STABILITY OF DETERMINACY AND INVERSE
SPECTRAL PROBLEMS FOR JACOBI OPERATORS

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(*Communicated by F. Gesztesy*)

Abstract. This work studies the interplay between Green functions, the index of determinacy of spectral measures and interior finite rank perturbations of Jacobi operators. The index of determinacy quantifies the stability of uniqueness of solutions of the moment problem. We give results on the constancy of this index in terms of perturbations of the corresponding Jacobi operators. The permanence of the N-extremality of a measure is also studied. A measure μ is N-extremal when the polynomials are dense in $L_2(\mathbb{R}, \mu)$. As a by-product, we give a characterization of the index in terms of cyclic vectors. We consider a new inverse problem for Jacobi operators in which information on the place where the interior perturbation occurs is obtained from the index of determinacy.

1. Introduction

Given a sequence $\{s_k\}_{k=0}^{\infty}$ of real numbers, the problem of finding a Borel measure μ in $\mathbb{R}$ such that

$$s_k = \int_{\mathbb{R}} t^k d\mu \quad \text{for all } k = 0, 1, 2, \dots$$

is called the Hamburger moment problem.

Denote by $\mathcal{M}$ the set of Borel measures on $\mathbb{R}$ with infinite support and all their moments finite. For a positive sequence $\{s_k\}_{k=0}^{\infty}$ (see definition in [2, Chap. 1, Sec 1]), the corresponding Hamburger moment problem has always a solution $\mu \in \mathcal{M}$ [2, Thm. 2.1.1]. $\mathcal{M}$ splits into two sets, one corresponding to the case when the Hamburger moment problem has a unique solution and the other when it has various solutions. In the first case, it is said that the moment problem is determinate, whereas, in the second case, the problem is called indeterminate. If a moment problem is determinate (indeterminate), then the corresponding solution, that is the measure, is also called determinate (indeterminate).

The problem of finding conditions under which a relevant class of functions is dense in the spaces $L_p(\mathbb{R}, \mu)$ is classical in analysis. In particular, conditions which guarantee density of polynomials go back at least to the work of Hamburger [23]. For related work see for example [1, 3, 5, 6, 7, 8, 9, 10, 12, 17, 18] (see in [26, Sec. 4.8]

a brief compilation of results on the matter). A fundamental result characterizing the measures $\mu \in \mathcal{M}$ for which the polynomials are dense in $L_2(\mathbb{R}, \mu)$ is due to M. Riesz [2, Thm. 2.3.2], [29]. It establishes that for the polynomials to be dense in $L_2(\mathbb{R}, \mu)$ it is necessary and sufficient that μ be N-extremal (see definition in [2, Pag. 43]). In [31, Pag. 86], N-extremal solutions are called von Neumann solutions whereas in [13, Pag. 2796] N-extremal means Nevanlinna extremal. Note that in contrast to the definition given in [13], here all determinate solutions are N-extremal [2, Cor. 2.3.3].

A concept related to the determinacy and N-extremality of a measure is the concept of the *index of determinacy* introduced by Berg and Durán in [13]. The index of determinacy of $\mu \in \mathcal{M}$ quantifies the stability of μ to be the unique solution of a moment problem under perturbations of it (see Definition 4.3). This index also gives information on how a measure can be perturbed and maintain the property of being N-extremal. The fact that a measure $\mu \in \mathcal{M}$ is indeterminate N-extremal or determinate may be changed by adding or substracting the mass at only one point (see Proposition 4.1 below) or by modifying the weights without changing the support (see Proposition 4.14).

Jacobi operators, i. e., self-adjoint extensions of operators having a tridiagonal matrix representation (see (2.2)), naturally appear in the theory of the Hamburger moment problem. It turns out that every N-extremal solution of a Hamburger moment problem normalized so that $s_0 = 1$ is the spectral measure of a Jacobi operator (see Theorem 2.4). Thus, the study of measures μ such that the polynomials are dense in $L_2(\mathbb{R}, \mu)$ is the study of self-adjoint extensions of operators having a semi-infinite Jacobi matrix as its matrix representation.

We study the constancy of the index of determinacy after changing the weights and support of the measure. The permanence of the N-extremality of a given measure is also considered. Similar questions on stability are studied in [25] and [33]. Our approach to this matter is mainly based on Jacobi operators and Green functions. This allows us to give results on the stability of the index of determinacy of the spectral measure of a Jacobi operator under finite rank perturbations of the operator. Jacobi operators model linear mass-spring systems and the perturbations considered here correspond to changing one mass and spring constant in some place of the chain.

