J)$, and in that case $U_a^{-1} = U_{a^{-1}}$ (see, for example [9, §4.1.1]).

The notion of invertibility in the Jordan setting provides an adequate point of view to study regularity. More concretely, it is shown in [20], [21, Lemma 3.2] and [4, Proposition 2.2 and proof of Theorem 3.4] that an element $a$ in a JB$^*$-triple $E$ is von Neumann regular if and only if there exists a tripotent $v \in E$ such that $a$ is a positive and invertible element in the JB$^*$-algebra $E_2(e)$. It is further known that $a^\wedge$ is precisely the (Jordan) inverse of $a$ in $E_2(v)$.

## 2. Pointwise-generalized-inverses

Our first lemma gathers some basic properties of pg-inverses.

LEMMA 1. *Let $\Phi : A \to B$ be a linear map between complex Banach algebras admitting a pg-inverse $\Psi$. Then the following statements hold:*

$(a)$ *$\Phi$ maps regular elements in $A$ to regular elements in $B$, that is, $\Phi$ is a weak regular preserver. More concretely, if $b$ is a generalized inverse of $a$ then $\Psi(b)$ is a generalized inverse of $\Phi(a)$;*

$(b)$ *If $A$ is unital and $(\Phi,\Psi)$ is Jordan-triple multiplicative, then* $\ker(\Phi) = \ker(\Psi)$;

$(c)$ *If $A$ is unital and $(\Phi,\Psi)$ is Jordan-triple multiplicative, then $\Phi$ is norm continuous if and only if $\Psi$ is norm continuous;*

$(d)$ *If $A$ and $B$ are unital and $\Phi(1) \in B^{-1}$, then $\Psi = R_{\Phi(1)^{-1}} \circ L_{\Phi(1)^{-1}} \circ \Phi$ is the unique pg-inverse of* $\Phi$;

$(e)$ *Let $\Phi_1 : C \to A$ and $\Phi_2 : B \to C$ be linear maps admitting a pg-inverse, where $C$ is a Banach algebra, then $\Phi_2\Phi$ and $\Phi\Phi_1$ admit a pg-inverse too. In particular, if $A$ and $B$ are $C^*$-algebras, then the maps $x \mapsto \Phi(x)^*$, $x \mapsto \Phi(x^*)$, and $x \mapsto \Phi(x^*)^*$ admit pg-inverses.*

*Proof.* $(a)$ Suppose that $a$ is a regular element in $A$ and let $b$ be a generalized inverse of $a$. Then $\Phi(a) = \Phi(aba) = \Phi(a)\Psi(b)\Phi(a)$, and hence $\Phi(a)$ is regular too.

$(b)$ The conclusion follows from the identities $\Phi(x) = \Phi(1)\Psi(x)\Phi(1)$ and $\Psi(x) = \Psi(1)\Phi(x)\Psi(1)$ $(x \in A)$. Statement $(c)$ can be proved from the same identities.

$(d)$ Suppose $A$ and $B$ are unital and $\Phi(1) \in B^{-1}$. Let $\Psi : A \to B$ be an arbitrary pg-inverse of $\Phi$. Since the identity $\Phi(b) = \Phi(1)\Psi(b)\Phi(1)$ holds for every $b \in A$, we deduce that $\Psi = R_{\Phi(1)^{-1}} \circ L_{\Phi(1)^{-1}} \circ \Phi$.

$(e)$ Suppose that $\Phi_1 : C \to A$ admits a pg-inverse $\Psi_1$. Then

$$\Phi\Phi_1(aba) = \Phi(\Phi_1(a)\Psi_1(b)\Phi_1(a)) = \Phi(\Phi_1)(a)\Psi(\Psi_1(b))\Phi(\Phi_1)(a),$$

for all $a,b \in C$. The rest is clear. $\square$

In the hypothesis of the above lemma, let us observe that a pg-inverse of a continuous linear operator $\Phi : A \to B$ need not be, in general, continuous. Take, for example, two infinite dimensional Banach algebras $A$ and $B$, a continuous homomorphism $\pi : A \to B$ and an unbounded linear mapping $F : A \to B$. We define $\Phi, \Psi : A \oplus^\infty A \to B \oplus^\infty B$, $\Phi(a_1,a_2) = (\pi(a_1),0)$ and $\Psi(\pi(a_1),F(a_2))$. Clearly, $\Psi$ is unbounded and $\Phi(a_1,a_2)\Psi(b_1,b_2)\Phi(a_1,a_2) = \Phi((a_1,a_2)(b_1,b_2)(a_1,a_2))$.

We have just seen that every linear map admitting a pg-inverse is a weak regular preserver. The Example 1 below shows that the reciprocal implication is not always true.

The following technical lemma isolates an useful property of linear maps admitting a pg-inverse.

LEMMA 2. *Let $\Phi : A \to B$ be a linear map between complex Banach algebras, where $A$ is unital. Suppose that $\Psi : A \to B$ is a pg-inverse of $\Phi$. Then we have*

$$\Phi = L_{(\Phi(1)\Psi(1))} \circ \Phi = R_{(\Psi(1)\Phi(1))} \circ \Phi.$$

*Proof.* Since $\Phi(1) = \Phi(1)\Psi(1)\Phi(1)$, we deduce that $\Psi(1)$ is a generalized inverse of $\Phi(1)$, and consequently the elements $\Phi(1)\Psi(1)$ and $\Psi(1)\Phi(1)$ are idempotents. For each $x \in A$ we have

$$2\Phi(x) = \Phi(11x + x11) = \Phi(1)\Psi(1)\Phi(x) + \Phi(x)\Psi(1)\Phi(1).$$

Since $\Phi(1)\Psi(1)$ and $\Psi(1)\Phi(1)$ are idempotents we deduce that $\Phi(1)\Psi(1)\Phi(x) = \Phi(1)\Psi(1)\Phi(x)\Psi(1)\Phi(1) = \Phi(x)\Psi(1)\Phi(1)$, and

$$\Phi(x) = (\Phi(1)\Psi(1))\Phi(x) = \Phi(x)(\Psi(1)\Phi(1)). \quad \square$$

It is not obvious that a linear map admitting a pg-inverse also admits a normalized-pg-inverse. We can conclude now that if the domain is a unital Banach algebra then the desired statement is always true.

PROPOSITION 1. Suppose that $A$ is a unital Banach algebra. Let $\Phi : A \to B$ a linear map admitting a pointwise-generalized-inverse. Then $\Phi$ has a normalized-pg-inverse. More concretely, if $\Psi$ is pg-inverse of $\Phi$, then the mapping $\Theta = L_{\Psi(1)} \circ R_{\Psi(1)} \circ \Phi$ is a normalized-pg-inverse of $\Phi$.

*Proof.* Since $\Psi$ is a pg-inverse of $\Phi$, we deduce that $\Psi(1)$ is a generalized inverse of $\Phi(1)$. We set $\Theta = L_{\Psi(1)} \circ R_{\Psi(1)} \circ \Phi$. By applying Lemma 2, we get

$$\begin{aligned}
\Theta(aba) &= \Psi(1)\Phi(aba)\Psi(1) = \Psi(1)\Phi(a)\Psi(b)\Phi(a)\Psi(1) \\
&= \Psi(1)\Big(\Phi(a)\Psi(1)\Phi(1)\Big)\Psi(b)\Big(\Phi(1)\Psi(1)\Phi(a)\Big)\Psi(1) \\
&= \Big(\Psi(1)\Phi(a)\Psi(1)\Big)\Big(\Phi(1)\Psi(b)\Phi(1)\Big)\Big(\Psi(1)\Phi(a)\Psi(1)\Big) = \Theta(a)\Phi(b)\Theta(a).
\end{aligned}$$

On the other hand, by Lemma 2 we also have

$$\begin{aligned}
\Phi(aba) &= \Phi(a)\Psi(b)\Phi(a) = \Big(\Phi(a)\Psi(1)\Phi(1)\Big)\Psi(b)\Big(\Phi(1)\Psi(1)\Phi(a)\Big) \\
&= \Phi(a)\Big(\Psi(1)\Big(\Phi(1)\Psi(b)\Phi(1)\Big)\Psi(1)\Big)\Phi(a) = \Phi(a)\Big(\Psi(1)\Phi(b)\Psi(1)\Big)\Phi(a) \\
&= \Phi(a)\Theta(b)\Phi(a). \quad \square
\end{aligned}$$

Let $A$ and $B$ be complex Banach algebras. We recall that a linear map $T: A \to B$ is called a Jordan homomorphism if $T(a^2) = T(a)^2$ for every $a \in A$, or equivalently, $T(a \circ b) = T(a) \circ T(b)$, where $\circ$ denotes the natural Jordan product defined by $x \circ y := \frac{1}{2}(xy + yx)$. For each $a$ in $A$ the mapping $U_a : A \to A$ is given by $U_a(x) := 2(a \circ x) \circ a - a^2 \circ x = axa$. It is well known that a Jordan homomorphism satisfies the identity $T(aba) = T(U_a(b)) = U_{T(a)}(T(b)) = T(a)T(b)T(a)$, for all $a, b \in A$.

