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REMARKS ON NEARLY EQUIVALENT OPERATORS

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Abstract. An operator $S \in \mathscr{L}(\mathscr{H})$ is said to be nearly equivalent to T if there exists an invertible operator $V \in \mathscr{L}(\mathscr{H})$ such that $S^*S = V^{-1}T^*TV$. In this paper, we study several properties of nearly equivalent operators, and investigate their local spectral properties and invariant subspaces.

1. Introduction

Let $\mathscr{H}$ be a separable complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. As usual, we write $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, the point spectrum, and the approximate point spectrum of T, respectively.

A subspace $\mathscr{M}$ of $\mathscr{H}$ is called an *invariant subspace* for an operator $T \in \mathscr{L}(\mathscr{H})$ if $T\mathscr{M} \subset \mathscr{M}$. An operator T in $\mathscr{L}(\mathscr{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $ker(U) = ker(|T|) = ker(T)$ and $ker(U^*) = ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the *Aluthge transform* of T, denoted throughout this paper by $\tilde{T}$ (see [6] for more details).

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a p-*hyponormal* operator if $(T^*T)^p \geqslant (TT^*)^p$, where $0 < p < \infty$. If $p = 1$, T is called *hyponormal*. An operator X in $\mathscr{L}(\mathscr{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator S in $\mathscr{L}(\mathscr{H})$ if there is a quasiaffinity X in $\mathscr{L}(\mathscr{H})$ such that $XT = SX$, and this relation of S and T is denoted by $T \prec S$. If both $T \prec S$ and $S \prec T$, then we say that S and T are *quasisimilar*.

An operator $S \in \mathscr{L}(\mathscr{H})$ is said to be nearly equivalent to T if there exists an invertible operator $V \in \mathscr{L}(\mathscr{H})$ such that $S^*S = V^{-1}T^*TV$ (see Example 1). In this paper, we study several properties of nearly equivalent operators, and investigate their local spectral properties and invariant subspaces.

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2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the *resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T-z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Using local spectra, we define the *local spectral subspace* of T by $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known from [8] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

It can be shown that the converse implications do not hold in general as can be seen from [5] and [8]. For an operator $T \in \mathscr{L}(\mathscr{H})$, we define *a spectral maximal space* of T to be a closed T-invariant subspace $\mathscr{M}$ of $\mathscr{H}$ with the property that $\mathscr{M}$ contains any closed T-invariant subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\sigma(T|_{\mathscr{N}}) \subset \sigma(T|_{\mathscr{M}})$, where $T|_{\mathscr{M}}$ denotes the restriction of T to $\mathscr{M}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *decomposable* if for every finite open covering $\{U_1, \cdots, U_n\}$ of $\mathbb{C}$ there exists a system $\{X_1, \cdots, X_n\}$ of spectral maximal subspaces of T such that $\mathscr{H} = X_1 + \cdots + X_n$ and $\sigma(T|_{X_i}) \subset U_i$ for every $1 \leqslant i \leqslant n$.

3. Main results

Let S and T be in $\mathscr{L}(\mathscr{H})$. Recall that $S \in \mathscr{L}(\mathscr{H})$ is said to be nearly equivalent to T if there exists an invertible operator $V \in \mathscr{L}(\mathscr{H})$ such that $S^*S = V^{-1}T^*TV$, or equivalently, $S^*S = |S|^2$ and $T^*T = |T|^2$ are unitarily equivalent, i.e., $W|S|^2 = |T|^2W$ for some unitary operator W on $\mathscr{H}$. Since $|S|$ and $|T|$ are positive operators, $W|S|^\alpha = |T|^\alpha W$ holds for some $\alpha \in (0,1]$ with the same W.

EXAMPLE 1. Let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ and $S = \begin{pmatrix} |A| & 0 \\ 0 & |B| \end{pmatrix}$ be in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where $|R| = (R^*R)^{\frac{1}{2}}$. Then $S^*S = W^*T^*TW$ where $W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is unitary. Hence S and T are nearly equivalent.

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathscr{H}$ and let $\{\alpha_n\}_{n=1}^\infty$ be a bounded sequence of complex numbers. An operator $W \in \mathscr{L}(\mathscr{H})$ is called a unilateral weighted shift with weights $\{\alpha_n\}$ if $We_n = \alpha_n e_{n+1}$ for all positive integers n.

