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# EIGENVALUE INTERLACING FOR FIRST ORDER DIFFERENTIAL SYSTEMS WITH PERIODIC $2 \times 2$ MATRIX POTENTIALS AND QUASI–PERIODIC BOUNDARY CONDITIONS

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*Abstract.* The self-adjoint first order system, $JY' + QY = \lambda Y$, with locally integrable, real, symmetric, $\pi$-periodic, $2 \times 2$ matrix potential $Q$ is considered, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By means of a unitary transformation applied to the boundary value problem considered in [6], it is shown that all eigenvalues to the above equation with boundary conditions $Y(\pi) = \pm R(\theta)Y(0)$, where $R(\theta)$ is the rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, occur when the discriminant $\Delta_\theta = Tr(\mathbb{Y}(\pi)^T R(\theta))$ is equal to $\pm 2$. Here $\mathbb{Y}$ is the solution of the first order system obeying the initial condition $\mathbb{Y}(0) = \mathbb{I}$. In addition, an expression for the $\lambda$-derivative of the discriminant $\Delta_\theta$ is given and some monotonicity results are obtained. Interlacing/indexing properties for the eigenvalues of various operator eigenvalue problems are proved.

## 1. Introduction

Quasiperiodic eigenvalue problems fall into the following two categories:
(i) The potential is quasiperiodic, see [1, 7].
(ii) The boundary conditions are quasiperiodic, see [11, 12].

The problems that are investigated in this work are of the second type. In [8] eigenvalue problems with quasiperiodic boundary conditions were studied. In particular, boundary conditions of the form $y(\pi) = \omega y(0)$ with $|\omega| = 1$ and $arg(\omega) \neq k\pi$, were considered. We note that periodic and antiperiodic boundary value problems are in fact special cases of these. Quasiperiodic boundary value problems have in addition been referred to as $\omega$-twisted boundary value problems, see [4, p. 21]. For recent work done in this area see [2, 13, 14, 16].

Here we consider the differential equation

$$\ell Y := JY' + QY = \lambda Y, \tag{1.1}$$

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where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} q_1 & q \\ q & q_2 \end{pmatrix}$. Here $q, q_1, q_2$ are real valued, integrable and $\pi$-periodic.

Let $\mathbb{Y} = [Y_1\ Y_2] = \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix}$ be the solution of (1.1) obeying the initial condition $\mathbb{Y}(0) = \mathbb{I}$. The $\lambda$-intervals on the real line for which all solutions are bounded will be called the intervals of stability while the intervals for which at least one solutions is unbounded will be called instability intervals.

In Section 2 we define a unitary transformation which enables us to obtain the following (as a direct application of this unitary transformation to the boundary value problem considered in [6]):

(i) The eigenvalues of (1.1) with boundary conditions $Y(\pi) = \pm R(\theta)Y(0)$ occur precisely where $\Delta_\theta := Tr(\mathbb{Y}(\pi)^T R(\theta)) = \pm 2$.

(ii) An explicit form for the $\lambda$-derivative of $\Delta_\theta$.

(iii) Monotonicity results concerning the first and second $\lambda$-derivatives of $\Delta_\theta$.

Section 3 contains the main results, namely an interlacing structure for the eigenvalues of (1.1) with certain separated boundary conditions. This relates to the indexing of eigenvalues and hence also to [6, 9, 11, 12, 15].

## 2. Unitary transformation

Let $Y$ be a solution of (1.1) obeying the boundary conditions $Y(\pi) = \pm R(\theta)Y(0)$. If the unitary transformation $V$ of $Y$ is defined as follows

$$V(x) = \begin{pmatrix} \cos\frac{\theta x}{\pi} & -\sin\frac{\theta x}{\pi} \\ \sin\frac{\theta x}{\pi} & \cos\frac{\theta x}{\pi} \end{pmatrix} Y(x), \tag{2.1}$$

for $x \in [0, \pi]$, then $V$ is a solution of the boundary value problem

$$JV' + \tilde{Q}V = \left(\lambda + \frac{\theta}{\pi}\right)V, \tag{2.2}$$

satisfying $V(\pi) = \pm V(0)$. Here

$$\tilde{Q}(x) = \begin{pmatrix} \cos\frac{\theta x}{\pi} & -\sin\frac{\theta x}{\pi} \\ \sin\frac{\theta x}{\pi} & \cos\frac{\theta x}{\pi} \end{pmatrix} Q(x) \begin{pmatrix} \cos\frac{\theta x}{\pi} & \sin\frac{\theta x}{\pi} \\ -\sin\frac{\theta x}{\pi} & \cos\frac{\theta x}{\pi} \end{pmatrix}. \tag{2.3}$$

Let $\mathbb{V} = [V_1\ V_2] = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}$ be the solution of (2.2) obeying the initial condition $\mathbb{V}(0) = \mathbb{I}$. Note that $[v_{11}v_{22} - v_{21}v_{12}](\pi) = 1$.