We can now add some additional information to the statement in the above proposition. If $\Psi : A \to B$ is normalized-pg-inverse of a linear mapping $\Phi : A \to B$, by Proposition 1, $\Psi(1)$ is a generalized inverse of $\Phi(1)$, and we clearly have

$$\Psi(x) = \Psi(1)\Phi(x)\Psi(1),$$

for all $x \in A$.

LEMMA 3. *Let $\Phi, \Psi : A \to B$ be linear maps between Banach algebras, with $A$ unital. Suppose that $(\Phi, \Psi)$ is Jordan-triple multiplicative. Then the following statements hold:*

$(a)$ *The identities*

$$\Psi(1)\Phi(a) = \Psi(a)\Phi(1), \quad \Phi(a)\Psi(1) = \Phi(1)\Psi(a),$$

$$\Phi(a)\Psi(b) = \Phi(1)\Psi(a)\Phi(b)\Psi(1), \text{ and } \Psi(1)\Phi(a)\Psi(b)\Phi(1) = \Psi(a)\Phi(b),$$

*hold for all $a, b \in A$;*

$(b)$ *The linear maps $T = L_{\Psi(1)} \circ \Phi$ and $S = R_{\Psi(1)} \circ \Phi$ are Jordan homomorphisms satisfying:*

$$\Phi(a)\Psi(b) = S(a)S(b), \text{ and } \Psi(a)\Phi(b) = T(a)T(b),$$

*for all $a, b \in A$.*

*Proof.* $(a)$ We know from previous results that $\Phi(a) = \Phi(1)\Psi(a)\Phi(1)$, $\Psi(a) = \Psi(1)\Phi(a)\Psi(1)$, for all $a \in A$, and $\Phi(1)$ is a normalized generalized inverse of $\Psi(1)$. We conclude from Lemma 2 that

$$\Psi(1)\Phi(a) = \Psi(1)\Phi(1)\Psi(a)\Phi(1) = \Psi(a)\Phi(1),$$

and

$$\Phi(a)\Psi(1) = \Phi(1)\Psi(a)\Phi(1)\Psi(1) = \Phi(1)\Psi(a),$$

for all $a \in A$. Consequently,

$$\Phi(a)\Psi(b) = \Phi(1)\Psi(a)\Phi(1)\Psi(1)\Phi(b)\Psi(1) = \Phi(1)\Psi(a)\Phi(b)\Psi(1).$$

The remaining identity follows by symmetry.

$(b)$ With the notation above, $T(a)T(b) = \Psi(1)\Phi(a)\Psi(1)\Phi(b) = \Psi(a)\Phi(b)$, and consequently,

$$\begin{aligned}2T(a^2) &= 2\Psi(1)\Phi(a^2) = \Psi(1)\Phi(aa1+1aa)\\ &= \Psi(1)\Phi(a)\Psi(a)\Phi(1)+\Psi(1)\Phi(1)\Psi(a)\Phi(a) = 2\Psi(a)\Phi(a) = 2T(a)^2.\end{aligned}$$

The rest is left to the reader. $\square$

The previous properties now result in an equivalence.

PROPOSITION 2. Let $\Phi : A \to B$ be a linear map between complex Banach algebras with $A$ unital. Then the following statements are equivalent:

$(a)$ $\Phi$ admits a normalized-pg-inverse;

$(b)$ There exists a Jordan homomorphism $T : A \to B$ such that $\Phi = R_{\Phi(1)} \circ T$ and $\Phi(1)B = T(1)B$;

$(c)$ There exists a Jordan homomorphism $S : A \to B$ such that $\Phi = L_{\Phi(1)} \circ S$ and $B\Phi(1) = BS(1)$.

*Proof.* $(a) \Rightarrow (b)$ Suppose that $\Phi$ admits a normalized-pg-inverse $\Psi : A \to B$. By Lemma 3 the mapping $T = L_{\Phi(1)} \circ \Psi$ is a Jordan homomorphism and $R_{\Phi(1)} \circ T(a) = \Phi(1)\Psi(a)\Phi(1) = \Phi(a)$, or every $a \in A$. On the other hand, $T(1) = \Phi(1)\Psi(1)$ is an idempotent in $B$ and $T(1)\Phi(1) = \Phi(1)$, which implies that $T(1)B = \Phi(1)B$.

$(b) \Rightarrow (a)$ Let $T : A \to B$ be a Jordan homomorphism such that $\Phi = R_{\Phi(1)} \circ T$ and $\Phi(1)B = T(1)B$. Under these hypothesis, there exists $c \in B$ such that $T(1) = T(1)^2 = \Phi(1)c$. The element $T(1)$ is an idempotent in $B$ with $T(a) \circ T(1) = T(a)$, for every $a \in A$. Thus, $T(a) = T(1)T(a) = T(a)T(1) = T(1)T(a)T(1)$, for every $a$ in $A$. If we set $\Psi = L_c \circ T$, by applying Lemma 2, we obtain

$$\begin{aligned}\Phi(aba) &= T(aba)\Phi(1) = T(a)T(b)T(a)\Phi(1) = T(a)T(1)T(b)T(a)\Phi(1)\\ &= T(a)[\Phi(1)c]T(b)T(a)\Phi(1) = \Phi(a)\Psi(b)\Phi(a); \quad \forall\, a,b \in A.\end{aligned}$$

The implications $(a) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ follow by similar arguments. $\square$

EXAMPLE 1. [7, Remark 5.10] Let $H$ be an infinite dimensional complex Hilbert space, let $v,w$ be (maximal) partial isometries such that $v^*v = 1 = w^*w$ and $vv^* \perp ww^*$. We set $A = \mathbb{C} \oplus^\infty \mathbb{C}$, and consider the operator $T : A \to B(H)$ given by

$$T(\lambda,\mu) = \frac{\lambda}{2}(v+w) + \frac{\mu}{2}(v-w).$$

It is shown in [7, Remark 5.10] that $T$ maps extreme point of the closed unit ball of $A$ to extreme point of the closed unit ball of $B(H)$, but $T$ does not preserves Moore-Penrose inverses strongly, that is, $T(a^\dagger) \neq T(a)^\dagger$ for every Moore-Penrose invertible element $a \in A$.

Let us show that $T$ is a weak preserver, that is, $T$ maps regular elements to regular elements. It is easy to check that an element $a=(\lambda,\mu)\in A$ is regular if and only if it is Moore-Penrose invertible if and only if $|\lambda|+|\mu|\neq 0$ (i.e. $a\neq 0$), and in such a case $a^{\dagger}=(\lambda^{-1},0)$ if $\mu=0$, $a^{\dagger}=(0,\mu^{-1})$ if $\lambda=0$ and $a^{\dagger}=a^{-1}$ otherwise. Given $\lambda,\mu\in\mathbb{C}$ we have

$$\begin{aligned}T(a)^*T(a)&=\left(\frac{\overline{\lambda}}{2}(v+w)^*+\frac{\overline{\mu}}{2}(v-w)^*\right)\left(\frac{\lambda}{2}(v+w)+\frac{\mu}{2}(v-w)\right)\\&=\left(\frac{|\lambda|^2}{4}+\frac{|\mu|^2}{4}\right)(v^*v+w^*w)=\left(\frac{|\lambda|^2}{4}+\frac{|\mu|^2}{4}\right)1,\end{aligned}$$

which assures that $T(a)$ admits a Moore-Penrose inverse.