EXAMPLE 2. Let S and T be the unilateral weighted shifts in $\mathscr{L}(\mathscr{H})$ with the weight sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{e^{i\theta_n}\alpha_n\}_{n=1}^{\infty}$, respectively. Then S and T are nearly equivalent. Indeed, $S^*S = W^*T^*TW$ where W is a unitary operator defined by $We_n = \gamma_n e_n$, where $\gamma_n = e^{i\theta_n}$ for all $n \geqslant 1$.

REMARK 1. *We note that* $W|T|$ *in Theorem 1 is not the polar decomposition* $U|T|$ *of* T *and* $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ *is not the Aluthge transform* $\tilde{T}$ *of* T*, i.e.,* $\tilde{T} \neq |T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$.

We next give an example about Remark 1.

EXAMPLE 3. Let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where $A = V_A|A|$ and $B = V_B|B|$ are the polar decompositions of A and B, respectively, $A, B \neq 0, I$, and let $S = \begin{pmatrix} |A| & 0 \\ 0 & |B| \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$. Then $T^*T = \begin{pmatrix} |B|^2 & 0 \\ 0 & |A|^2 \end{pmatrix}$. Hence S is nearly equivalent to $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. In fact, $S^*S = W^*T^*TW$ where $W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is unitary. Let $T = V_T|T|$ be the polar decomposition of T. Then $|T| = \begin{pmatrix} |B| & 0 \\ 0 & |A| \end{pmatrix}$ and $V_T = \begin{pmatrix} 0 & V_A \\ V_B & 0 \end{pmatrix}$. On the other hand, $W|T| = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} |B| & 0 \\ 0 & |A| \end{pmatrix} = \begin{pmatrix} 0 & |A| \\ |B| & 0 \end{pmatrix} \neq T$. Hence $W|T|$ is not the polar decomposition of T. Similarly, the Aluthge transform $\tilde{T}$ of T is

$$\tilde{T} = |T|^{\frac{1}{2}} V_T |T|^{\frac{1}{2}} = \begin{pmatrix} 0 & |B|^{\frac{1}{2}} V_A |A|^{\frac{1}{2}} \\ |A|^{\frac{1}{2}} V_B |B|^{\frac{1}{2}} & 0 \end{pmatrix}.$$

On the other hand,

$$|T|^{\frac{1}{2}} W |T|^{\frac{1}{2}} = \begin{pmatrix} 0 & |B|^{\frac{1}{2}} |A|^{\frac{1}{2}} \\ |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\tilde{T} \neq |T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$, in general.

We next state some properties about nearly equivalent operators.

PROPOSITION 1. *Let* S *and* T *be in* $\mathscr{L}(\mathscr{H})$*. Suppose that* S *is nearly equivalent to* T *such that* $S^*S = W^*T^*TW$ *for some unitary* W*. If* $|S| \geqslant |T|$*, then* $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ *is hyponormal. In particular, if* $|S| = |T|$*, then* $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ *is normal. Conversely, if* $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ *is hyponormal and* ran $|T|^{\frac{1}{2}}$ *is dense in* $\mathscr{H}$*, then* $|S| \geqslant W|T|W^*$.

Proof. Since $S^*S = W^*T^*TW$, $|S| = W^*|T|W$. Since $|S| \geqslant |T|$, $W^*|T|W \geqslant |T| \geqslant W|T|W^*$. Thus

$$\begin{aligned}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})^*(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) &= |T|^{\frac{1}{2}}W^*|T|U|T|^{\frac{1}{2}} \\ &\geqslant |T|^{\frac{1}{2}}W|T|W^*|T|^{\frac{1}{2}} \\ &= (|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})^*.\end{aligned}$$

Hence $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is hyponormal. In particular, if $|S| = |T|$, then

$$W^*|T|W = |T| = W|T|W^*.$$

Hence $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is normal. Conversely, if $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is hyponormal, then

$$|T|^{\frac{1}{2}}(W^*|T|W - W|T|W^*)|T|^{\frac{1}{2}} \geqslant 0.$$

Since *ran* $|T|^{\frac{1}{2}}$ is dense on $\mathscr{H}$, $|S| = W^*|T|W \geqslant W|T|W^*$. $\square$

We turn now to the intimate connection between invariant subspaces of operators $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ and $W|S|$.