Our findings on the stability of the index of determinacy and the N-extremality of the spectral measures of Jacobi operators shed light on the inverse spectral problem of reconstructing an operator from its spectrum and the spectrum of an interior finite rank perturbation of it. It turns out that the aforehand knowledge of the index of determinacy of the spectral measure of the Jacobi operator determines the place where the interior perturbation occurs. Remarkably, for finite Jacobi matrices, as well as in the case of infinite index of determinacy, one cannot recover the place of the perturbation.

As a by-product of our research, we give a new characterization of the index of determinacy in terms of the cyclicity of vectors generated by polynomials functions of Jacobi operators.

The paper is organized as follows. In the next section we give some preparatory facts on Jacobi operators. In Section 3, the Weyl and Green functions associated to Jacobi operators are introduced and we prove a criterion for a Green function to be a Weyl function (Theorem 3.7). This result is interpreted later in terms of the index

of determinacy (Corollary 4.5). Section 4 presents a characterization of the index of determinacy (Corollary 4.13) and establishes stability results for the index. We provide conditions for two measures with the same support and different weights to have the same index (Theorems 4.16 and 5.2). Moreover, conditions for two measures with different supports to have the same index are found (Corollary 4.20). We show that finite-rank perturbations of Jacobi operators do not modify the index of determinacy of the corresponding measures. Section 5 presents a new development in the inverse spectral analysis of interior perturbations of Jacobi operators. We consider a two-spectra inverse problem where the information of the index of determinacy is given in advance. This section connects the results of previous sections to the inverse spectral problem studied in [19]. To the best of our knowledge, this is the first time that the index of determinacy is used in inverse spectral theory.

2. Jacobi operators

For a sequence $f = \{f_k\}_{k=1}^{\infty}$ of complex numbers, consider the second order difference expressions

$$(\Upsilon f)_k := b_{k-1}f_{k-1} + q_k f_k + b_k f_{k+1} \quad k \in \mathbb{N} \setminus \{1\}, \tag{2.1a}$$

$$(\Upsilon f)_1 := q_1 f_1 + b_1 f_2, \tag{2.1b}$$

where $q_k \in \mathbb{R}$ and $b_k > 0$ for any $k \in \mathbb{N} = \{1, 2, \ldots\}$. We remark that (2.1b) can be seen as a boundary condition.

DEFINITION 2.1. Let $l_2(\mathbb{N})$ be the space of square summable complex sequences. In this Hilbert space, define the operator J_0 whose domain is the set of sequences having a finite number of non-zero elements and is given by $J_0 f := \Upsilon f$.

Clearly, the operator J_0 is symmetric and therefore closable, so one can consider the operator $\overline{J_0}$ being its closure. By the definition of the matrix representation of an unbounded symmetric operator given in [4, Sec. 47], $\overline{J_0}$ is the operator whose matrix representation with respect to the canonical basis $\{e_n\}_{n=1}^{\infty}$ in $l_2(\mathbb{N})$ is

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}. \tag{2.2}$$

Recall that the element e_n of the canonical basis is the sequence whose elements are zero except for the n-th entry which is 1. Thus $\overline{J_0}$ is the minimal closed symmetric operator such that $\langle e_j, \overline{J_0} e_k \rangle$ is the j, k entry of the matrix above.

REMARK 2.2. The deficiency indices of the symmetric operator $\overline{J_0}$ are either $(1,1)$ or $(0,0)$ [2, Chap. 4, Sec. 1.2], [11, Chap. 7 Thm. 1.1]. When $\overline{J_0}$ has deficiency indices $(1,1)$, respectively $(0,0)$, the matrix (2.2) is said to be in the limit circle case, respectively limit point case [2, Def. 1.3.2]. Thus, if J is a self-adjoint extension of J_0, then either J is a proper closed symmetric extension of $\overline{J_0}$ or $J = \overline{J_0}$.

DEFINITION 2.3. Given the matrix (2.2), we consider J to be a fixed self-adjoint extension of J_0 and refer to it as the Jacobi operator associated with (2.2).

When (2.2) is in the limit circle case, there are more than one Jacobi operators associated with the matrix (2.2).