Then, from [6, Section 3],

$$\begin{aligned} \Delta_\theta &= v_{11}(\pi,\lambda) + v_{22}(\pi,\lambda) \\ &= y_{11}(\pi)\cos\theta - y_{12}(\pi)\sin\theta + y_{21}(\pi)\sin\theta + y_{22}(\pi)\cos\theta. \end{aligned}$$

For each $\lambda \in \mathbb{C}$ consider the problem of solving

$$V(\pi) = \rho_\theta(\lambda)V(0), \tag{2.4}$$

where $V$ is a non-trivial solution of (2.2) and $\rho_\theta(\lambda) \in \mathbb{C}$. This results in $\rho_\theta(\lambda)$ being multivalued and $V$ can be represented as $V = \mathbb{V}(x)\underline{c}$, for some $\underline{c} \in \mathbb{C}^2 \setminus \{0\}$. Thus solving (2.4) is equivalent to solving

$$(\mathbb{V}(\pi) - \rho_\theta(\lambda)I)\underline{c} = 0 \tag{2.5}$$

for $\underline{c} \in \mathbb{C}^2 \setminus \{0\}$ and $\rho_\theta(\lambda) \in \mathbb{C}$. A necessary and sufficient condition for the existence of solutions $\underline{c} \in \mathbb{C}^2 \setminus \{0\}$ of (2.5) is $\det(\mathbb{V}(\pi) - \rho_\theta I) = 0$. This can be expressed as

$$\rho_\theta^2 - \rho_\theta \Delta_\theta + 1 = 0. \tag{2.6}$$

Hence $\Delta_\theta$ is called the $R(\theta)$-discriminant of (1.1) and

$$\rho_\theta^\pm = \frac{\Delta_\theta \pm \sqrt{\Delta_\theta^2 - 4}}{2}, \tag{2.7}$$

are called the Floquet multipiers of (1.1). Note that $\rho_\theta^+ \rho_\theta^- = 1$. It thus follows that $\lambda$ is an eigenvalue of (1.1) with boundary condition $Y(\pi) = R(\theta)Y(0)$ if and only if $\Delta_\theta(\lambda) = 2$ and $\lambda$ is an eigenvalue of (1.1) with boundary condition $Y(\pi) = -R(\theta)Y(0)$ if and only if $\Delta_\theta(\lambda) = -2$. The eigenvalue problems under consideration are self-adjoint, so we can restrict our attention to $\lambda \in \mathbb{R}$ and these eigenvalues are the boundary points of the sets $\pm\Delta_\theta \geqslant 2$.

Let

$$\Sigma_\theta := \{\lambda \in \mathbb{R} : |\Delta_\theta(\lambda)| \geqslant 2\}. \tag{2.8}$$

The maximally connected subsets of $\Sigma_\theta$ are referred to as regions of $R(\theta)$-instability and the maximally connected subsets of $\mathbb{R} \setminus \Sigma_\theta$ are referred to as regions of $R(\theta)$-stability. The $R(0)$-instability intervals and the $R(0)$-discriminant were studied in [6].

We now focus our attention on the discriminant $\Delta_\theta$. Note that for real $\lambda$, $\mathbb{Y}$ has real entries and hence $\Delta_\theta$ is real valued.

THEOREM 2.1. *Let $\lambda \in \mathbb{R}$. The $\lambda$-derivative of $\Delta_\theta$ is given by*

$$\begin{aligned}
\frac{d\Delta_\theta}{d\lambda} &= (y_{21}(\pi)\cos\theta - y_{22}(\pi)\sin\theta)\int_0^\pi Y_1^T Y_1\,dt \\
&\quad + [(y_{22}(\pi) - y_{11}(\pi))\cos\theta + (y_{12}(\pi) + y_{21}(\pi))\sin\theta]\int_0^\pi Y_1^T Y_2\,dt \\
&\quad + (-y_{11}(\pi)\sin\theta - y_{12}(\pi)\cos\theta)\int_0^\pi Y_2^T Y_2\,dt,
\end{aligned} \tag{2.9}$$

*which can expressed as*

$$\frac{d\Delta_\theta}{d\lambda} = -(y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta)\left\{\|Y_2 - AY_1\|_2^2 + B\|Y_1\|_2^2\right\}, \tag{2.10}$$

$$\textit{for} \quad y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta \neq 0,$$

$$\frac{d\Delta_\theta}{d\lambda} = -(-y_{21}(\pi)\cos\theta + y_{22}(\pi)\sin\theta)\left\{\|Y_1 - CY_2\|_2^2 + D\|Y_2\|_2^2\right\}, \tag{2.11}$$

$$\textit{for} \quad y_{21}(\pi)\cos\theta - y_{22}(\pi)\sin\theta \neq 0.$$

*where*

$$A = \frac{(y_{22}(\pi) - y_{11}(\pi))\cos\theta + (y_{12}(\pi) + y_{21}(\pi))\sin\theta}{2(y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta)},$$
$$B = \frac{4 - \Delta_\theta^2}{4(y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta)^2},$$
$$C = \frac{(y_{22}(\pi) - y_{11}(\pi))\cos\theta + (y_{12}(\pi) + y_{21}(\pi))\sin\theta}{2(y_{22}(\pi)\sin\theta - y_{21}(\pi)\cos\theta)},$$
$$D = \frac{4 - \Delta_\theta^2}{4(y_{21}(\pi)\cos\theta - y_{22}(\pi)\sin\theta)^2}.$$