LEMMA 1. *Let S and T be in $\mathscr{L}(\mathscr{H})$. Suppose that S is nearly equivalent to T such that $S^*S = W^*T^*TW$ for some unitary W and $|T|^{\frac{1}{2}}$ is a quasiaffinity. If $\mathscr{M}$ is a nontrivial invariant subspace for $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$, then $\overline{|T|^{\frac{1}{2}}\mathscr{M}}$ is a nontrivial invariant subspace for $W|S|$. Moreover, if $\mathscr{N}$ is a nontrivial invariant subspace for $W|S|$, then $\overline{|T|^{\frac{1}{2}}W\mathscr{N}}$ is a nontrivial invariant subspace for $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$.*

Proof. If $|T|^{\frac{1}{2}}$ is a quasiaffinity, then $|S|$ is a quasiaffinity. Since $|S| = W^*|T|W$ and W is unitary,

$$\begin{aligned}W|S|(|T|^{\frac{1}{2}}\mathscr{M}) &= W(W^*|T|W)|T|^{\frac{1}{2}}\mathscr{M} \\ &= |T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}\mathscr{M}) \\ &\subseteq |T|^{\frac{1}{2}}\mathscr{M}.\end{aligned}$$

Hence $W|S|(\overline{|T|^{\frac{1}{2}}\mathscr{M}}) \subseteq \overline{|T|^{\frac{1}{2}}\mathscr{M}}$. Since $|T|^{\frac{1}{2}}$ is a quasiaffinity and $\mathscr{M}$ is nontrivial, $\overline{|T|^{\frac{1}{2}}\mathscr{M}}$ is a nontrivial invariant subspace for $W|S|$. Moreover, if $\mathscr{N}$ is a nontrivial invariant subspace for $W|S|$, then $|T|W\mathscr{N} \subseteq \mathscr{N}$ since $W|S| = WW^*|T|W = |T|W$. Hence

$$|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}W\mathscr{N}) = |T|^{\frac{1}{2}}W(|T|W\mathscr{N}) \subseteq |T|^{\frac{1}{2}}W\mathscr{N}.$$

Thus $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}(\overline{|T|^{\frac{1}{2}}W\mathscr{N}}) \subseteq \overline{|T|^{\frac{1}{2}}W\mathscr{N}}$. Since $|T|^{\frac{1}{2}}$ is a quasiaffinity, U is unitary, and $\mathscr{N}$ is nontrivial, $\overline{|T|^{\frac{1}{2}}W\mathscr{N}}$ is nontrivial $\square$

As some applications of Lemma 1, we get the following theorem.

THEOREM 1. *Let S and T be in $\mathscr{L}(\mathscr{H})$. Suppose that S is nearly equivalent to T such that $S^*S = W^*T^*TW$ for a unitary operator W. Then the following statements hold.*

(i) *If $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ has a nontrivial invariant subspace, then so does $W|S|$.*

(ii) *If $|S| \geqslant |T|$, then there exists a positive integer K such that for all positive integers $k \geqslant K$, $(W|S|)^k$ has a nontrivial invariant subspace.*

Proof. (i) If $W|S|$ is not a quasiaffinity, then $0 \in \sigma_p(W|S|) \cup \sigma_p(|S|W^*)$. Hence $W|S|$ has a nontrivial invariant subspace. If $W|S|$ is a quasiaffinity, then $|S|$ is a quasiaffinity since W is unitary. Since $|S| = W^*|T|W$, $|T|$ is also quasiaffinity. If $\mathscr{M}$ is a nontrivial invariant subspace for $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$, then $\overline{|T|^{\frac{1}{2}}\mathscr{M}}$ is a nontrivial invariant subspace for $W|S|$ from Lemma 1.