We shall finally show that $T$ does not admit a pg-inverse. Arguing by contradiction, we assume that $T$ admits a pg-inverse. Proposition 1 assures that $T$ admits a normalized-pg-inverse and Proposition 2$(c)$ implies the existence of a Jordan homomorphism $J:A\to B(H)$ such that $T(a)=T(1)J(a)$, for every $a\in A$. Having in mind that $T(1)=T(1,1)=v$, we have $T(\lambda,\mu)=vJ(\lambda,\mu)$, for every $\lambda,\mu\in\mathbb{C}$. Therefore $J(\lambda,\mu)=v^*vJ(\lambda,\mu)=v^*T(\lambda,\mu)$, for every $\lambda,\mu\in\mathbb{C}$, and thus

$$\begin{aligned}\frac{\lambda^2+\mu^2}{2}1&=v^*\left(\frac{\lambda^2+\mu^2}{2}v+\frac{\lambda^2-\mu^2}{2}w\right)=v^*T(\lambda^2,\mu^2)=v^*T((\lambda,\mu)^2)\\&=(v^*T(\lambda,\mu))(v^*T(\lambda,\mu))\\&=v^*\left(\frac{\lambda+\mu}{2}v+\frac{\lambda-\mu}{2}w\right)v^*\left(\frac{\lambda+\mu}{2}v+\frac{\lambda-\mu}{2}w\right)\\&=\frac{\lambda+\mu}{2}1\frac{\lambda+\mu}{2}1=\frac{(\lambda+\mu)^2}{4}1,\end{aligned}$$

for every $\lambda,\mu\in\mathbb{C}$, which is impossible.

It is known that we can find an infinite dimensional complex Banach algebra $A$ and an unbounded homomorphism $\pi:A\to\mathbb{C}$. Clearly $\pi$ admits a normalized-pg-inverse but it is not continuous. However, every homomorphism $\pi$ from an arbitrary complex Banach algebra $A$ into a C$^*$-algebra $B$ whose image is a $^*$-subalgebra of $B$ is automatically continuous (see [29, Theorem 4.1.20]).

In Proposition 2 we can relax the hypothesis of $A$ being unital at the cost of assuming the continuity of $\Phi$ and $\Psi$. Henceforth, the bidual of a Banach space $X$ will be denoted by $X^{**}$.

LEMMA 4. *Let $\Phi,\Psi:A\to B$ be continuous linear maps between $C^*$-algebras. Suppose that $\Psi$ is a (normalized-)pg-inverse of $\Phi$. Then $\Psi^{**}:A^{**}\to B^{**}$ is a (normalized-)pg-inverse of $\Phi^{**}$.*

*Proof.* The maps $\Phi^{**},\Psi^{**}:A^{**}\to B^{**}$ are weak$^*$-to-weak$^*$ continuous operators between von Neumann algebras. We recall that, by Sakai's theorem (see [30, Theorem 1.7.8]), the products of $A^{**}$ and $B^{**}$ are separately weak$^*$-continuous. Let us fix

$a,b,c \in A^{**}$. By Goldstine's theorem we can find three bounded nets $(a_\lambda)$, $(b_\mu)$ and $(c_\delta)$ in $A$ converging in the weak$^*$-topology of $A^{**}$ to $a,b$ and $c$, respectively. By hypothesis,

$$\Phi(a_\lambda b_\mu c_\delta + c_\delta a_\lambda b_\mu) = \Phi(a_\lambda)\Psi(b_\mu)\Phi(c_\delta) + \Phi(c_\delta)\Phi(a_\lambda)\Psi(b_\mu),$$

for every $\lambda, \mu$ and $\delta$. Taking weak$^*$-limits in $\lambda$, $\mu$ and $\delta$ we get

$$\Phi^{**}(abc+cba) = \Phi^{**}(a)\Psi^{**}(b)\Phi^{**}(c) + \Phi^{**}(c)\Phi^{**}(a)\Psi^{**}(b).$$

Combining Proposition 2 with Lemma 4 we get the following.

COROLLARY 1. *Let $\Phi : A \to B$ be a continuous linear operator between $C^*$-algebras. Then the following statements are equivalent:*

$(a)$ *$\Phi$ admits a continuous normalized-pg-inverse;*

$(b)$ *There exists a continuous Jordan homomorphism $T : A^{**} \to B^{**}$ such that $\Phi = R_{\Phi^{**}(1)} \circ T$ and $\Phi^{**}(1)B^{**} = T(1)B^{**}$;*

$(c)$ *There exists a continuous Jordan homomorphism $S : A^{**} \to B^{**}$ such that $\Phi = L_{\Phi^{**}(1)} \circ S$ and $B^{**}\Phi^{**}(1) = B^{**}S(1)$.*

Let $A$ and $B$ be C$^*$-algebras. We recall that a linear mapping $T : A \to B$ strongly preserves Moore-Penrose invertibility (respectively, invertibility) if for each Moore-Penrose invertible (respectively, invertible) element $a \in A$, the element $T(a)$ is Moore-Penrose invertible (respectively, invertible) and we have $T(a^\dagger) = T(a)^\dagger$ (respectively, $T(a^{-1}) = T(a)^{-1}$). Hua's theorem (see [18]) affirms that every unital additive map between skew fields that strongly preserves invertibility is either an isomorphism or an anti-isomorphism. Suppose $A$ is unital. In this case M. Burgos, A. C. Márquez-García and A. Morales-Campoy establish in [6, Theorem 3.5] that a linear map $T : A \to B$ strongly preserves Moore-Penrose invertibility if, and only if, $T$ is a Jordan $^*$-homomorphism $S$ multiplied by a partial isometry $e$ in $B$ such that $T(a) = ee^*T(a)e^*e$ for all $a \in A$, if and only if, $T$ is a triple homomorphism (i.e. $T$ preserves triple products of the form $\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*a)$). The problem for linear maps strongly preserving Moore-Penrose invertibility between general C$^*$-algebras remains open.

Let $T : A \to B$ be a triple homomorphism between C$^*$-algebras. In this case

$$T(aba) = T(\{a,b^*,a\}) = \{T(a),T(b^*),T(a)\} = T(a)T(b^*)^*T(a),$$

and

$$\begin{aligned} T(a^*)^*T(b)T(a^*)^* &= \{T(a^*)^*,T(b)^*,T(a^*)^*\} = \{T(a^*),T(b),T(a^*)\}^* \\ &= T(\{a^*,b,a^*\})^* = T(a^*b^*a^*)^* = T((aba)^*)^*, \end{aligned}$$

for all $a,b \in A$. These identities show that $x \mapsto T(x^*)^*$ is a normalized-pg-inverse of $T$. So, when $A$ is unital, it follows from the results by Burgos, Márquez-García and

Morales-Campoy that every linear map $T: A \to B$ strongly preserving Moore-Penrose invertibility admits a normalized-pg-inverse. However, the class of linear maps admitting a normalized-pg-inverse is strictly bigger than the class of linear maps strongly preserving Moore-Penrose invertibility. For example, let $z$ be an invertible element in $B(H)$ with $z^* \neq z$, the mapping $T: B(H) \to B(H)$, $T(x) = zxz^{-1}$ is a homomorphism, and hence a Jordan homomorphism and does not strongly preserve Moore-Penrose invertibility.

We recall that an element $e$ in a C$^*$-algebra $A$ is a *partial isometry* if $ee^*e = e$. Let us observe that a C$^*$-algebra might not contain a single partial isometry. However, a famous result due to Kadison shows that the extreme points of the closed unit ball of a unital C$^*$-algebra $A$ are precisely the maximal partial isometries in $A$ (see [30, Proposition 1.6.1 and Theorem 1.6.4]). Therefore, every von Neumann algebra contains an abundant set of partial isometries. When a C$^*$-algebra $A$ is a regarded as a JB$^*$-triple with respect to the product given by $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$, partial isometries in $A$ are exactly the fixed points of this triple product and are called *tripotents*.

Suppose that $e$ and $v$ are non-zero partial isometries in a C$^*$-algebra $A$ such that $eve = e$ and $v = vev$. Then $e = (ee^*)v^*(e^*e)$ and $v = (vv^*)e^*(v^*v)$. This implies, in the terminology of [13], that $P_2(e)(v^*) = (ee^*)v^*(e^*e) = e$. Since $v$ is a norm-one element, we can conclude from [13, Lemma 1.6 or Corollary 1.7] that $v^* = e + (1 - ee^*)v^*(1 - e^*e)$. However the identity $v = vev$ implies that $v = e^*$.

