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SHARPENING SOME CLASSICAL NUMERICAL RADIUS INEQUALITIES

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Abstract. New upper and lower bounds for the numerical radii of Hilbert space operators are given. Among our results, we prove that if $A \in \mathscr{B}(\mathscr{H})$ is a hyponormal operator, then for all non-negative non-decreasing operator convex f on $[0,\infty)$, we have

$$f(\omega(A)) \leqslant \frac{1}{2}\left\| f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A|\right) + f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A^*|\right)\right\|,$$

where $\xi_{|A|} = \inf_{\|x\|=1}\left\{\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle}\right\}$. Our results refine and generalize earlier inequalities for hyponormal operator.

1. Introduction

Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, we denote by $|A|$ the absolute value operator of A, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A. A continuous real-valued function f defined on an interval I is said to be operator convex if $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$ for all self-adjoint operators A,B with spectra contained in I and all $\lambda \in [0,1]$.

The numerical range of an operator A in $\mathscr{B}(\mathscr{H})$ is defined as $W(A) = \{\langle Ax,x\rangle : \|x\| = 1\}$. For any $A \in \mathscr{B}(\mathscr{H})$, $\overline{W(A)}$ is a convex subset of the complex plane containing the spectrum of A (see [5, Chapter 2]).

Recall that $\omega(A) = \sup_{\|x\|=1}|\langle Ax,x\rangle|$ and $\|A\| = \sup_{\|x\|=1}\|Ax\|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, for $A \in \mathscr{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{1.1}$$

Other facts about the numerical radius that we use can be found in [6].

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The inequalities in (1.1) have been improved considerably by many authors, (see, e.g., [1, 8, 9, 15, 16, 17]), Kittaneh [12, 14] has shown the following precise estimates of $\omega(A)$ by using several norm inequalities and ingenious techniques:

$$\omega(A) \leqslant \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right), \tag{1.2}$$

and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \leqslant \omega^2(A) \leqslant \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.3}$$

In [3], Dragomir gave the following estimate of the numerical radius which refines the second inequality in (1.1): For every $AA \in \mathscr{B}(\mathscr{H})$,

$$\omega^2(A) \leqslant \frac{1}{2}\left(\omega\left(A^2\right) + \|A\|^2\right).$$

In this paper, we establish a considerable improvement of the second inequality in (1.3). We also propose a new upper bound for $\omega(\cdot)$ for the hyponormal operators. Next, we will give a refinement of the first inequality in (1.1).

2. Upper bounds for the numerical radii

The following lemma is known as the mixed Schwarz inequality (see [7, pp. 75–76]).

LEMMA 2.1. *If $A \in \mathscr{B}(\mathscr{H})$, then*

$$|\langle Ax, y\rangle| \leqslant \langle |A|x, x\rangle^{\frac{1}{2}} \langle |A^*|y, y\rangle^{\frac{1}{2}},$$

for all $x, y \in \mathscr{H}$.

The second lemma is a norm inequality for the sum of two positive operators, which can be found in [13].

LEMMA 2.2. *If A and B are positive operators in $\mathscr{B}(\mathscr{H})$, then*

$$\|A+B\| \leqslant \max(\|A\|, \|B\|) + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|.$$

The following lemma contains a simple inequality, which will be needed in the sequel.

LEMMA 2.3. *For each $\alpha \geqslant 1$, we have*

$$\frac{\alpha-1}{\alpha+1} \leqslant \ln\alpha. \tag{2.1}$$

Proof. Taking $f(\alpha) \equiv \ln\alpha - \frac{\alpha-1}{\alpha+1}$, where $\alpha \geqslant 1$. By an elementary computation we have $f'(\alpha) \geqslant 0$, so $f(\alpha)$ is an increasing function for $\alpha \geqslant 1$. On the other hand $f(\alpha) \geqslant f(1) = 0$. $\square$

Now, we are ready to present our new improvement of the second inequality in (1.3). Recall that, an operator A defined on a Hilbert space $\mathscr{H}$ is said to be hyponormal if $A^*A - AA^* \geqslant 0$, or equivalently if $\|A^*x\| \leqslant \|Ax\|$ for every $x \in \mathscr{H}$.

THEOREM A. *Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Then, for all non-negative non-decreasing operator convex f on $[0,\infty)$, we have*

$$f(\omega(A)) \leqslant \frac{1}{2}\left\| f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A|\right) + f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A^*|\right)\right\|, \tag{2.2}$$

where $\xi_{|A|} = \inf\limits_{\|x\|=1}\left\{\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle}\right\}$.