(ii) If $W|S|$ is not a quasiaffinity, then $0 \in \sigma_p(W|S|) \cup \sigma_p(|S|W^*)$. Hence $W|S|$ has a nontrivial invariant subspace. Then $(W|S|)^k$ has a nontrivial invariant subspace. Assume $W|S|$ is a quasiaffinity. If $|S| \geqslant |T|$, then $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is hyponormal for a unitary operator W from Proposition 1. By C. Berger's theorem(see [3]), there exists a positive integers K such that for all positive integers $k \geqslant K$, $(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})^k$ has a nontrivial invariant subspace $\mathscr{M}$. Since $|S| = W^*|T|W$ and W is unitary,

$$\begin{aligned}(W|S|)^k|T|^{\frac{1}{2}}\mathscr{M} &= (W|S|)^{k-1}|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}\mathscr{M}) \\ &\subseteq (W|S|)^{k-1}|T|^{\frac{1}{2}}\mathscr{M}.\end{aligned}$$

By induction, we get that $(W|S|)^k|T|^{\frac{1}{2}}\mathscr{M} \subseteq |T|^{\frac{1}{2}}\mathscr{M}$. Hence $(W|S|)^k(\overline{|T|^{\frac{1}{2}}\mathscr{M}}) \subseteq \overline{|T|^{\frac{1}{2}}\mathscr{M}}$. Since $W|S|$ is a quasiaffinity and $\mathscr{M}$ is nontrivial, $\overline{|T|^{\frac{1}{2}}\mathscr{M}}$ is a nontrivial invariant subspace for $(W|S|)^k$. $\square$

As some applications of Theorem 1, we get the following corollary.

COROLLARY 1. *Under the same hypotheses with Theorem 1, the following statements hold.*

(i) *If $|S| = |T|$, then $W|S|$ has a nontrivial invariant subspace.*

(ii) *If $|S| \geqslant |T|$ and $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$ has nonempty interior, then $W|S|$ has a nontrivial invariant subspace.*

Proof. (i) Since $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is normal from Proposition 1, $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ has a nontrivial invariant subspace. Hence $W|S|$ has a nontrivial invariant subspace from Theorem 1.

(ii) Since $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is hyponormal from Proposition 1 and $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$ has nonempty interior in $\mathbb{C}$, $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ has a nontrivial invariant subspace from theorem of S. Brown([4]). Thus $W|S|$ has a nontrivial invariant subspace from Theorem 1. $\square$

The operator $W|S| = |T|W$ and $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ are of the form AB and BA with $A = |T|^{\frac{1}{2}}$ and $B = |T|^{\frac{1}{2}}U$ where W is a unitary operator. From now on, we consider properties of AB and BA. We begin with the following elementary lemma.

LEMMA 2. *Let X be a vector space and let $A, B, C : X \to X$ be linear mappings where C commutes with A and B.*

(i) *If C is injective, then $AB + C$ is injective if and only if $BA + C$ is injective.*
(ii) *If C is surjective, then $AB + C$ is surjective if and only if $BA + C$ is surjective.*
(iii) *If C is bijective, then $AB + C$ is bijective if and only if $BA + C$ is bijective.*

Proof. (i) Let $AB+C$ be injective. If $x \in X$ with $(BA+C)x = 0$, then $0 = A(BA+C)x = (AB+C)Ax$ and hence $Ax = 0$. Thus $BAx = 0$. As C is injective we obtain $x = 0$. The converse is obtained by interchanging the role of A and B.

(ii) is obtained by applying (i) to the algebraic transposed operators and (iii) follows from (i) and (ii). □

Recall an operator $T \in \mathscr{L}(\mathscr{H})$ has the single valued extension property, respectively, Bishop's property (β) modulo a closed set $S \subset \mathbb{C}$ if for all open subsets $V \subseteq \mathbb{C} \setminus S$ the mapping

$$\mathscr{O}(V, \mathscr{H}) \to \mathscr{O}(V, \mathscr{H}), \quad f \mapsto (T - z)f$$

is injective, respectively injective with closed range on the space $\mathscr{O}(V, \mathscr{H})$ of all analytic functions on V with values in $\mathscr{H}$. If these conditions are satisfied with $S = \emptyset$, the T will be said to possess the single valued extension property or Bishop's property (β), respectively. We say that T has *property* (δ) *modulo* S if for every open cover $\{U, V\}$ of $\mathbb{C}$, the decomposition $\mathscr{H} = H_T(\overline{V}) + H_T(\mathbb{C} \setminus U)$ holds for $S \subset U \subset \overline{U} \subset V$.