THEOREM 1. *Let $\Phi, \Psi : A \to B$ be linear maps between $C^*$-algebras. Suppose that $(\Phi, \Psi)$ is Jordan-triple multiplicative. Then the following are equivalent:*

$(a)$ *$\Phi$ and $\Psi$ are contractive;*

$(b)$ *$\Psi(a) = \Phi(a^*)^*$, for every $a \in A$;*

$(c)$ *$\Phi$ and $\Psi$ are triple homomorphisms.*

*Proof.* $(a) \Rightarrow (b)$ Clearly $\Phi^{**}$ and $\Psi^{**}$ are contractive operators and by Lemma 4, $\Psi^{**}$ is a normalized-pg-inverse of $\Phi^{**}$. Let $e$ be a partial isometry in $A^{**}$. Since

$$\Phi^{**}(e) = \Phi^{**}(e)\Psi^{**}(e^*)\Phi^{**}(e), \text{ and } \Psi^{**}(e^*) = \Psi^{**}(e^*)\Phi^{**}(e)\Psi^{**}(e^*), \tag{1}$$

we deduce that $\Psi^{**}(e^*)$ is a generalized inverse of $\Phi^{**}(e)$. Applying that $\Phi^{**}$ and $\Psi^{**}$ are contractions, it follows that $\Phi^{**}(e)$ and $\Psi^{**}(e)$ lie in the closed unit ball of $B^{**}$ and admit normalized generalized inverses in the closed unit ball of $B^{**}$. Corollary 3.6 in [4] implies that $\Phi^{**}(e)$ and $\Psi^{**}(e)$ are partial isometries in $B^{**}$. We can now deduce from (1) and the comments preceding this theorem that $\Psi^{**}(e^*) = \Phi^{**}(e)^*$. In particular, $\Psi(p) = \Phi(p)^*$, for every projection $p \in A^{**}$. Since in a von Neumann algebra every self-adjoint element can be approximated in norm by a finite linear combination of mutually orthogonal projections, we get $\Psi^{**}(a) = \Phi^{**}(a)^*$, for every $a \in A^{**}_{sa}$, and by linearity we have $\Phi^{**}(a)^* = \Psi^{**}(a^*)$, for every $a \in A^{**}$.

$(b) \Rightarrow (c)$ Let us assume that $\Psi(a^*) = \Phi(a)^*$, for every $a \in A$. In this case

$$\Phi\{abc\} = \frac{1}{2}\Phi(ab^*c + cb^*a) = \frac{1}{2}(\Phi(a)\Psi(b^*)\Phi(c) + \Phi(c)\Psi(b^*)\Phi(a))$$
$$= \frac{1}{2}(\Phi(a)\Phi(b)^*\Phi(c) + \Phi(c)\Phi(b)^*\Phi(a)) = \{\Phi(a), \Phi(b), \Phi(c)\},$$

which shows that $\Phi$ (and hence $\Psi$) is a triple homomorphism.

The implication $(c) \Rightarrow (a)$ follows form the fact that triple homomorphisms are contractive (see, for example, [14, Proposition 3.4] or [1, Lemma 1 $(a)$]). $\square$

The fact that every contractive representation of a C$^*$-algebra (equivalently, every contractive homomorphism between C$^*$-algebras) is a $^*$-homomorphism seems to be part of the folklore in C$^*$-algebra theory (see, for example, the last lines in the proof of [3, Theorem 1.7]). Actually, every contractive Jordan homomorphism between C$^*$-algebras is a Jordan $^*$-homomorphism. However, we do not know an explicit reference for this fact. We present next an explicit argument derived from our results. A generalization for Jordan homomorphisms between JB$^*$-algebras will be established in Corollary 4.

COROLLARY 2. *Let $A$ and $B$ be $C^*$-algebras and let $\Phi : A \to B$ be a Jordan homomorphism. Then the following statements are equivalent:*

$(a)$ $\Phi$ *is a contraction;*

$(b)$ $\Phi$ *is a symmetric map* (*i.e.* $\Phi$ *is a Jordan* $^*$*-homomorphism*)*;*

$(c)$ $\Phi$ *is a triple homomorphism.*

*If $A$ is unital, then the above statements are also equivalent to the following:*

$(d)$ $\Phi$ strongly preserves regularity.

*Proof.* The implication $(a) \Rightarrow (b)$ is given by Theorem 1. It is known that every Jordan $^*$-homomorphism is a triple homomorphism, then $(b)$ implies $(c)$. Every triple homomorphism is continuous and contractive (see [1, Lemma 1 $(a)$]), and hence $(c) \Rightarrow (a)$.

The final statement follows from [6, Theorem 3.5]. $\square$

It seems appropriate to clarify the connections between Corollary 2 and previous results. It is known that every triple homomorphism between general C$^*$-algebras strongly preserves regularity (compare [6] and [7]). Actually, if $A$ and $B$ are C$^*$-algebras with $A$ unital, and $T : A \to B$ is a linear map, then by [6, Theorem 3.5], $T$ strongly preserves regularity if, and only if, $T$ is a triple homomorphism. So, if $A$ is unital the equivalence $(c) \Leftrightarrow (d)$ in Corollary 2 can be established under weaker hypothesis. For a non-unital C$^*$-algebra $A$ the continuity of a linear mapping $T : A \to B$ strongly preserving regularity does not follow automatically. For example, by [7, Remark 4.2], we know the existence of an unbounded linear mapping $T : c_0 \to c_0$ which strongly preserves regularity. According to our knowledge, it is an open problem whether every continuous linear map strongly preserving regularity between general C$^*$-algebras is a triple homomorphism.

## 3. Orthogonality preservers and non-unital versions

Let $A$ be a C$^*$-algebra. We recall that an *approximate unit* of $A$ is a net $(u_\lambda)$ such that $0 \leqslant u_\lambda \leqslant 1$ for every $\lambda$, $u_\lambda \leqslant u_\mu$ for every $\lambda \leqslant \mu$, and

$$\lim_\lambda \|x - xu_\lambda\| = \lim_\lambda \|x - u_\lambda x\| = \lim_\lambda \|x - u_\lambda x u_\lambda\| = 0,$$

for every $x \in A$. Every C$^*$-algebra admits an approximate unit (see [28, Theorem 3.1.1]).

Let $(u_\lambda)$ be an approximate unit in a C$^*$-algebra $A$, and let us regard $A$ as a C$^*$-subalgebra of $A^{**}$. Having in mind that a functional $\phi$ in $A^*$ is positive if and only if $\|\phi\| = \lim_\lambda \phi(u_\lambda)$ (see [28, Theorem 3.3.3]), we can easily see that $(u_\lambda) \to 1$ in the weak$^*$ topology of $A^{**}$.

LEMMA 5. *Let $\Phi, \Psi : A \to B$ be linear maps between $C^*$-algebras. Suppose that $\Phi$ is continuous and $(\Phi, \Psi)$ is Jordan-triple multiplicative. Then the following statements hold:*

$(a)$ $\Phi^{**}(abc + cba) = \Phi^{**}(a)\Psi(b)\Phi^{**}(c) + \Phi^{**}(c)\Psi(b)\Phi^{**}(a)$ *for every $a, c$ in $A^{**}$, and every $b$ in $A$;*

$(b)$ $\Phi(b) = \Phi^{**}(1)\Psi(b)\Phi^{**}(1)$ *for every $b$ in $A$;*

$(c)$ *The mapping $T : A \to B^{**}$, $T(x) = \Phi^{**}(1)\Psi(x)$ satisfies $T(a)T(b) = \Phi(a)\Psi(b)$, and $\Phi(a) = T(a)\Phi^{**}(1)$, for every $a, b \in A$;*

$(d)$ *The mapping $S : A \to B^{**}$, $S(x) = \Psi(x)\Phi^{**}(1)$ satisfies $S(a)S(b) = \Psi(a)\Phi(b)$, and $\Phi(a) = \Phi^{**}(1)S(a)$, for every $a, b \in A$;*

$(e)$ *Suppose that $p$ and $q$ are projections in $A$ with $pq = 0$, then $T(p)T(q) = S(p)S(q) = 0$, where $T$ and $S$ are the maps defined in previous items.*

*Proof.* $(a)$ Applying that $\Phi$ is continuous, the bitransposed map $\Phi^{**} : A^{**} \to B^{**}$ is weak$^*$-continuous. Let $a$ and $c$ be elements in $A^{**}$, and let $b \in A$. By Goldstine's theorem we can find bounded nets $(a_\lambda)$ and $(c_\mu)$ in $A$ converging, in the weak$^*$ topology of $A^{**}$, to $a$ and $c$, respectively. By hypothesis

$$\Phi(a_\lambda b c_\mu + c_\mu b a_\lambda) = \Phi(a_\lambda)\Psi(b)\Phi(c_\mu) + \Phi(c_\mu)\Psi(b)\Phi(a_\lambda),$$

for every $\lambda, \mu$. Since the product of $A^{**}$ is separately weak$^*$ continuous, the weak$^*$-continuity of $\Phi^{**}$ implies that

$$\Phi^{**}(abc + cba) = \Phi^{**}(a)\Psi(b)\Phi^{**}(c) + \Phi^{**}(c)\Psi(b)\Phi^{**}(a).$$

$(b)$ Follows from $(a)$ with $a = c = 1$.