*Proof.* From [6, Lemma 3.2] we obtain that the $\lambda$-derivative of $\Delta_\theta$ is given by

$$\frac{d\Delta_\theta}{d\lambda} = v_{21}(\pi)\int_0^\pi V_1^T V_1 dt + (v_{22}(\pi) - v_{11}(\pi))\int_0^\pi V_1^T V_2\, dt - v_{12}(\pi)\int_0^\pi V_2^T V_2\, dt \tag{2.12}$$

which can also be expressed as

$$\frac{d\Delta_\theta}{d\lambda} = v_{12}(\pi)\left\{\frac{\Delta_\theta^2 - 4}{4v_{12}^2(\pi)}\|V_1\|_2^2 - \left\|V_2 - \frac{v_{22}(\pi) - v_{11}(\pi)}{2v_{12}(\pi)}V_1\right\|_2^2\right\},\ v_{12}(\pi) \neq 0, \tag{2.13}$$
$$\frac{d\Delta_\theta}{d\lambda} = v_{21}(\pi)\left\{\left\|V_1 + \frac{v_{22}(\pi) - v_{11}(\pi)}{2v_{21}(\pi)}v_2\right\|_2^2 - \frac{\Delta_\theta^2 - 4}{4v_{21}^2(\pi)}\|V_2\|_2^2\right\},\ v_{21}(\pi) \neq 0. \tag{2.14}$$

Using the transformation

$$V_i(x) = \begin{pmatrix} \cos\frac{\theta x}{\pi} & -\sin\frac{\theta x}{\pi} \\ \sin\frac{\theta x}{\pi} & \cos\frac{\theta x}{\pi} \end{pmatrix} Y_i(x),$$

for $i = 1, 2$ we obtain, by means of straightforward calculations, equations (2.9), (2.10) and (2.11). $\square$

As a consequence of the above theorem we obtain the following three corollaries.

COROLLARY 2.2. *If* $\Delta_\theta(\lambda) = \pm 2$ *and* $\frac{d\Delta_\theta}{d\lambda}(\lambda) = 0$ *then* $y_{11}(\pi) = \pm\cos\theta = y_{22}(\pi)$ *and* $\pm\frac{d^2\Delta_\theta}{d^2\lambda}(\lambda) < 0$.

*Proof.* Again directly from [6, Lemma 3.2] we have that if $\Delta_\theta(\lambda) = \pm 2$ and $\frac{d\Delta_\theta}{d\lambda}(\lambda) = 0$ then $\mp\frac{d^2\Delta_\theta}{d\lambda^2}(\lambda) > 0$ and $v_{12}(\pi) = 0 = v_{21}(\pi)$ . Thus

$$y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta = 0 = y_{21}(\pi)\cos\theta - y_{22}(\pi)\sin\theta.$$

The above equations together with $[y_{11}y_{22} - y_{21}y_{12}](\pi) = 1$ and $\Delta_\theta = \pm 2$ give $y_{21}(\pi) = \pm\sin\theta$, $y_{12}(\pi) = \mp\sin\theta$ and $y_{11}(\pi) = \pm\cos\theta = y_{22}(\pi)$ so that $\mathbb{Y}(\pi) = \pm R(\theta)$. $\square$

COROLLARY 2.3. *If* $|\Delta_\theta| \leqslant 2$ *then for* $y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta \neq 0$

$$\frac{1}{y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta}\frac{d\Delta_\theta}{d\lambda} < 0, \tag{2.15}$$

*and for* $-y_{21}(\pi)\cos\theta + y_{22}(\pi)\sin\theta \neq 0$

$$\frac{1}{-y_{21}(\pi)\cos\theta + y_{22}(\pi)\sin\theta}\frac{d\Delta_\theta}{d\lambda} < 0. \tag{2.16}$$

*Proof.* If $|\Delta_\theta| \leqslant 2$ then by [6, Lemma 3.2]

$$\frac{1}{v_{12}(\pi)}\frac{d\Delta_\theta}{d\lambda} < 0, \quad \text{for} \quad v_{12}(\pi) \neq 0, \tag{2.17}$$

$$\frac{1}{v_{21}(\pi)}\frac{d\Delta_\theta}{d\lambda} > 0, \quad \text{for} \quad v_{21}(\pi) \neq 0. \tag{2.18}$$

Since $v_{12}(\pi) = y_{11}(\pi)\sin\theta + y_{12}(\pi)\cos\theta$ and $v_{21}(\pi) = -y_{21}(\pi)\cos\theta + y_{22}(\pi)\sin\theta$ the result follows. $\square$

COROLLARY 2.4. *If* $\sin\theta y_{11}(\pi) + \cos\theta y_{12}(\pi) = 0$ *or* $\cos\theta y_{21}(\pi) - \sin\theta y_{22}(\pi) = 0$*, then* $\Delta_\theta \cdot \operatorname{sgn}(\cos\theta y_{11}(\pi) - \sin\theta y_{12}(\pi)) \geqslant 2$.