By means of Lemma 2, one now obtains the following results:

PROPOSITION 2. *Let T_1 and T_2 be in $\mathscr{L}(\mathscr{H})$. If $S \subset \mathbb{C}$ is a closed set, then T_1T_2 has the single valued extension property modulo S if and only if T_2T_1 has this property.*

Proof. Assume that T_1T_2 has the single valued extension property modulo S. Let open set $V \subseteq \mathbb{C} \setminus S$ and let f be a sequence in $\mathscr{O}(V, \mathscr{H})$ with the mapping

$$\mathscr{O}(V, \mathscr{H}) \to \mathscr{O}(V, \mathscr{H}), \quad f \mapsto (T_2T_1 - z)f$$

is injective, i.e.,

$$(T_2T_1 - z)f(z) \equiv 0 \tag{1}$$

in $\mathscr{O}(V, \mathscr{H})$. Multiplying both sides by T_1, we get that

$$(T_1T_2 - z)T_1 f(z) \equiv 0$$

in $\mathscr{O}(V, \mathscr{H})$. Since T_1T_2 has the single valued extension property modulo S, we have that

$$T_1 f(z) \equiv 0$$

in $\mathscr{O}(V, \mathscr{H})$. By (1), $zf(z) \equiv 0$ in $\mathscr{O}(V, \mathscr{H})$. Hence T_2T_1 has the single valued extension property modulo S. The converse implication is similar. □

PROPOSITION 3. *Let T_1 and T_2 be in $\mathscr{L}(\mathscr{H})$. If $S \subset \mathbb{C}$ is a closed set, then T_1T_2 has the Bishop's property (β) modulo S if and only if T_2T_1 has this property.*

Proof. Fix an arbitrary open set $V \subseteq \mathbb{C} \setminus S$ and let now X be the quotient of the space $w(\mathbb{N}, \mathscr{O}(V,\mathscr{H}))$ of all sequences in $\mathscr{O}(V,\mathscr{H})$ modulo the subspace $c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H}))$ of all sequences that tend to 0 in $\mathscr{O}(V,E)$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{O}(V,\mathscr{H})$. We can choice the following maps

$$\begin{aligned} A &: (f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})) \mapsto (T_1 f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})), \\ B &: (f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})) \mapsto (T_2 f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})), \\ C &: (f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})) \mapsto (z f_n) + c_0(\mathbb{N}, \mathscr{O}(V,\mathscr{H})). \end{aligned} \tag{2}$$

Assume that T_1T_2 has the Bishop's property (β) modulo S. Let open set $V \subseteq \mathbb{C} \setminus S$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{O}(V,\mathscr{H})$ with

$$\lim_{n\to\infty} (T_2T_1 - z) f_n(z) = 0. \tag{3}$$

Then $\lim_{n\to\infty}(T_1T_2 - z)T_1 f_n(z) = 0$ in $\mathscr{O}(V,\mathscr{H})$. Since T_1T_2 has the Bishop's property (β) modulo S, we have that

$$\lim_{n\to\infty} T_1 f_n(z) = 0$$

in $\mathscr{O}(V,\mathscr{H})$. By (3), $\lim_{n\to\infty} z f_n(z) = 0$ in $\mathscr{O}(V,\mathscr{H})$. Hence T_2T_1 has the Bishop's property (β) modulo S. The converse implication is similar. □

By Theorems 8 and 21 in [2], a bounded linear operator $T \in \mathscr{L}(\mathscr{H})$ is decomposable modulo a closed set $S \subseteq \mathbb{C}$ if and only if T and its adjoint $T^* \in \mathscr{L}(\mathscr{H}^*)$ both have the Bishop's property (β) modulo S. Hence we get from Proposition 2 the following corollary.