By setting $f_1 = 1$, a solution of the equations

$$(\Upsilon f)_1 := z f_1, \tag{2.3a}$$
$$(\Upsilon f)_k := z f_k, \quad k \in \mathbb{N} \setminus \{1\}, \tag{2.3b}$$

can be found uniquely by recurrence. This solution, denoted by

$$\pi(z) = \{\pi_k(z)\}_{k=1}^{\infty}, \tag{2.4}$$

is such that $\pi_k(z)$ is a polynomial of degree $k-1$. The elements of the sequence $\pi(z)$ are referred to as the polynomials of the first kind associated to the matrix (2.2). By comparing (2.1) with (2.3), one concludes that for $\pi(z)$ to be in $\ker(J_0^* - zI)$, it is necessary and sufficient that $\pi(z)$ be an element of $l_2(\mathbb{N})$. Of course, $\pi(z) \in \ker(J - zI)$, if and only if $\pi(z) \in \mathrm{dom}(J)$.

Observe that

$$Je_k = b_{k-1}e_{k-1} + q_k e_k + b_k e_{k+1} \quad k \in \mathbb{N} \setminus \{1\},$$
$$Je_1 = q_1 e_1 + b_1 e_2,$$

Thus, by the definition of $\pi_k(z)$, one has

$$e_k = \pi_k(J)e_1 \quad \forall k \in \mathbb{N}. \tag{2.5}$$

This implies that J is simple and e_1 is a cyclic vector (see [4, Sec. 69]). Therefore, if one defines

$$\rho(t) := \langle e_1, E_J(t)e_1 \rangle, \quad t \in \mathbb{R}, \tag{2.6}$$

where E_J is the resolution of the identity given by the spectral theorem, then, by [4, Sec. 69, Thm. 2]), there is a unitary map $\Phi : L_2(\mathbb{R}, \rho) \to l_2(\mathbb{N})$ such that $\Phi^{-1}J\Phi$ is the multiplication by the independent variable defined in its maximal domain. Henceforth we identify the function $\rho(t)$ with the Borel measure ρ which it uniquely determines and call it spectral measure of J (with respect to e_1). Moreover, due to [4, Sec. 69, Thm. 2]), it follows from (2.5) that the function $\pi_k \restriction_{\mathbb{R}}$ belongs to $L_2(\mathbb{R}, \rho)$ for all $k \in \mathbb{N}$, i.e., all moments of ρ are finite (see also [2, Thm. 4.1.3]). The equation (2.5) means that

$$\Phi\pi_k = e_k, \quad \forall k \in \mathbb{N}, \tag{2.7}$$

which implies that the polynomials are dense in $L_2(\mathbb{R},\rho)$ since Φ is unitary. Note also that, due to (2.6), $\int_\mathbb{R} d\rho = 1$ holds.

Now, assume that one is given a measure ρ satisfying $\int_\mathbb{R} d\rho = 1$ and such that all the polynomials are in $L_2(\mathbb{R},\rho)$ and they are dense in this space. Consider the operator of multiplication by the independent variable A in $L_2(\mathbb{R},\rho)$ with

$$\text{dom}(A) = \{f \in L_2(\mathbb{R},\rho) : \int_\mathbb{R} t^2 |f|^2 d\rho(t) < +\infty\}$$

This operator is self-adjoint and $E_A(\Delta) = \chi_\Delta$, where χ_Δ is the characteristic function of the Borel set $\Delta \subset \mathbb{R}$. Therefore, similar to (2.6),

$$\rho(\Delta) = \langle 1, \chi_\Delta 1 \rangle$$

for any Borel set $\Delta \subset \mathbb{R}$. Note that 1 is a cyclic vector for A since the polynomials are dense in $L_2(\mathbb{R},\rho)$. Applying the Gram-Schmidt procedure to the sequence $\{1, t, t^2, \ldots\}$, one obtains an orthonormal basis $\{p_1 = 1, p_2, p_3 \ldots\}$ contained in the domain of A. It can be verified [4, Sec. 69] (cf. [31, Pags. 92, 93]) that the matrix

$$a_{jk} = \langle p_j, A p_k \rangle \quad \forall j, k \in \mathbb{N}. \tag{2.8}$$

is a Jacobi matrix. According to [4, Sec. 47], A is a self-adjoint extension of an operator whose matrix representation is (2.8).