*Proof.* The determinant of $\mathbb{V}$ being 1 gives, $v_{11}(\pi)v_{22}(\pi) = 1$, so that $\Delta_\theta = v_{11}(\pi) + \frac{1}{v_{11}(\pi)}$. Thus if $v_{11}(\pi) > 0$ then $\Delta_\theta \geqslant 2$ and if $v_{11}(\pi) < 0$ then $\Delta_\theta \leqslant -2$. The results follows since $v_{11}(\pi) = \cos\theta y_{11}(\pi) - \sin\theta y_{12}(\pi)$. $\square$

## 3. Interlacing of eigenvalues

Let $\mathbb{H} = \mathscr{L}_2(0,\pi) \times \mathscr{L}_2(0,\pi)$ be the Hilbert space with inner product

$$\langle Y, Z\rangle = \int_0^\pi Y(t)^T \overline{Z}(t)\,dt \qquad \text{for } Y, Z \in \mathbb{H},$$

and norm $\|Y\|_2 := \sqrt{\langle Y, Y\rangle}$. The Wronskian of $Y$ and $Z$ is given by $[Y,Z]_W = Y^T R(\theta) Z$.

We consider the self-adjoint operator eigenvalue problems

$$L_i Y = \lambda Y, \tag{3.1}$$

where $L_i = \ell|_{\mathscr{D}(L_i)}$ and

$$\mathscr{D}(L_i) = \{Y \in \mathbb{H} : Y \in \mathrm{AC}, \ell Y \in \mathbb{H}, \mathrm{Y} \text{ obeys } (BC_i)\},$$

for $i = 1,\dots,8$. Here

$$\begin{aligned} Y(0) &= Y(\pi), && (BC_1)\\ Y(0) &= -Y(\pi), && (BC_2)\\ y_1(0) &= 0 = y_1(\pi), && (BC_3)\\ y_2(0) &= 0 = y_2(\pi), && (BC_4)\\ R(\theta)Y(0) &= Y(\pi), && (BC_5)\\ -R(\theta)Y(0) &= Y(\pi), && (BC_6)\\ y_1(0) &= 0 = y_2(\pi), && (BC_7)\\ y_2(0) &= 0 = y_1(\pi). && (BC_8) \end{aligned}$$

For $\lambda, \gamma \in \mathbb{R}$, let $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ be the solution of (1.1) satisfying the initial condition $\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$. Here $\psi_1$ and $\psi_2$ are real valued. Define $P(x,\lambda,\gamma)$ and $\varphi(x,\lambda,\gamma)$ by

$$\Psi(x) = \begin{pmatrix} P(x,\lambda,\gamma)\cos\varphi(x,\lambda,\gamma) \\ P(x,\lambda,\gamma)\sin\varphi(x,\lambda,\gamma) \end{pmatrix}, \tag{3.2}$$

where $P(x,\lambda,\gamma) > 0$ and $\varphi(x,\lambda,\gamma)$ is a continuous function of $x$ with $\varphi(0,\lambda,\gamma) = \gamma$. From now on $\varphi$ will be referred to as the angular part of $\Psi$. The function $P(x,\lambda,\gamma)$ is differentiable in $x,\lambda,\gamma$, and $\varphi(x,\lambda,\gamma)$ is real analytic in $\lambda$ and $\gamma$ for fixed $x$, and differentiable in $x$ for fixed $\lambda$ and $\gamma$. Here $\varphi(x,\lambda,\gamma)$ is the solution to a first order initial value problem

$$\varphi' = \lambda - q\sin 2\varphi - q_1\cos^2\varphi - q_2\sin^2\varphi, \tag{3.3}$$
$$\varphi(0) = \gamma. \tag{3.4}$$

where $\varphi' = \frac{\partial\varphi}{\partial x}$. This initial value problem obeys the conditions of [10, Section 69.1], from which it follows that $\varphi(x,\lambda,\gamma)$ is jointly continuous in $(x,\lambda,\gamma)$. Moreover, for fixed $x > 0$, $\varphi(x,\lambda,\gamma)$ is strictly increasing in $\gamma$ and $\lambda$, see Weidmann [15, p. 242], with $\varphi(x,\lambda,\gamma) \to \pm\infty$ as $\lambda \to \pm\infty$, see [3]. Thus the eigenvalues, $\nu_n$, $\mu_n$, $\beta_n$ and $\zeta_n$, $n \in \mathbb{Z}$, of $L_3$, $L_4$, $L_7$ and $L_8$, respectively, are simple and determined uniquely by the equations