COROLLARY 2. *If $S \subset \mathbb{C}$ is a closed set, then T_1T_2 is decomposable modulo S if and only if T_2T_1 is decomposable modulo S. In particular, if $S = \emptyset$, then T_1T_2 is decomposable in sense of Foiąs if and only if T_2T_1 is decomposable.*

Proof. By Theorems 8 in [2], both T_1T_2 has the Bishop's property (β) modulo S and T_1T_2 has the property (δ) modulo S. From Proposition 2, T_2T_1 has the Bishop's property (β) modulo S. Since T_1T_2 has the property (δ) modulo S, adjoint of T_1T_2 has the Bishop's property (β) modulo S by Theorems 21 in [2]. Hence adjoint of T_2T_1 has the Bishop's property (β) modulo S by Proposition 3. Thus T_2T_1 is decomposable modulo S. The converse implication is similar. □

The following corollary is an immediate consequences of Proposition 2, 3, and Corollary 2. The proofs follow with appropriate choices of T_1 and T_2 in these two propositions and the corollary.

COROLLARY 3. *Let P and V be in $\mathscr{L}(\mathscr{H})$ with $P \geqslant 0$. For $0 \leqslant \alpha \leqslant 1$, we write $\widetilde{T}_\alpha := P^\alpha V P^{1-\alpha}$. If $S \subset \mathbb{C}$ is a closed set, then the following statements hold.*

(i) *$\widetilde{T}_\alpha$ has the single valued extension property modulo S for some $\alpha \in [0,1]$ if and only if $\widetilde{T}_\alpha$ has this property for all $\alpha \in [0,1]$.*

(ii) *$\widetilde{T}_\alpha$ has the Bishop's property (β) modulo S for some $\alpha \in [0,1]$ if and only if $\widetilde{T}_\alpha$ has this property for all $\alpha \in [0,1]$.*

(iii) $\widetilde{T}_\alpha$ *is decomposable modulo* S *for some* $\alpha \in [0,1]$ *if and only if* $\widetilde{T}_\alpha$ *is decomposable modulo* S *for all* $\alpha \in [0,1]$.

From Corollary 3, we observe that this result includes and improves Theorem 1.1, Corollary 1.13, and Theorem 1.14 in [7].

Recall that given $x \in \mathscr{H}$ and $T \in \mathscr{L}(\mathscr{H})$, $r_T(x) = \limsup_{n\to\infty} ||T^n x||^{\frac{1}{n}}$ is called the local spectral radius of T at x. As some applications, we get the following corollaries.

COROLLARY 4. *Let* $S \subset \mathbb{C}$ *be a closed set. If* T_2T_1 *has the Bishop's property* (β) *modulo* S, *then the following statements hold.*

(i) T_1T_2 *has the Dunford's property* (C) *modulo* S *and the single-valued extension property modulo* S.

(ii) $r_{T_1T_2}(x) = \lim_{n\to\infty} \|(T_1T_2)^n x\|^{\frac{1}{n}}$ *for all* $x \in \mathscr{H}$.

(iii) $\mathscr{H}_{T_1T_2}(E)$ *is the spectral maximal space of* T_1T_2 *and* $\sigma(T_1T_2|_{\mathscr{H}_{T_1T_2}}(E)) \subset \sigma(T_1T_2) \cap E$ *for any closed subset* E *in* $\mathbb{C} \setminus S$.

Proof. (i) Since T_1T_2 has the Bishop's property (β) modulo S by Proposition 3, the proof follows from [1, Theorem 2.77 and Theorem 6.18].

(ii) The proof follows from Proposition 3 and [8, Proposition 3.3.17].

(iii) Since T_1T_2 has the Bishop's property (β) modulo S by Proposition 3, $\mathscr{H}_{T_1T_2}(E)$ is closed for any closed set E in $\mathbb{C} \setminus S$. Hence the proof follows from [2, Lemma 1]. $\square$

COROLLARY 5. *Let* $S \subset \mathbb{C}$ *be a closed set. If* T_1T_2 *has the single-valued extension property modulo* S, *then the following statements hold.*

(i) $\sigma_{T_1T_2}(T_1x) \subset \sigma_{T_2T_1}(x)$ *and* $\sigma_{T_2T_1}(T_2x) \subset \sigma_{T_1T_2}(x)$.