By constructing an isometry Ψ between $L_2(\mathbb{R},\rho)$ and $l_2(\mathbb{N})$ such that $\Psi p_k = e_k$, one arrives at the following central assertion (cf. [2, Thms. 2.3.3 and 4.1.4]).

THEOREM 2.4. *A measure ρ is the spectral measure of a Jacobi operator if and only if $\int_\mathbb{R} d\rho = 1$, all the polynomials are in $L_2(\mathbb{R},\rho)$ and they are dense in this space.*

REMARK 2.5. Any probability measure with finite support is the spectral measure of the operator associated with some finite Jacobi matrix.

DEFINITION 2.6. The Weyl m-function is defined as follows

$$m(z) := \langle e_1, (J - zI)^{-1} e_1 \rangle, \qquad z \notin \sigma(J). \tag{2.9}$$

Here and henceforth, for a given operator T, $\sigma(T)$ denotes the spectrum of it.

Using the map Φ, one concludes from this definition that

$$m(z) = \int_\mathbb{R} \frac{d\rho(t)}{t - z}. \tag{2.10}$$

Thus, by the Nevanlinna representation theorem (see [30, Thm. 5.3]), $m(z)$ is a Herglotz function. Recall that a function f is Herglotz (also called Pick or Nevanlinna-Pick function) when it is holomorphic in the upper half-plane and $\text{Im} f(z) \geqslant 0$ whenever $\text{Im} z > 0$.

3. Green functions for Jacobi operators

DEFINITION 3.1. We use the following notation

$$G(z,k) := \langle e_k, (J - zI)^{-1} e_k \rangle \qquad z \notin \sigma(J)$$

and call $G(z,k)$ the k-th Green function of the Jacobi operator J. Observe that $G(z,1) = m(z)$ (see Definition 2.6).

In view of (2.5) and (2.6), one has

$$G(z,n) = \int_{\mathbb{R}} \frac{\pi_n^2(t) d\rho(t)}{t - z}. \tag{3.1}$$

Thus, for any $n \in \mathbb{N}$, $G(\cdot, n)$ is a Herglotz function. This function is extended analytically to the eigenvalues of J which are simultaneously zeros of π_n since these points are removable singularities.

Using the von Neumann expansion for the resolvent (cf. [32, Chap. 6, Sec. 6.1])

$$(J - zI)^{-1} e_n = -\sum_{k=0}^{N-1} \frac{J^k e_n}{z^{k+1}} + \frac{J^N}{z^N} (J - zI)^{-1} e_n, \quad N \in \mathbb{N},$$

where $z \in \mathbb{C} \setminus \sigma(J)$, one obtains the following asymptotic formula

$$G(z,n) = -\frac{1}{z} + O(z^{-2}) \tag{3.2}$$

as $z \to \infty$ along any ray intersecting the real axis only at 0.

The following definition is taken from [20, Def. 2.1].

DEFINITION 3.2. For a subspace $\mathscr{G} \subset l_2(\mathbb{N})$ (therefore $\mathscr{G}$ is closed), let $P_{\mathscr{G}}$ be the orthogonal projection onto $\mathscr{G}$. Also, define $\mathscr{G}^\perp := \{\phi \in l_2(\mathbb{N}) : \langle \phi, \psi \rangle = 0 \, \forall \psi \in \mathscr{G}\}$ and the subspace $\mathscr{F}_n := \text{span}\{e_k\}_{k=1}^n$. For the operator J given in Definition 2.3, consider the operators

$$J_n^+ := P_{\mathscr{F}_n^\perp} J \restriction_{\mathscr{F}_n^\perp} \quad n \in \mathbb{N}, \quad J_n^- := P_{\mathscr{F}_{n-1}} J \restriction_{\mathscr{F}_{n-1}} \quad n \in \mathbb{N} \setminus \{1\}. \tag{3.3}$$

Here, we have used the notation $J \restriction_{\mathscr{G}}$ for the restriction of J to the set $\mathscr{G}$, that is, $\text{dom}(J \restriction_{\mathscr{G}}) = \text{dom}(J) \cap \mathscr{G}$. Consider also the corresponding m-Weyl functions

$$m_n^+(z) := \langle e_{n+1}, (J_n^+ - zI)^{-1} e_{n+1} \rangle, \qquad m_n^-(z) := \langle e_{n-1}, (J_n^- - zI)^{-1} e_{n-1} \rangle. \tag{3.4}$$

REMARK 3.3. The operator J_n^+ is a self-adjoint extension of the operator whose matrix representation with respect to the basis $\{e_k\}_{k=n+1}^\infty$ of the Hilbert space $(\text{span}\{e_k\}_{k=1}^n)^\perp$ is (2.2) with the first n rows and n columns removed. When J_0 is not essentially self-adjoint, J_n^+ has the same boundary conditions at infinity as the operator J. Clearly, the operator J_n^- lives in an $n-1$-dimensional space.