$$\varphi(\pi,\nu_n,\pi/2) = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}, \tag{3.5}$$
$$\varphi(\pi,\mu_n,0) = n\pi, \quad n \in \mathbb{Z}. \tag{3.6}$$
$$\varphi(\pi,\beta_n,\pi/2) = (n+1)\pi, \quad n \in \mathbb{Z}, \tag{3.7}$$
$$\varphi(\pi,\zeta_n,0) = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}. \tag{3.8}$$

As a consequence of the above observation it follows that $\mu_n, \nu_n, \beta_n, \zeta_n \to \pm\infty$ as $n \to \pm\infty$.

For $\Sigma_0$, as defined in equations (2.8), it has been shown, [6], that

$$\Sigma_0 = \bigcup_{k=-\infty}^{\infty} [\lambda_{2k-1}, \lambda_{2k}] \cup [\lambda'_{2k-1}, \lambda'_{2k}]. \tag{3.9}$$

Here $\lambda_n$ and $\lambda'_n$ are the eigenvalues of the periodic and anti-periodic eigenvalue problems with suitable indexing, see [6]. Also from [6] we have that:

$$\max\{\mu_n, \nu_n\} < \min\{\mu_{n+1}, \nu_{n+1}\}, \quad n \in \mathbb{Z}; \tag{3.10}$$

$$(-1)^n \Delta'_0(\lambda) < 0, \quad \text{for } \lambda \in (\min\{\nu_n, \mu_n\}, \max\{\nu_{n+1}, \mu_{n+1}\}) \text{ with } |\Delta_0(\lambda)| \leqslant 2; \tag{3.11}$$

the set $|\Delta_0(\lambda)| \geqslant 2$ consists of a countable union of disjoint closed finite intervals, each of which contains precisely one of the sets $\{\nu_n, \mu_n\}, n \in \mathbb{Z}$. The end points of these intervals as the only points at which $|\Delta_0(\lambda)| = 2$.

We can now prove the following interlacing results for $0 < \theta \leqslant \frac{\pi}{2}$:

THEOREM 3.1. *For each $n \in \mathbb{Z}$,*

$$\nu_{n+1}, \mu_{n+1} \in (\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\}), \tag{3.12}$$
$$\beta_n, \zeta_n \in (\max\{\mu_n, \nu_n\}, \min\{\mu_{n+1}, \nu_{n+1}\}). \tag{3.13}$$

*Proof.* Since $\varphi(x, \lambda, \gamma)$ is strictly increasing in $\lambda$ we have

$$\begin{aligned} \varphi\left(\pi, \beta_n, \frac{\pi}{2}\right) &= (n+1)\pi \\ &< (n+1)\pi + \frac{\pi}{2} = \varphi\left(\pi, \nu_{n+1}, \frac{\pi}{2}\right) \\ &< (n+2)\pi = \varphi\left(\pi, \beta_{n+1}, \frac{\pi}{2}\right), \end{aligned}$$

giving

$$\beta_n < \nu_{n+1} < \beta_{n+1}, \tag{3.14}$$

furthermore

$$\begin{aligned} \varphi(\pi, \zeta_n, 0) &= n\pi + \frac{\pi}{2} \\ &< (n+1)\pi = \varphi(\pi, \mu_{n+1}, 0) \\ &< (n+1)\pi + \frac{\pi}{2} = \varphi(\pi, \zeta_{n+1}, 0), \end{aligned}$$

showing that

$$\zeta_n < \mu_{n+1} < \zeta_{n+1}. \tag{3.15}$$

Suppose $\beta_n \geqslant \mu_{n+1}$, then from the monotonicity of $\varphi$ in the eigenparameter and initial value we have

$$(n+1)\pi = \varphi(\pi, \mu_{n+1}, 0) \leqslant \varphi(\pi, \beta_n, 0) < \varphi\left(\pi, \beta_n, \frac{\pi}{2}\right) = (n+1)\pi,$$

a contradiction, so $\beta_n < \mu_{n+1}$. Suppose $\beta_{n+1} \leqslant \mu_{n+1}$, then

$$(n+2)\pi = \varphi\left(\pi, \beta_{n+1}, \frac{\pi}{2}\right) \leqslant \varphi\left(\pi, \mu_{n+1}, \frac{\pi}{2}\right) < \varphi(\pi, \mu_{n+1}, \pi) = (n+2)\pi,$$

a contradiction, so $\beta_{n+1} > \mu_{n+1}$ and thus

$$\beta_n < \mu_{n+1} < \beta_{n+1}. \tag{3.16}$$

Suppose $\zeta_n \geqslant \nu_{n+1}$ from the monotonicity of $\varphi$ in the eigenparameter and initial condition we have