$(c)$ By definition and $(b)$ we have

$$T(a)T(b) = \Phi^{**}(1)\Psi(a)\Phi^{**}(1)\Psi(b) = \Phi(a)\Psi(b),$$

and $T(a)\Phi^{**}(1)=\Phi^{**}(1)\Psi(a)\Phi^{**}(1)=\Phi(a)$, for every $a,b\in A$. The proof of $(d)$ is very similar.

$(e)$ Let us take two projections $p,q\in A$ with $pq=0$. By definition and $(b)$ or $(c)$ we have

$$\begin{aligned}T(p)T(q) &= \Phi^{**}(1)\Psi(p)\Phi^{**}(1)\Psi(q)=\Phi(p)\Psi(q)\\ &= \Phi(p)\Psi(q)\Phi(q)\Psi(q)=(\Phi(pqq+qqp)-\Phi(q)\Psi(q)\Phi(p))\Psi(q)\\ &= -\Phi(q)\Psi(q)\Phi(p)\Psi(q)=-\Phi(q)\Psi(qpq)=0. \quad \square\end{aligned}$$

Let us explore some of the questions posed before. In our first proposition we shall prove that the normalized-pg-inverse of a continuous linear map on $c_0$ is automatically continuous.

PROPOSITION 3. Let $\Phi,\Psi : c_0\to c_0$ be linear maps such that $\Phi$ is continuous and $(\Phi,\Psi)$ is Jordan-triple multiplicative. Then $\Psi$ is continuous.

*Proof.* We can assume that $\Phi,\Psi\neq 0$. Let $(e_n)$ be the canonical basis of $c_0$. Applying the previous Lemma 5 $(c)$, the mapping $T:c_0\to c_0^{**}=\ell_\infty$, $T(x)=\Phi^{**}(1)\Psi(x)$ satisfies $T(a)T(b)=\Phi(a)\Psi(b)$, and $\Phi(a)=T(a)\Phi^{**}(1)$, for every $a,b\in c_0$. By the just quoted lemma, $T(p)T(q)=0$ for every pair of projections $p,q\in c_0$ with $pq=0$, and consequently,

$$\Phi(p)\Phi(q)=T(p)\Phi^{**}(1)T(q)\Phi^{**}(1)=T(p)T(q)\Phi^{**}(1)\Phi^{**}(1)=0.$$

We can therefore conclude that $\Phi(e_n)\Phi(e_m)=0$ for every $n\neq m$ in $\mathbb{N}$. Since $\Phi(e_n)=\Phi(e_n)\Psi(e_n)\Phi(e_n)$ and $\Psi(e_n)=\Psi(e_n)\Phi(e_n)\Psi(e_n)$, we deduce that $\Phi(e_n)$ and $\Psi(e_n)$ both are regular elements in $c_0$ and $\Phi(e_n)$ is a normalized generalized inverse of $\Psi(e_n)$. Therefore, for each natural $n$ with $\Phi(e_n)\neq 0$ there exists a finite subset $\mathrm{supp}(\Phi(e_n))=\{k_1^n,\ldots,k_{m_n}^n\}\subset\mathbb{N}$ and non-zero complex numbers $\{\lambda_j^n : j\in\mathrm{supp}(\Phi(e_n))\}$ with the following properties: $|\lambda_j^n|\leqslant\|\Phi\|$ for every $j\in\mathrm{supp}(\Phi(e_n))$ and every natural $n$,

$$\mathrm{supp}(\Phi(e_n))\cap\mathrm{supp}(\Phi(e_m))=\emptyset, \text{ for all } n\neq m,$$

and

$$\Phi(e_n)=\sum_{j\in\mathrm{supp}(\Phi(e_n))}\lambda_j^n e_j, \text{ and } \Psi(e_n)=\sum_{j\in\mathrm{supp}(\Phi(e_n))}\frac{1}{\lambda_j^n}e_j, \quad \forall n\in\mathbb{N}.$$

Let us observe that $\|\Psi(e_n)\|=\max\{\frac{1}{|\lambda_j^n|} : j\in\mathrm{supp}(\Phi(e_n))\}$. To simplify the notation, let $j(n)\in\mathrm{supp}(\Phi(e_n))$ be an element satisfying $\frac{1}{|\lambda_{j(n)}^n|}=\|\Psi(e_n)\|$.

We claim that the set $\{\|\Psi(e_n)\| : n\in\mathbb{N}\}$ must be bounded. Otherwise, we can find a subsequence $(\|\Psi(e_{\sigma(n)})\|)$ satisfying $\frac{1}{|\lambda_{j(\sigma(n))}^{\sigma(n)}|}=\|\Psi(e_{\sigma(n)})\|>n$ for every natural $n$. Let $\pi_2:c_0\to c_0$ be the natural projection of $c_0$ onto the C$^*$-subalgebra generated by the elements $\{e_{j(\sigma(n))} : n\in\mathbb{N}\}$, and let $\iota : c_0=\overline{span}\{e_{\sigma(n)} : n\in\mathbb{N}\}\to c_0$ denote the natural inclusion. The maps $\Phi_1 = \pi_2\Phi\iota, \Psi_1 = \pi_2\Psi\iota : c_0 \to c_0$ are linear maps, $\Psi$ is a normalized-pg-inverse of $\Phi$, the latter is continuous, $\Psi_1(e_{\sigma(n)}) = \frac{1}{\lambda_{j(\sigma(n))}^{\sigma(n)}} e_{j(\sigma(n))}$, and $\Phi_1(e_{\sigma(n)}) = \lambda_{j(\sigma(n))}^{\sigma(n)} e_{j(\sigma(n))}$. The element $a = \sum_{m\in\mathbb{N}} \lambda_{j(\sigma(m))}^{\sigma(m)} e_{j(\sigma(m))}$ lies in $c_0$ and $\|\Psi_1(a)\| < \infty$. Therefore $\Psi_1(a) = \sum_{m\in\mathbb{N}} \mu_m e_{j(\sigma(m))}$ for a unique sequence $(\mu_m) \to 0$. Let us write $j(\sigma(n)) = j_1(n)$. Under these conditions

$$\begin{aligned}
\lambda_{j_1(n)}^{\sigma(n)}\mu_n e_{j_1(n)} &= \Psi_1(a)\Phi_1(e_{j_1(n)}) = \Psi_1(a)\Phi_1(e_{j_1(n)})\Psi_1(e_{j_1(n)})\Phi_1(e_{j_1(n)})\\
&= (\Psi_1(e_{j_1(n)}e_{j_1(n)}a + ae_{j_1(n)}e_{j_1(n)}) - \Psi_1(e_{j_1(n)})\Phi_1(e_{j_1(n)})\Psi_1(a))\Phi_1(e_{j_1(n)})\\
&= \Psi_1(2\lambda_{j_1(n)}^{\sigma(n)}e_{j_1(n)})\Phi_1(e_{j_1(n)}) - \Psi_1(e_{j_1(n)})\Phi_1(e_{j_1(n)})\Psi_1(a)\Phi_1(e_{j_1(n)})\\
&= 2\lambda_{j_1(n)}^{\sigma(n)}\Psi_1(e_{j_1(n)})\Phi_1(e_{j_1(n)}) - \Psi_1(e_{j_1(n)})\Phi_1(e_{j_1(n)}ae_{j_1(n)})\\
&= 2\lambda_{j_1(n)}^{\sigma(n)}e_{j_1(n)} - \Psi_1(e_{j_1(n)})\Phi_1(\lambda_{j_1(n)}^{\sigma(n)}e_{j_1(n)}) = \lambda_{j_1(n)}^{\sigma(n)}e_{j_1(n)},
\end{aligned}$$

which proves that $\mu_n = 1$ for all $n$, leading to a contradiction.