(ii) $T_1\mathscr{H}_{T_2T_1}(E) \subset \mathscr{H}_{T_1T_2}(E)$ *and* $T_2\mathscr{H}_{T_1T_2}(E) \subset \mathscr{H}_{T_2T_1}(E)$ *for any closed subset* E *in* $\mathbb{C} \setminus S$.

Proof. (i) Let open set $V \subseteq \mathbb{C} \setminus S$. If $\lambda \notin \sigma_{T_2T_1}(x)$, then there exists an analytic function f in $\mathscr{O}(V, \mathscr{H})$ such that

$$(T_2T_1 - \lambda)f(\lambda) \equiv x.$$

Multiplying both sides by T_1, we get that

$$T_1x \equiv T_1(T_2T_1 - \lambda)f(\lambda) = (T_1T_2 - \lambda)T_1f(\lambda). \tag{4}$$

Hence $\lambda \notin \sigma_{T_1T_2}(T_1x)$. Thus $\sigma_{T_1T_2}(T_1x) \subset \sigma_{T_2T_1}(x)$.

Similarly, if $\lambda \notin \sigma_{T_1T_2}(x)$, then there exists an analytic function f in $\mathscr{O}(V, \mathscr{H})$ such that

$$(T_1T_2 - \lambda)f(\lambda) \equiv x.$$

Multiplying both sides by T_2, we get that

$$T_2x \equiv (T_1T_2 - \lambda)T_2f(\lambda). \tag{5}$$

Hence $\lambda \notin \sigma_{T_1T_2}(T_2x)$. Thus $\sigma_{T_1T_2}(T_2x) \subset \sigma_{T_2T_1}(x)$.

(ii) If $x \in \mathscr{H}_{T_1T_2}(E)$ for any closed set $E \subset \mathbb{C} \setminus S$, then $\sigma_{T_1T_2}(x) \subset E$. Since $\sigma_{T_2T_1}(T_2x) \subset \sigma_{T_1T_2}(x)$ from (i), we have that $\sigma_{T_2T_1}(T_2x) \subset E$, i.e., $T_2x \in \mathscr{H}_{T_2T_1}(E)$. Hence $T_2\mathscr{H}_{T_1T_2}(E) \subset \mathscr{H}_{T_2T_1}(E)$.

Similarly, if $x \in \mathscr{H}_{T_2T_1}(E)$, then $\sigma_{T_2T_1}(x) \subset E$. Since $\sigma_{T_1T_2}(T_1x) \subset \sigma_{T_2T_1}(x)$ from (i), we have that $\sigma_{T_1T_2}(T_1x) \subset E$, i.e., $T_1x \in \mathscr{H}_{T_1T_2}(E)$. Hence $T_1\mathscr{H}_{T_2T_1}(E) \subset \mathscr{H}_{T_1T_2}(E)$. □

COROLLARY 6. *Let T_1 and T_2 be in $\mathscr{L}(\mathscr{H})$ and let $S \subset \mathbb{C}$ be a closed set. Suppose that T_1 is nearly equivalent to T_2 such that $T_1^*T_1 = W^*T_2^*T_2W$ for a unitary operator W. If $|T_1| \geqslant |T_2|$, then $W|T_1|$ has the Bishop's property (β) modulo S.*

Proof. If $|T_1| \geqslant |T_2|$, then $|T_2|^{\frac{1}{2}}W|T_2|^{\frac{1}{2}}$ is hyponormal from Proposition 1. Hence $|T_2|^{\frac{1}{2}}W|T_2|^{\frac{1}{2}}$ has the Bishop's property (β) modulo S. Let the operator $|T_2|^{\frac{1}{2}}W|T_2|^{\frac{1}{2}}$ be of the form AB with $A = |T_2|^{\frac{1}{2}}W$ and $B = |T_2|^{\frac{1}{2}}$. Hence $W|T_1| = BA$ has the Bishop's property (β) modulo S by Proposition 3. □

Let T_1 and T_2 in $\mathscr{L}(\mathscr{H})$. It is well known that $\sigma(T_1T_2) \setminus \{0\} = \sigma(T_2T_1) \setminus \{0\}$, $\sigma_{ap}(T_1T_2) \setminus \{0\} = \sigma_{ap}(T_2T_1) \setminus \{0\}$, and $\sigma_p(T_1T_2) \setminus \{0\} = \sigma_p(T_2T_1) \setminus \{0\}$. Using these facts, we give some spectral relations between $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ and $W|S|$.