$$(n+1)\pi + \frac{\pi}{2} = \varphi\left(\pi, \nu_{n+1}, \frac{\pi}{2}\right) \leqslant \varphi\left(\pi, \zeta_n, \frac{\pi}{2}\right) < \varphi(\pi, \zeta_n, \pi) = (n+1)\pi + \frac{\pi}{2},$$

a contradiction, so $\zeta_n < \nu_{n+1}$. Suppose $\nu_{n+1} \geqslant \zeta_{n+1}$, then

$$(n+1)\pi + \frac{\pi}{2} = \varphi(\pi, \zeta_{n+1}, 0) \leqslant \varphi(\pi, \nu_{n+1}, 0) < \varphi\left(\pi, \zeta_{n+1}, \frac{\pi}{2}\right) = (n+1)\pi + \frac{\pi}{2},$$

so $\nu_{n+1} < \zeta_{n+1}$ and thus

$$\zeta_n < \nu_{n+1} < \zeta_{n+1}. \tag{3.17}$$

Combining (3.14), (3.15), (3.16) and (3.17) gives (3.12) from which (3.13) follows. □

THEOREM 3.2. *If* $\lambda \in (\min\{\mu_{n+1}, \nu_{n+1}\}, \max\{\beta_{n+1}, \zeta_{n+1}\})$ *and* $|\Delta_\theta(\lambda)| \leqslant 2$ *then*

$$(-1)^n \frac{d\Delta_\theta}{d\lambda}(\lambda) > 0. \tag{3.18}$$

*Proof.* From the monotonicity of $\varphi(\pi, \lambda, \frac{\pi}{2})$ in $\lambda$, we have for $\lambda \in (\beta_n, \beta_{n+1})$ that

$$(n+1)\pi = \varphi\left(\pi, \beta_n, \frac{\pi}{2}\right) < \varphi\left(\pi, \lambda, \frac{\pi}{2}\right) < \varphi\left(\pi, \beta_{n+1}, \frac{\pi}{2}\right) = (n+2)\pi,$$

thus

$$(-1)^n y_{22}(\pi, \lambda) = (-1)^n P\left(\pi, \lambda, \frac{\pi}{2}\right) \sin\varphi\left(\pi, \lambda, \frac{\pi}{2}\right) < 0. \tag{3.19}$$

While for $\lambda \in (\nu_{n+1}, \nu_{n+2})$ we have that

$$(n+1)\pi + \tfrac{\pi}{2} = \varphi\left(\pi, \nu_{n+1}, \tfrac{\pi}{2}\right) < \varphi\left(\pi, \lambda, \tfrac{\pi}{2}\right) < \varphi\left(\pi, \nu_{n+2}, \tfrac{\pi}{2}\right) = (n+2)\pi + \tfrac{\pi}{2},$$

and so

$$(-1)^n y_{21}(\pi, \lambda) = (-1)^n P\left(\pi, \lambda, \frac{\pi}{2}\right) \cos\varphi\left(\pi, \lambda, \frac{\pi}{2}\right) > 0. \tag{3.20}$$

Thus for $\lambda \in (\beta_n, \beta_{n+1}) \cap (\nu_n, \nu_{n+1}) = (\nu_{n+1}, \beta_{n+1})$ and $|\Delta_\theta \leqslant 2|$ we have by Corollary 2.3 that $(-1)^n \frac{d\Delta_\theta}{d\lambda}(\lambda) > 0$.

Moreover for $\lambda \in (\zeta_n, \zeta_{n+1})$ we have

$$n\pi + \tfrac{\pi}{2} = \varphi(\pi, \zeta_n, 0) < \varphi(\pi, \lambda, 0) < \varphi(\pi, \zeta_{n+1}, 0) = (n+1)\pi + \tfrac{\pi}{2},$$

thus

$$(-1)^n y_{11}(\pi, \lambda) = (-1)^n P(\pi, \lambda, 0)\cos\varphi(\pi, \lambda, 0) < 0. \tag{3.21}$$

Lastly, for $\lambda \in (\mu_{n+1}, \mu_{n+2})$ we have

$$(n+1)\pi = \varphi(\pi, \mu_{n+1}, 0) < \varphi(\pi, \lambda, 0) < \varphi(\pi, \mu_{n+2}, 0) = (n+2)\pi,$$

resulting in

$$(-1)^n y_{12}(\pi, \lambda) = (-1)^n P(\pi, \lambda, 0)\sin\varphi(\pi, \lambda, 0) < 0. \tag{3.22}$$

So for $\lambda \in (\zeta_n, \zeta_{n+1}) \cap (\mu_n, \mu_{n+1}) = (\mu_{n+1}, \zeta_{n+1})$ and $|\Delta_\theta \leqslant 2|$ we have by Corollary 2.3 that $(-1)^n \frac{d\Delta_\theta}{d\lambda}(\lambda) > 0$.