Let $M$ be a positive bound of the set $\{\|\Psi(e_n)\| : n \in \mathbb{N}\}$. For each natural $n$, we set $q_n := \sum_{k=1}^{n} e_k$. Clearly, $(q_n)$ is an approximate unit in $c_0$. Since for each $n \neq m$ we have $\Phi(e_n)\Phi(e_m) = 0$ (i.e., $\operatorname{supp}(\Phi(e_n)) \cap \operatorname{supp}(\Phi(e_m)) = \emptyset$), and, for each natural $j$, $\Phi(e_j)$ is a normalized generalized inverse of $\Psi(e_j)$, we deduce that $\Psi(e_n)\Psi(e_m) = 0$ (i.e., $\operatorname{supp}(\Psi(e_n)) \cap \operatorname{supp}(\Psi(e_m)) = \emptyset$) for every $n \neq m$. Consequently, for each finite subset $F \subseteq \mathbb{N}$ we have

$$\left\|\Psi\left(\sum_{j\in F} e_j\right)\right\| = \max\left\{\left\|\Psi(e_j)\right\| : j \in F\right\} \leqslant M, \tag{2}$$

and consequently $\|\Psi(q_n)\| \leqslant M$, for every natural $n$.

We shall prove next that for each $x \in c_0$ we have

$$\lim_n (\Psi(x - q_n x))_n = 0.$$

Indeed, let us take $y, z, w \in c_0$ such that $x = yzw$ (in the case of $c_0$ the existence of such $y, z, w$ is almost obvious but we can always allude to Cohen's factorization theorem [16, Theorem VIII.32.22]). By assumptions

$$\Psi(x - q_n x) = \Psi(y(1 - q_n)zw) = \Psi(y)\Phi(z - q_n z)\Psi(w).$$

Since $\Phi$ is continuous and $((1 - q_n)z)$ tends in norm to $0$, we deduce that $\lim_n(\Psi(x - q_n x))_n = 0$ as we claimed.

Finally, for an arbitrary $x$ in the closed unit ball of $c_0$ we have

$$\Psi(q_n x) = \Psi(q_n x q_n) = \Psi(q_n)\Phi(x)\Psi(q_n),$$

and hence $\|\Psi(q_n x)\| \leqslant M^2 \|\Phi\|$. The norm convergence of $\Psi(q_n x)$ to $\Psi(x)$, assures that $\|\Psi(x)\| \leqslant M^2 \|\Phi\|$. The arbitrariness of $x$ proves the continuity of $\Psi$. $\square$

The previous proposition remains valid if $c_0$ is replaced with $c_0(\Gamma)$.

Our next goal is to extend the previous Proposition 3 to linear maps on $K(H)$. For that purpose we isolate first a technical result which is implicit in the proof of the just commented proposition.

LEMMA 6. *Let $\Phi, \Psi : A \to B$ be linear maps between $C^*$-algebras such that $\Phi$ is continuous and $(\Phi, \Psi)$ is Jordan-triple multiplicative. Then the following are equivalent:*

(1) *$\Phi$ admits a continuous normalized-pg-inverse $\Psi : A \to B^{**}$;*

(2) *$\Phi^{**}(1)$ is a regular element in $B^{**}$.*

*Proof.* $(1) \Rightarrow (2)$ Suppose that $\Phi$ admits a continuous normalized-pg-inverse $\Psi : A \to B$. By Lemma 4, the mapping $\Psi^{**} : A^{**} \to B^{**}$ is a normalized-pg-inverse of $\Phi^{**}$. In particular $\Phi^{**}(1) = \Phi^{**}(1)\Psi^{**}(1)\Phi^{**}(1)$.

$(2) \Rightarrow (1)$ Let $v \in B^{**}$ such that $\Phi^{**}(1) = \Phi^{**}(1)v\Phi^{**}(1)$. The mapping $\Psi' = L_v \circ R_v \circ \Phi : A \to B^{**}$ is continuous, and by Lemma 5 $(b)$, we have

$$\Phi(b) = \Phi^{**}(1)\Psi(b)\Phi^{**}(1), \quad \forall\, b \in A,$$

and consequently

$$\Phi(b)v\Phi^{**}(1) = \Phi^{**}(1)\Psi(b)\Phi^{**}(1)v\Phi^{**}(1) = \Phi(b),$$

and

$$\Phi^{**}(1)v\Phi(b) = \Phi^{**}(1)v\Phi^{**}(1)\Psi(b)\Phi^{**}(1) = \Phi(b), \quad \forall\, b \in A$$

Now, for arbitrary $a$, $b \in A$, we get:

$$\begin{aligned}\Phi(aba) &= \Phi(a)\Psi(b)\Phi(a) = \Phi(a)v\Phi^{**}(1)\Psi(b)\Phi^{**}(1)v\Phi(a)\\ &= \Phi(a)v\Phi(b)v\Phi(a) = \Phi(a)\Psi'(b)\Phi(a)\end{aligned}$$

and

$$\begin{aligned}\Psi'(aba) &= v\Phi(aba)v = v\Phi(a)\Psi(b)\Phi(a)v\\ &= v\Phi(a)v\Phi^{**}(1)\Psi(b)\Phi^{**}(1)v\Phi(a)v = \Psi'(a)\Phi(b)\Psi'(a). \quad \square\end{aligned}$$

We can now extend our study to linear maps between $K(H)$ spaces.

THEOREM 2. *Let $\Phi, \Psi : K(H_1) \to K(H_2)$ be linear maps such that $\Phi$ is continuous and $(\Phi, \Psi)$ is Jordan-triple multiplicative. Then $\Phi$ admits a continuous normalized-pg-inverse.*

*Proof.* We may assume that $H_1$ is infinite dimensional.

We shall first prove that for every infinite family $\{p_j : j \in \Lambda\}$ of mutually orthogonal projections in $K(H_1)$ the set

$$\{\Psi(p_j) : j \in \Lambda\} \text{ is bounded.} \tag{3}$$

Arguing by contradiction, we assume that the above set is unbounded. Then we can find a countable subset $\Lambda_0$ in $\Lambda$ such that $\|\Psi(p_n)\| \geqslant n^3$, for every natural $n$. Since the projections in the sequence $(p_n)$ are mutually orthogonal, the element $x_0 = \sum_{k=1}^{\infty} \frac{1}{n} p_n \in K(H_1)$, and by hypothesis,

$$\Psi(x_0)\Phi(p_n)\Psi(x_0) = \Psi(x_0 p_n x_0) = \frac{1}{n^2}\Psi(p_n),$$

and hence

$$n = \frac{1}{n^2}n^3 < \left\| \frac{1}{n^2}\Psi(p_n) \right\| \leqslant \|\Psi(x_0)\|^2 \|\Phi(p_n)\| \leqslant \|\Psi(x_0)\|^2 \|\Phi\|,$$

for every natural $n$, which is impossible.

Now, let $\{p_j : j \in \Lambda\}$ be a maximal set of mutually orthogonal (minimal) projections in $K(H_1)$. By (3) there exists a positive $R$ such that $\|\Psi(p_j)\| \leqslant R$, for every $j \in \Lambda$. Let $\mathscr{F}(\Lambda)$ denote the collection of all finite subsets of $\Lambda$, ordered by inclusion. For each $F \in \mathscr{F}(\Lambda)$ we set $q_F := \sum_{j \in F} p_j \in K(H_1)$. It is known that $(q_F)_{F \in \mathscr{F}(\Lambda)})$ is an approximate unit in $K(H_1)$. Clearly for each $F \in \mathscr{F}(\Lambda)$ we have $\|\Psi(q_F)\| \leqslant (\sharp F)\, R$.

We shall now prove that

$$\{\Psi(q_F) : F \in \mathscr{F}(\Lambda)\} \text{ is bounded.} \tag{4}$$

Suppose, contrary to our goal, that the above set is unbounded.