PROPOSITION 4. *Let S and T be in $\mathscr{L}(\mathscr{H})$. If S and T are nearly equivalent such that $S^*S = W^*T^*TW$ for a unitary operator W, then $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma(W|S|)$, $\sigma_{ap}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma_{ap}(W|S|)$, and $\sigma_p(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma_p(W|S|)$.*

Proof. Since $W|S| = |T|W$ and $(|T|^{\frac{1}{2}}W)|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}W)$, $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) \setminus \{0\} = \sigma(|T|W) \setminus \{0\}$, $\sigma_{ap}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) \setminus \{0\} = \sigma_{ap}(W|S|) \setminus \{0\}$, and $\sigma_p(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) \setminus \{0\} = \sigma_p(W|S|) \setminus \{0\}$ hold. So it suffices to show that the equalities hold about 0.

If $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is invertible, then $|T|^{\frac{1}{2}}$ is invertible. Since $|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})|T|^{-\frac{1}{2}} = |T|W = W|S|$, it follows that $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ and $W|S|$ are similar. Hence $W|S|$ is invertible, i.e., $\sigma(W|S|) \subseteq \sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$. By the similar argument, $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) \subseteq \sigma(W|S|)$. Thus $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma(W|S|)$.

If there exists a sequence $\{x_n\}$ with unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} \||T|Wx_n\| = 0,$$

then

$$\lim_{n\to\infty} \|(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}Wx_n)\| = 0.$$

If $\{|T|^{\frac{1}{2}}Wx_n\}$ does not tend to zero in norm, $0 \in \sigma_{ap}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$. Otherwise, $\{|T|^{\frac{1}{2}}Wx_n\}$ tends to zero in norm. Hence $\lim_{n\to\infty} \|(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})Wx_n\| = 0$. Since $\{Wx_n\}$ cannot converge to zero in norm, $0 \in \sigma_{ap}(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$.

If there exists a sequence $\{y_n\}$ with unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} \||T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}y_n\| = 0,$$

then

$$0 = \lim_{n\to\infty} \||T|W(|T|^{\frac{1}{2}}y_n)\| = \lim_{n\to\infty} \|W|S|(|T|^{\frac{1}{2}}y_n)\|,$$

which gives $0 \in \sigma_{ap}(W|S|)$ if $\{|T|^{\frac{1}{2}}y_n\}$ does not tend to zero in norm. Otherwise, $\{|T|^{\frac{1}{2}}y_n\}$ tends to zero in norm. Hence

$$\begin{aligned}\lim_{n\to\infty} \|W|S|W^*y_n\| &= \lim_{n\to\infty} \||T|WW^*y_n\| = \lim_{n\to\infty} \||T|y_n\| \\ &= \lim_{n\to\infty} \||T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}y_n)\| = 0.\end{aligned}$$

Since $\{W^*y_n\}$ cannot converge to zero in norm, $0 \in \sigma_{ap}(W|S|)$.

The same argument hold for the point spectrum $\sigma_p(\cdot)$. □

Let us recall that an operator T is said to be isoloid if for any $\lambda \in \text{iso}\,\sigma(T)$, $\lambda \in \mathbb{C}$ is an eigenvalue of T, where $\text{iso}\,\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$ $(i.e., \text{iso}\,\sigma(T) \subseteq \sigma_p(T))$.

COROLLARY 7. *Let S and T be in $\mathscr{L}(\mathscr{H})$ and S is nearly equivalent to T such that $S^*S = W^*T^*TW$ for a unitary operator W. If $|S| \geqslant |T|$, then $W|S|$ is isoloid.*

Proof. If $|S| \geqslant |T|$, then $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is hyponormal from Proposition 1. Since $|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}$ is isoloid, $\text{iso}\,\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) \subseteq \sigma_p(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}})$. Since $\sigma(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma(W|S|)$ and $\sigma_p(|T|^{\frac{1}{2}}W|T|^{\frac{1}{2}}) = \sigma_p(W|S|)$ from Proposition 4, $W|S|$ is isoloid. □

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