Now, $(\nu_{n+1}, \beta_{n+1}) \cap (\mu_{n+1}, \zeta_{n+1}) = (\max\{\mu_{n+1}, \nu_{n+1}\}, \min\{\zeta_{n+1}, \beta_{n+1}\}) \neq \phi$, by Theorem 3.1. Therefore for

$$\begin{aligned}\lambda &\in (\nu_{n+1}, \beta_{n+1}) \cup (\mu_{n+1}, \zeta_{n+1})\\ &= (\min\{\mu_{n+1}, \nu_{n+1}\}, \max\{\beta_{n+1}, \zeta_{n+1}\}),\end{aligned}$$

with $|\Delta_\theta(\lambda)| \leqslant 2$ we have that $(-1)^n \frac{d\Delta_\theta}{d\lambda}$ is positive. $\square$

THEOREM 3.3. *For* $\lambda \in (\max\{\beta_n, \zeta_n\}, \min\{\mu_{n+1}, \nu_{n+1}\}) \neq \phi$, $(-1)^n \Delta_\theta(\lambda) < 0$ *and* $\Delta_\theta$ *has precisely one zero in* $[\min\{\mu_n, \nu_n\}, \max\{\beta_n, \zeta_n\}]$.

*For* $\lambda \in (\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\})$, $(-1)^n \Delta_0(\lambda) < 0$ *and* $\Delta_0$ *has precisely one zero in* $[\min\{\beta_n, \zeta_n\}, \max\{\beta_n, \zeta_n\}]$.

*Proof.* From (3.22) for $\lambda \in (\mu_n, \mu_{n+1})$ we have

$$(-1)^n y_{12}(\pi, \lambda) > 0. \tag{3.23}$$

Similarly from (3.20) for $\lambda \in (\nu_n, \nu_{n+1})$ we have

$$(-1)^n y_{21}(\pi, \lambda) < 0. \tag{3.24}$$

Combining (3.10), (3.19), (3.21), (3.23) and(3.24),

$$(-1)^n \Delta_\theta(\lambda) < 0, \tag{3.25}$$

for $\lambda \in (\max\{\beta_n, \zeta_n\}, \min\{\mu_{n+1}, \nu_{n+1}\})$. Thus there must be at least one zero of $\Delta_\theta$ in $[\min\{\mu_n, \nu_n\}, \max\{\beta_n, \zeta_n\}]$. Combining Theorems 3.1 and 3.2 shows that there is no more than one zero of $\Delta_\theta$ in $[\min\{\mu_n, \nu_n\}, \max\{\beta_n, \zeta_n\}]$.

Using (3.19), (3.21) and (3.13), for

$$\lambda \in (\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\}), \tag{3.26}$$

we have that $(-1)^n \Delta_0 < 0$. Hence $\Delta_0$ has at least one zero in $[\min\{\beta_n, \zeta_n\}, \max\{\beta_n, \zeta_n\}]$. Now Theorem 3.1 and (3.11) show that $\Delta_0$ has at most one zero in

$$[\min\{\beta_n, \zeta_n\}, \max\{\beta_n, \zeta_n\} \subset (\min\{\mu_n, \nu_n\}, \max\{\mu_{n+1}, \nu_{n+1}\})]. \quad \square$$

THEOREM 3.4. *The set* $\Sigma'_\theta := \{\lambda \in \mathbb{R} : |\Delta_\theta| \geqslant 2\sin\theta\}$ *consists of a countable union of disjoint closed finite intervals, each of which contains precisely one zero of* $\Delta_0$*. The zeros of* $\Delta_0$ *and* $\Delta_\theta$ *interlace each other.*

*Proof.* Since $\Delta_\theta$ is continuous, $\Sigma'_\theta$ consists of closed intervals. If $\lambda_0$ is a zero of $\Delta_0$ then $y_{11}(\pi)+y_{22}(\pi)=0$. Since $y_{11}y_{22}-y_{12}y_{21}=1$ we have $y_{12}(\pi)y_{21}(\pi) = -(1+y_{22}^2(\pi))$ which gives $y_{12}(\pi)y_{21}(\pi) < -1$. If $y_{12}(\pi) > 0$ then $y_{21}(\pi) < 0$ and $y_{21}(\pi) < -1/y_{12}(\pi)$ so $\Delta_\theta(\lambda) = (y_{21}(\pi)-y_{12}(\pi))\sin\theta < -\left(y_{12}(\pi)+\frac{1}{y_{12}(\pi)}\right)\sin\theta < -2\sin\theta$, while if $y_{12}(\pi) < 0$ then $y_{21}(\pi) > 0$ and $y_{12}(\pi) < -1/y_{21}(\pi)$ so $\Delta_\theta(\lambda) = (y_{21}(\pi)-y_{12}(\pi))\sin\theta > \left(y_{21}(\pi)+\frac{1}{y_{21}(\pi)}\right)\sin\theta > 2\sin\theta$. Thus $\lambda_0 \in \Sigma'_\theta$. Let $n$ be such that $\lambda_0 \in [\min\{\beta_n,\zeta_n\},\max\{\beta_n,\zeta_n\}]$.