Now, we shall establish the following property: for each $F \in \mathscr{F}(\Lambda)$, and each positive $\delta$ there exists $G \in \mathscr{F}(\Lambda)$ with $G \cap F = \emptyset$ and $\|\Psi(q_G)\| > \delta$. Indeed, if that is not the case, there would exist $F \in \mathscr{F}(\Lambda)$ and $\delta > 0$ such that $\|\Psi(q_G)\| \leqslant \delta$, for every $G \in \mathscr{F}(\Lambda)$ with $G \cap F = \emptyset$. In such a case, for each $H \in \mathscr{F}(\Lambda)$ we have

$$\|\Psi(q_H)\| \leqslant \|\Psi(q_{(H\cap F)})\| + \|\Psi(q_{(H\cap F^c)})\| \leqslant (\sharp F)\, R + \delta,$$

which contradicts the unboundedness of the set $\{\Psi(q_F) : F \in \mathscr{F}(\Lambda)\}$.

Applying the above property, we find a sequence $(F_n) \subset \mathscr{F}(\Lambda)$ with $F_n \cap F_m = \emptyset$ for every $n \neq m$ and $\|\Psi(q_{F_n})\| > n^3$, for every natural $n$. We take $y_0 := \sum_{n=1}^{\infty} \frac{1}{n} q_{F_n} \in K(H_1)$. By hypothesis, $\Psi(y_0)\Phi(q_{F_n})\Psi(y_0) = \Psi(y_0 q_{F_n} y_0) = \frac{1}{n^2}\Psi(q_{F_n})$, and hence

$$n = \frac{1}{n^2}n^3 < \|\Psi(y_0)\Phi(q_{F_n})\Psi(y_0)\| \leqslant \|\Psi(y_0)\|^2 \|\Phi\|,$$

for every natural $n$, leading to the desired contradiction. This concludes the proof of (4).

Now, by (4) the net $(\Psi(q_F))_{F\in\mathscr{F}(\Lambda)}$ is bounded in $K(H_2)\subseteq B(H_2)$, and by the weak$^*$-compactness of the closed unit ball of the latter space, we can find a subnet $(\Psi(q_j))_{j\in\Lambda'}$ converging to some $w\in B(H_2)$ in the weak$^*$ topology of this space. We observe that $(q_F)_{F\in\mathscr{F}(\Lambda)}\to 1$ in the weak$^*$ topology of $B(H_1)$, and by the weak$^*$ continuity of $\Phi^{**}$ we also have $(\Phi(q_j))_{j\in\Lambda'}\to\Phi^{**}(1)$ in the weak$^*$ topology of $B(H_2)$. Lemma 5 implies that

$$\Phi(q_j)=\Phi^{**}(1)\Psi(q_j)\Phi^{**}(1)$$

for every $j\in\Lambda'$. Taking weak$^*$ limits in the above equality we get

$$\Phi^{**}(1)=\Phi^{**}(1)w\Phi^{**}(1),$$

and hence $\Phi^{**}(1)$ is regular in $B(H_2)$.

Finally, an application of Lemma 6 gives the desired statement. □

We can now obtain an improved version of Corollary 1 for linear maps between $K(H)$ spaces.

COROLLARY 3. *Let $\Phi,\Upsilon : K(H_1)\to K(H_2)$ be linear maps such that $\Phi$ is continuous and $(\Phi,\Upsilon)$ is Jordan-triple multiplicative. Then the following statements hold:*

(a) *There exists a continuous Jordan homomorphism $T : K(H_1)\to B(H_2)$ such that $\Phi(a)=T(a)\Phi^{**}(1)$, for every $a\in K(H_1)$, and $\Phi^{**}(1)B(H_2)=T(1)B(H_2)$;*

(b) *There exists a continuous Jordan homomorphism $S : K(H_1)\to B(H_2)$ such that $\Phi(a)=\Phi^{**}(1)S(a)$, for every $a\in K(H_1)$, and $B(H_2)\Phi^{**}(1)=B(H_2)S(1)$.*

*Proof.* By Theorem 2 $\Phi$ admits a continuous normalized-pg-inverse $\Psi : K(H_1)\to B(H_2)$. Applying Lemma 5 we deduce that the mappings $T,S : K(H_1)\to B(H_2)$, $T(a)=\Phi^{**}(1)\Psi(a)$ and $S(a)=\Psi(a)\Phi^{**}(1)$ ($a\in K(H_1)$), are linear and continuous and the identities

$$T(a)T(b)=\Phi(a)\Psi(b),\ \Phi(a)=T(a)\Phi^{**}(1),$$

and

$$S(a)S(b)=\Psi(a)\Phi(b),\ \Phi(a)=\Phi^{**}(1)T(a),$$

hold for every $a,b\in K(H_1)$.

Let $(u_\lambda)$ be an approximate unit in $K(H_1)$. Applying the separate weak$^*$ continuity of the product of $B(H_2)$ we have

$$\begin{aligned}\Psi(a)\Phi^{**}(1)\Psi(a)&=\text{weak}^*\text{-}\lim_\lambda\Psi(a)\Phi(u_\lambda)\Psi(a)\\&=\text{weak}^*\text{-}\lim_\lambda\Psi(au_\lambda a)=\Psi^{**}(a^2)=\Psi(a^2),\end{aligned}$$

for all $a\in K(H_1)$. Finally, by Lemma 5 we get

$$T(a)^2=\Phi^{**}(1)\Psi(a)\Phi^{**}(1)\Psi(a)=\Phi^{**}(1)\Psi(a^2)=T(a^2),$$

for all $a$ in $K(H_1)$. The statement for $S$ follows by similar arguments. □

Let $\Phi : K(H_1)\to K(H_2)$ be a bounded linear map. We do not know if any normalized-pg-inverse of $\Phi$ is automatically continuous.

## 4. Pointwise-generalized-inverses of linear maps between JB$^*$-triples

In this section we explore a version of pointwise-generalized inverse in the setting of JB$^*$-triples.

DEFINITION 2. Let $\Phi : E \to F$ be a linear mapping between JB$^*$-triples. We shall say that $T$ admits a pointwise-generalized-inverse (pg-inverse) if there exists a linear mapping $\Psi : E \to F$ satisfying

$$\Phi\{a,b,c\} = \{\Phi(a),\Psi(b),\Phi(c)\},$$

for every $a,b,c \in E$. If $\Phi$ also is a pg-inverse of $\Psi$ we shall say that $\Psi$ is a normalized-pg-inverse of $\Phi$ or that $(\Phi,\Psi)$ is JB$^*$-triple multiplicative.

Let $\Phi,\Psi : A \to B$ be linear maps between C$^*$-algebras. The pair $(\Phi,\Psi)$ is Jordan-triple multiplicative if $\Phi(aba) = \Phi(a)\Psi(b)\Phi(a)$ and $\Psi(aba) = \Psi(a)\Phi(b)\Psi(a)$. C$^*$-algebras can be regarded as JB$^*$-triples and in such a case, the couple $(\Phi,\Psi)$ is JB$^*$-triple multiplicative if $\Phi(ab^*a) = \Phi(a)\Psi(b)^*\Phi(a)$ and $\Psi(ab^*a) = \Psi(a)\Phi(b)^*\Psi(a)$. We should remark, that these two notions are, in principle, independent.

Every triple homomorphism between JB$^*$-triples is a normalized-pg-inverse of itself. The next lemma gathers some basic properties of linear maps between JB$^*$-triples admitting a pg-inverse.

LEMMA 7. *Let $\Phi : E \to F$ be a linear map between JB$^*$-triples admitting a pg-inverse $\Psi$. Then the following statements hold:*

- $(a)$ *$\Phi$ maps von Neumann regular elements in $E$ to von Neumann regular elements in $F$, that is, $\Phi$ is a weak regular preserver, More concretely, if $b$ is a generalized inverse of $a$ then $\Psi(b)$ is a generalized inverse of $\Phi(a)$;*
- $(b)$ *Let $\Phi_1 : A \to E$ and $\Phi_2 : F \to B$ be linear maps between JB$^*$-triples admitting a pg-inverse, then $\Phi_2\Phi$ and $\Phi\Phi_1$ admit a pg-inverse too;*
- $(c)$ *If $\Phi$ and $\Psi$ are continuous then $\Psi^{**} : E^{**} \to F^{**}$ is a pg-inverse of $\Phi^{**}$.*

*Proof.* $(a)$ If $a$ is von Neumann regular the there exists $b \in E$ such that $Q(a)(b) = \{a,b,a\} = a$. By hypothesis, $\Phi(a) = \Phi\{a,b,a\} = \{\Phi(a),\Psi(b),\Phi(a)\}$, which shows that $\Phi(a)$ is von Neumann regular.