Proof. Since A is a hyponormal operator we have $1 \leqslant \frac{\langle|A|x,x\rangle}{\langle|A^*|x,x\rangle}$, for each $x \in \mathscr{H}$. On choosing $\alpha = \frac{\langle|A|x,x\rangle}{\langle|A^*|x,x\rangle}$ in (2.1) we get

$$(0 \leqslant)\ \frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle} \leqslant \ln\frac{\langle|A|x,x\rangle}{\langle|A^*|x,x\rangle}.$$

Whence

$$\inf_{\|x\|=1}\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle} \leqslant \ln\frac{\langle|A|x,x\rangle}{\langle|A^*|x,x\rangle}. \tag{2.3}$$

We denote the expression on the left-hand side of (2.3) by $\xi_{|A|}$. On the other hand Zou et al. in [18] proved that for each $a,b > 0$,

$$\left(1+\frac{(\ln a - \ln b)^2}{8}\right)\sqrt{ab} \leqslant \frac{a+b}{2}.$$

By taking $a = \langle|A|x,x\rangle$ and $b = \langle|A^*|x,x\rangle$ and taking into account that $\xi_{|A|} \leqslant \ln\frac{\langle|A|x,x\rangle}{\langle|A^*|x,x\rangle}$, we infer that

$$\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle} \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}\langle(|A|+|A^*|)x,x\rangle.$$

By using Lemma 2.1, we get

$$|\langle Ax,x\rangle| \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}\langle(|A|+|A^*|)x,x\rangle.$$

Now, by taking supremum over $x \in \mathscr{H}, \|x\| = 1$, we get

$$\omega(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \||A|+|A^*|\| .$$

Therefore,

$$\begin{aligned} f(\omega(A)) &\leqslant f\left(\frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \||A|+|A^*|\|\right) \\ &= \left\| f\left(\frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} |A| + \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} |A^*|\right)\right\| \\ &\leqslant \frac{1}{2}\left\| f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}} |A|\right) + f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}} |A^*|\right)\right\| . \end{aligned}$$

This completes the proof. □

REMARK 2.1. Notice that, if A is a normal operator, then $\xi_{|A|} = 0$.

An important special case of Theorem A, which leads to an improvement and a generalization of inequality (1.3) for hyponormal operators, can be stated as follows.

COROLLARY 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Then, for all $1 \leqslant r \leqslant 2$ we have*

$$\omega^r(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r} \||A|^r+|A^*|^r\| ,$$

where $\xi_{|A|} = \inf_{\|x\|=1}\left\{\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle}\right\}$. *In particular,*

$$\omega(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \||A|+|A^*|\| , \tag{2.4}$$

and

$$\omega^2(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^2} \|A^*A+AA^*\| .$$

An operator norm inequality which will be used in next corollary says that for any positive operators $A, B \in \mathscr{B}(\mathscr{H})$, we have (see [2])

$$\|A^r B^r\| \leqslant \|AB\|^r, \qquad \text{for all } 0 \leqslant r \leqslant 1. \tag{2.5}$$

The following result refines and generalizes inequality (1.2) for hyponormal operators.

COROLLARY 2.2. *Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Then*

$$\omega^r(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r}\left(\|A\|^r + \left\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\right\|\right),$$

for all $1 \leqslant r \leqslant 2$. In particular

$$\omega^r(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r}\left(\|A\|^r + \|A^2\|^{\frac{r}{2}}\right),$$

for $1 \leqslant r \leqslant 2$.

Proof. Applying Corollary 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\omega^r(A) &\leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r}\||A|^r + |A^*|^r\| \\
&\leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r}\left(\max\left(\|A\|^r, \|A^*\|^r\right) + \left\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\right\|\right) \\
&= \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)^r}\left(\|A\|^r + \left\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\right\|\right).
\end{aligned}$$

For the particular applying inequality (2.5), we have

$$\left\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\right\| \leqslant \||A||A^*|\|^{\frac{r}{2}} = \|A^2\|^{\frac{r}{2}},$$

for $1 \leqslant r \leqslant 2$. □

Recently, Kian [11] improved Jensen's operator inequality via superquadratic functions. As an application, he showed that the following inequality is valid:

LEMMA 2.4. [11, Example 3.6] *Let $A_1, \ldots, A_n$ be positive operators, then*

$$\left\|\sum_{i=1}^n w_i A_i\right\|^r \leqslant \left\|\sum_{i=1}^n w_i A_i^r\right\| - \inf_{\|x\|=1}\left\{\sum_{i=1}^n w_i\left\langle\left|A_i - \sum_{j=1}^n w_j\langle A_j x, x\rangle\right|^r x, x\right\rangle\right\}, \qquad r \geqslant 2,$$

for each $w_1, \ldots, w_n$ with $\sum_{i=1}^n w_i = 1$.