Now by Theorem 3.2 for $\lambda \in (\max\{\mu_n,\nu_n\},\min\{\beta_n,\zeta_n\})$ with $|\Delta_\theta(\lambda)| \leqslant 2$ we have that $\frac{d\Delta_\theta}{d\lambda}$ has constant sign. Therefore $(\min\{\mu_n,\nu_n\},\max\{\beta_n,\zeta_n\}) \cap \Sigma'_\theta$ consists of at most one interval, on which $|\Delta_\theta| \geqslant 2\sin\theta$, but then $\lambda_0$ is in such an interval so there is precisely one such interval and $\lambda_0$ is in this interval. We now show that there is exactly one zero of $\Delta_0$ in each maximal connected subset of $\Sigma'_\theta$. Let $J$ be a maximal connected subset of $\Sigma'_\theta$ and suppose that there are $c,d \in J$ with $c<d, \Delta_0(c)=0=\Delta_0(d)$ and $\Delta_0(\lambda) \neq 0$ for all $\lambda \in (c,d)$. Given the above there is an $n \in \mathbb{Z}$ with

$$\min\{\beta_n,\zeta_n\} \leqslant c \leqslant \max\{\beta_n,\zeta_n\} < \min\{\beta_{n+1},\zeta_{n+1}\} \leqslant d \leqslant \max\{\beta_{n+1},\zeta_{n+1}\},$$

since the zeros of $\Delta_0$ are in the intervals $[\min\{\beta_j,\zeta_j\},\max\{\beta_j,\zeta_j\}], j \in \mathbb{Z}$, with precisely one in each such interval. But $(-1)^n\Delta_\theta(c) < 0$ and $(-1)^{n+1}\Delta_\theta(d) < 0$ so $\Delta_\theta$ has a zero in $(c,d)$ contradicting the definition of $J$. Thus there is precisely one zero of $\Delta_0$ in $J$.

To show the interlacing of the zeros of $\Delta_0$ and $\Delta_\theta$ we consider when $\Delta_0(\lambda)=0$. In this case $y_{11}=-y_{22}$ and from this together with the fact that $y_{11}y_{22}-y_{12}y_{21}=1$ we can conclude that $y_{21}$ and $y_{12}$ have opposite signs. Thus, since $\Delta_\theta(\lambda) = \sin\theta(y_{21}-y_{12})$ when $\Delta_0(\lambda)=0$, we have that $\Delta_\theta(\lambda)$ takes the sign of $y_{21}$. Now for $\lambda \in (\min\{\beta_n,\zeta_n\},\max\{\beta_{n+1},\zeta_{n+1}\})$ we have that $(-1)^n y_{21} < 0$ and hence $(-1)^n\Delta_\theta(\lambda) < 0$. However, from Theorem 3.3 above, $(-1)^n\Delta_\theta(\lambda) > 0$ for $\lambda \in (\max\{\beta_{n-1},\zeta_{n-1}\}, \min\{\mu_n,\nu_n\})$. Thus $\Delta_\theta$ has already changed sign before the zero of $\Delta_0$. Giving that the zeros of $\Delta_\theta$ and $\Delta_0$ interlace each other. □

COROLLARY 3.5. *The zeros of* $\Delta_0$ *are contained within* $\Sigma'_\theta$*, with each component of* $\Sigma'_\theta$ *containing exactly one zero of* $\Delta_0$ *and exactly one of the sets* $\{\beta_n,\zeta_n\}$*,* $n \in \mathbb{Z}$*.*

*Proof.* If $\Delta_0(\lambda)=0$, then from Theorem 3.4, $\lambda \in \Sigma'_\theta$ and every component of $\Sigma'_\theta$ contains precisely one zero of $\Delta_0$ and conversely each zero $\Delta_0$ lies in precisely one of the components of $\Sigma'_\theta$. Furthermore, each zero of $\Delta_0$ lies in precisely one of the intervals $[\min\{\beta_j,\zeta_j\},\max\{\beta_j,\zeta_j\}], j \in \mathbb{Z}$, and each such interval contains a zero of $\Delta_0$. Note that $\Delta_\theta$ does not change sign of this interval and can only obey the equality $|\Delta_\theta| = 2\sin\theta$ at at most one point of this interval. However $\Delta_\theta(\lambda) = -2(-1)^n$ for $\lambda = \beta_n,\zeta_n$ giving that $[\min\{\beta_n,\zeta_n\},\max\{\beta_n,\zeta_n\}] \subset \Sigma'_\theta, n \in \mathbb{Z}$ from which the result follows. □

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