$(b)$ Under these hypothesis, let $\Psi_1$ be a pg-inverse of $\Phi_1$. Then

$$\Phi_1\Phi\{a,b,a\} = \Phi_1\{\Phi(a),\Psi(b),\Phi(a)\} = \{\Phi_1\Phi(a),\Psi_1\Psi(b),\Phi_1\Phi(a)\},$$

which shows that $\Psi_1\Psi$ is a pg-inverse of $\Phi_1\Phi$. The rest of the statement follows from similar arguments.

$(c)$ Assuming that $\Phi$ and $\Psi$ are continuous, the maps $\Phi^{**}$, $\Psi^{**}$ are weak$^*$-continuous. The bidual $E^{**}$ of $E$ is a JBW$^*$-triple, and hence its triple product is separately weak$^*$ (see [2]). Then we can repeat the arguments in the proof of Lemma 4 to conclude, via Goldstine's theorem, that

$$\Phi^{**}\{a,b,c\} = \{\Phi^{**}(a), \Psi^{**}(b), \Phi^{**}(c)\},$$

for every $a,b,c \in E^{**}$. $\square$

Let us observe that the arguments in the proof of Theorem 1 are obtained with geometric tools which are not merely restricted to the setting of C$^*$-algebras. Our next result is a generalization of the just commented theorem, to clarify the parallelism, we recall that, by Kadison's theorem ([30, Proposition 1.6.1 and Theorem 1.6.4]), a C$^*$-algebra $A$ is unital if and only if its closed unit ball contains extreme points.

THEOREM 3. *Let $\Phi,\Psi : E \to F$ be linear maps between JB$^*$-triples. Suppose that $(\Phi,\Psi)$ is JB$^*$-triple multiplicative. Then the following are equivalent:*

$(a)$ *$\Phi$ and $\Psi$ are contractive;*

$(b)$ *$\Psi = \Phi$ is a triple homomorphism.*

*If the closed unit ball of $E$ contains extreme points, then the above statements are also equivalent to the following:*

$(c)$ $\Phi$ strongly preserves regularity, *that is,* $\Phi(x^\wedge) = \Phi(x)^\wedge$ *for every* $x \in E^\wedge$.

*Proof.* $(a) \Rightarrow (b)$ By Lemma 7$(c)$, $\Psi^{**}$ is a normalized-pg-inverse of $\Phi^{**}$. Let $e$ be a tripotent in $E^{**}$. The maps $\Psi^{**}$ and $\Phi^{**}$ are contractive, and by Lemma 7$(a)$, $\Psi^{**}(e)$ is a generalized inverse of $\Phi^{**}(e)$ and both lie in the closed unit ball of $F^{**}$. Corollary 3.6 in [4] assures that $\Phi^{**}(e)$ and $\Psi^{**}(e)$ both are tripotents in $F^{**}$. Let us assume that $\Phi^{**}(e)$ (equivalently, $\Psi^{**}(e)$) is non-zero. The identity

$$\Phi^{**}(e) = \{\Phi^{**}(e), \Psi^{**}(e), \Phi^{**}(e)\} \tag{5}$$

implies that $P_2(\Phi^{**}(e))(\Psi^{**}(e)) = \Phi^{**}(e)$. Lemma 1.6 in [13] assures that

$$\Psi^{**}(e) = \Phi^{**}(e) + P_0(\Phi^{**}(e))(\Psi^{**}(e))$$

and similarly

$$\Phi^{**}(e) = \Psi^{**}(e) + P_0(\Psi^{**}(e))(\Phi^{**}(e)).$$

We deduce from (5) that $\Phi^{**}(e) = \Psi^{**}(e)$, for every tripotent $e \in E^{**}$.

In a JBW$^*$-triple every element can be approximated in norm by a finite linear combination of mutually orthogonal tripotents (see [17, Lemma 3.11]). We can therefore guarantee that $\Phi^{**} = \Psi^{**}$ is a triple homomorphism.

The implication $(b) \Rightarrow (a)$ is established in [1, Lemma 1$(a)$].

The final statement follows from [7, Theorem 3.2]. $\square$

The next corollary, which is an extension of Corollary 2 for JB$^*$-algebras, is probably part of the folklore in JB$^*$-algebra theory but we do not know an explicit reference.

COROLLARY 4. *Let $A$ and $B$ be JB$^*$-algebras and let $\Phi : A \to B$ be a Jordan homomorphism. Then the following statements are equivalent:*

$(a)$ $\Phi$ *is a contraction;*

$(b)$ $\Phi$ *is a symmetric map* (*i.e.* $\Phi$ *is a Jordan* $^*$*-homomorphism*)*;*

$(c)$ $\Phi$ *is a triple homomorphism.*

*If the closed unit ball of $A$ contains extreme points, then the above statements are also equivalent to the following:*

$(d)$ $\Phi$ strongly preserves regularity, *that is,* $\Phi(x^\wedge) = \Phi(x)^\wedge$ *for every* $x \in A^\wedge$.

*Proof.* In the hypothesis of the Corollary, we observe that the identities

$$\Phi\{a,b,a\} = \Phi(U_a(b^*)) = U_{\Phi(a)}(\Phi(b^*)) = \{\Phi(a),\Phi(b^*)^*,\Phi(a)\},$$

$$\Phi(\{a,b,a\}^*)^* = \Phi(U_{a^*}(b))^* = U_{\Phi(a^*)^*}(\Phi(b)^*) = \{\Phi(a^*)^*,\Phi(b),\Phi(a^*)^*\},$$

hold for every $a,b \in A$. This shows that the mapping $x \mapsto \Psi(x) = \Phi(x^*)^*$ is a normalized-pg-inverse of $\Phi$.

$(a) \Rightarrow (b)$ If $\Phi$ is contractive then $\Psi$ is contractive too, and it follows from Theorem 3 that $\Psi = \Phi$, or equivalently, $\Phi(a^*) = \Phi(a)^*$ for every $a$. The other implications have been proved in Theorem 3. $\square$

Returning to Corollaries 2 and 4, in a personal communication, M. Cabrera and A. Rodríguez noticed that, though an explicit reference for these results seems to be unknown, they can be also rediscovered with arguments contained in their recent monograph [9]. We thank Cabrera and Rodríguez for bringing our attention to the lemma and arguments presented below, and for providing the appropriate connections with the results in [9].

LEMMA 8. *Let $A$ be a JB$^*$-algebra, and let $e$ be an idempotent in $A$ such that $\|e\| = 1$. Then $e^* = e$.*

*Proof.* By [9, Proposition 3.4.6], the closed subalgebra of $A$ generated by $\{e,e^*\}$ is a JC$^*$-algebra (i.e. a norm closed Jordan $^*$-subalgebra of a C$^*$-algebra). Therefore $e$ can be regarded as a norm-one idempotent in a C$^*$-algebra, so that, by [9, Corollary 1.2.50], we have $e^* = e$, as required. $\square$

The unital version of Corollary 4 is treated in [9, Corollary 3.3.17(a)]. The general statement needs a more elaborated argument to rediscover Corollary 4.

*New proof of Corollary 4.* Let $\Phi : A \to B$ be a contractive Jordan homomorphism between JB$^*$-algebras. If $A$ and $B$ are unital and $\Phi$ maps the unit in $A$ to the unit in $B$, then the result follows from [9, Corollary 3.3.17(a)].

We deal now with the general statement. We may assume that $\Phi \neq 0$. It is known that $A^{**}$ and $B^{**}$ are unital JB$^*$-algebras whose products and involutions extend those of $A$ and $B$, respectively (cf. [9, Proposition 3.5.26]), $\Phi^{**}: A^{**} \to B^{**}$ is a contractive Jordan algebra homomorphism (cf. [9, Lemma 3.1.17]), and $e := \Phi(1)$ is a norm-one idempotent in $B^{**}$. Therefore, by Lemma 8 and [9, Lemma 2.5.3], $U_e(B^{**})$ is a closed Jordan $*$-subalgebra of $B^{**}$ (hence a unital JB$^*$-algebra) containing $\Phi^{**}(A^{**})$. Then $\Phi^{**}$, regarded as a mapping from $A^{**}$ to $U_e(B^{**})$, becomes a unit-preserving contractive algebra homomorphism. By the first paragraph of this proof, $\Phi^{**}$ (and hence $\Phi$) is a $*$-mapping. $\square$

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