This, in turn, leads to the following:

THEOREM B. *Let $A \in \mathscr{B}(\mathscr{H})$, then*

$$\omega^2(A) \leqslant \frac{1}{2}\left(\left\||A|^2+|A^*|^2\right\| - \inf_{\|x\|=1} \xi(x)\right), \tag{2.6}$$

where $\xi(x) = \left\langle \left(\left||A| - \frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right|^2 + \left||A^*| - \frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right|^2\right)x,x\right\rangle$.

Proof. One can easily see that for each $A \in \mathscr{B}(\mathscr{H})$ we have

$$\omega(A) \leqslant \frac{1}{2}\||A|+|A^*|\|,$$

we can also write

$$\omega^2(A) \leqslant \frac{1}{4}\||A|+|A^*|\|^2. \tag{2.7}$$

Choosing $n, r = 2$, $w_1 = w_2 = \frac{1}{2}$, $A_1 = |A|$ and $A_2 = |A^*|$ in Lemma 2.4, we infer

$$\begin{aligned}\||A|+|A^*|\|^2 \leqslant 2\Bigg(\left\||A|^2+|A^*|^2\right\| - \inf_{\|x\|=1}\Bigg\{&\left\langle\left||A| - \frac{1}{2}(\langle|A|x,x\rangle + \langle|A^*|x,x\rangle)\right|^2 x,x\right\rangle \\ &+ \left\langle\left||A^*| - \frac{1}{2}(\langle|A|x,x\rangle + \langle|A^*|x,x\rangle)\right|^2 x,x\right\rangle\Bigg\}\Bigg).\end{aligned}$$

It now follows from (2.7) that

$$\begin{aligned}\omega^2(A) &\leqslant \frac{1}{4}\||A|+|A^*|\|^2 \\ &\leqslant \frac{1}{2}\Bigg(\left\||A|^2+|A^*|^2\right\| - \inf_{\|x\|=1}\Bigg\{\left\langle\left||A| - \frac{1}{2}(\langle|A|x,x\rangle + \langle|A^*|x,x\rangle)\right|^2 x,x\right\rangle \\ &\qquad + \left\langle\left||A^*| - \frac{1}{2}(\langle|A|x,x\rangle + \langle|A^*|x,x\rangle)\right|^2 x,x\right\rangle\Bigg\}\Bigg).\end{aligned}$$

The validity of this inequality is just Theorem B. □

REMARK 2.2. Notice that

$$\inf_{\|x\|=1}\xi(x) > 0 \Leftrightarrow 0 \notin \overline{W\left(\left||A| - \frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right|^2 + \left||A^*| - \frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right|^2\right)}.$$

To make things a bit clearer, we consider the following example:

EXAMPLE 2.1. Taking $A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$. By an easy computation we find that

$$\left| |A| - \frac{1}{2}\langle (|A| + |A^*|)x, x \rangle \right|^2 + \left| |A^*| - \frac{1}{2}\langle (|A| + |A^*|)x, x \rangle \right|^2 = \begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}.$$

It is well-known that, $A = \lambda I$ if and only if $W(A) = \{\lambda\}$ (see, e.g., [10, Section 18]). So we get $\inf_{\|x\|=1} \xi(x) = 4.5 > 0$.

This shows that the inequality (2.6) provides an improvement for the second inequality in (1.3).

3. Lower bounds for the numerical radii

The next theorem is slightly more intricate.

THEOREM C. *Let $A \in \mathscr{B}(\mathscr{H})$, then*

$$\|A\| \left(1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leqslant \omega(A). \tag{3.1}$$

Proof. It is easy to check that

$$1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \leqslant \frac{1}{\|x\| \, \|y\|} |\langle x, y \rangle|, \tag{3.2}$$

for every $x, y \in \mathscr{H}$.

If we choose $\|x\| = \|y\| = 1$ in (3.2) we get

$$1 - \frac{1}{2}\|x - y\|^2 \leqslant |\langle x, y \rangle|. \tag{3.3}$$

This is an interesting inequality in itself as well. Now taking $y = \frac{Ax}{\|Ax\|}$ in (3.3), we infer

$$\|Ax\| \left(1 - \frac{1}{2} \left\| x - \frac{Ax}{\|Ax\|} \right\|^2 \right) \leqslant |\langle Ax, x \rangle|. \tag{3.4}$$

Since $\|x\| = 1$, $\|Ax\|$ does not exceed $\|A\|$. Hence we get from (3.4) that

$$\|Ax\| \left(1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leqslant |\langle Ax, x \rangle|.$$

Now by taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we deduce the desired inequality (3.1). $\square$

REMARK 3.1. It is striking that if $\|A-\|A\|\| \leqslant \|A\|$, then inequality (3.1) provides an improvement for the first inequality in (1.1).

EXAMPLE 3.1. Taking $A=\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$. Then $\|A\| \simeq 4.1594$ and $\|A-\|A\|\| \simeq 2.3807$. We obtain by easy computation

$$\frac{1}{2}\|A\| \simeq 2.079, \qquad \|A\|\left(1-\frac{1}{2}\left\|I-\frac{A}{\|A\|}\right\|\right) \simeq 2.968, \qquad \omega(A) \simeq 4.118,$$

whence

$$\frac{1}{2}\|A\| \lneqq \|A\|\left(1-\frac{1}{2}\left\|I-\frac{A}{\|A\|}\right\|\right) \lneqq \omega(A),$$

which shows that if $\|A-\|A\|\| \leqslant \|A\|$, then inequality (3.1) is really an improvement of the first inequality in (1.1).

The following basic lemma is essentially known as in [4, Lemma 1], but our expression is a little bit different from those in [4]. For the sake of convenience, we give it a slim proof.

LEMMA 3.1. *Let $x, y, z_i, i=1,\ldots,n$ be nonzero vectors and $\langle z_j, z_i\rangle \neq 0$, then*

$$\left|\left\langle x-\sum_i \frac{\langle x, z_i\rangle}{\sum_j |\langle z_j, z_i\rangle|} z_i, y\right\rangle\right|^2 \leqslant \|y\|^2\left(\|x\|^2-\sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|}\right). \tag{3.5}$$

Proof. Define

$$u=x-\sum_i \frac{\langle x, z_i\rangle}{\sum_j |\langle z_j, z_i\rangle|} z_i.$$

Whence

$$\|u\|^2=\left\|x-\sum_i a_i z_i\right\|^2 \leqslant \|x\|^2-\sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|}. \tag{3.6}$$

By multiplying both sides (3.6) by $\|y\|^2$ and then utilizing the Cauchy Schwarz inequality we get

$$|\langle u, y\rangle|^2 \leqslant \|y\|^2\left(\|x\|^2-\sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|}\right),$$

which is exactly desired inequality (3.5). □

Finally, we state the last result.

THEOREM D. *Let $A \in \mathscr{B}(\mathscr{H})$ be an invertible operator, then*

$$\inf_{\|x\|=1} \xi^2(x)+\omega^2(A) \leqslant \|A\|^2,$$

where $\xi(x)=\frac{|\langle A^2 x, x\rangle-\langle Ax, x\rangle^2|}{\|A^* x\|}$.

Proof. Simplifying (3.5) for the case $n=1$, we find that

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle}{\|z\|^2}\langle z,y\rangle\right|^2+\frac{|\langle x,z\rangle|^2}{\|z\|^2}\|y\|^2\leqslant\|x\|^2\|y\|^2.$$

Apply these considerations to $x=Ax$, $y=A^*x$ and $z=x$ with $\|x\|=1$ we deduce

$$\left(\frac{\left|\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right|}{\|A^*x\|}\right)^2+|\langle Ax,x\rangle|^2\leqslant\|Ax\|^2. \tag{3.7}$$

We denote the first expression on the left-hand side of (3.7) by $\xi(x)$. Whence (3.7) implies that

$$\inf_{\|x\|=1}\xi^2(x)+|\langle Ax,x\rangle|^2\leqslant\|Ax\|^2.$$

Now, the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

REMARK 3.2. Of course, if A is a normal operator we must have $\xi(x)=0$. In this regard, we have:

(i) If A is a normal matrix and x is an eigenvector of A with the eigenvalue e, then $\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2=e^2-e^2=0$.

(ii) Let $\sigma(A)$ and $\sigma_{ap}(A)$ be the spectrum and approximate spectrum of A, respectively. It is well-known that the spectrum of a normal operator has a simple structure. More precisely, if A is normal, then we have $\sigma(A)=\sigma_{ap}(A)$. If we assume that e is in the approximate point spectrum of normal operator A, then there is a sequence $x_n\in\mathscr{H}$ with $\|x_n\|=1$ and $\langle Ax_n,x_n\rangle\to e$ as $n\to\infty$. Therefore $\lim_{n\to\infty}\left|\left\langle A^2x_n,x_n\right\rangle-\langle Ax_n,x_n\rangle^2\right|=0$.

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