REMARK 3.4. By [20, Cor. 2.3], the set of zeros of the polynomial π_n coincides with the spectrum of J_n^-.

REMARK 3.5. It follows from [20, Lem. 2.9, and Prop. 3.3] that

$$\sigma(J_n^-) \cap \sigma(J_n^+) = \sigma(J_n^-) \cap \sigma(J).$$

The next assertion is proven in [22, Thm. 2.8] and [20, Prop.,2.3].

PROPOSITION 3.6. *For any $n \in \mathbb{N}$*

$$G(z,n) = \frac{-1}{b_n^2 m_n^+(z) + b_{n-1}^2 m_n^-(z) + z - q_n}, \tag{3.5}$$

where we define $m_1^-(z) \equiv 0$.

NOTATION. Let us denote by μ_n and σ_n the measures given by the Nevanlinna representation of the function $m_n^-(z)$ and $m_n^+(z)$, respectively, that is, $m_n^\pm$ given in (3.4) are the Borel transforms of μ_n and σ_n. Also, denote by ρ_n the measure given by the Nevanlinna representation of the function $G(z,n)$. Thus

$$m_n^+(z) = \int_{\mathbb{R}} \frac{d\sigma_n(t)}{t - z} \tag{3.6}$$

$$m_n^-(z) = \int_{\mathbb{R}} \frac{d\mu_n(t)}{t - z} \tag{3.7}$$

$$G(z,n) = \int_{\mathbb{R}} \frac{d\rho_n(t)}{t - z}. \tag{3.8}$$

We denote by δ_λ the measure

$$\delta_\lambda(\Delta) := \begin{cases} 1 & \lambda \in \Delta \\ 0 & \lambda \notin \Delta \end{cases} \tag{3.9}$$

where $\Delta \subset \mathbb{R}$ is a Borel set.

THEOREM 3.7. *Fix $n \in \mathbb{N}$ and let $G(z,n)$ be the n-th Green function of the Jacobi operator J. If the polynomials are dense in $L_2(\mathbb{R}, \rho_n)$, then $G(z,n)$ is the l-th Green function of some other Jacobi operator for any $l \in \mathbb{N}$.*

Proof. We show that the measure ρ_n satisfies the conditions of Theorem 2.4. In view of (3.1) and (3.8), for any $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} d\rho_n = \int_{\mathbb{R}} \pi_n^2(t) d\rho = \|\pi_n(\cdot)\|_{L_2(\mathbb{R}, \rho)}^2 = 1,$$

where the last equality holds due to (2.7). Moreover, for any $m \in \mathbb{N} \cup \{0\}$,

$$\int_{\mathbb{R}} t^m d\rho_n(t) = \int_{\mathbb{R}} t^m \pi_n^2(t) d\rho(t) < \infty$$

since all the moments of ρ are finite. Thus all the polynomials are in $L_2(\mathbb{R}, \rho_n)$ and by hypothesis the polynomials are dense there. Therefore Theorem 2.4, taking into account (3.8) and (2.10), implies that $G(z, n)$ is the Weyl m-function of some Jacobi operator.

Let $m(z)$ be the Weyl m-function of some Jacobi operator J. We show that $m(z)$ is the l-th Green function for any $l \in \mathbb{N}$. By Proposition 3.6 one has

$$-m(z)^{-1} = b_1^2 m_1^+(z) + z - q_1 = z - q_1 + \sum_{k=1}^{\infty} \frac{\eta_k}{\alpha_k - z}.$$

Thus, since m_1^+ is the Weyl m-function of the Jacobi operator J_1^+, it follows from Theorem 2.4 that the measure

$$\sigma := \sum_{k=1}^{\infty} \eta_k \delta_{\alpha_k}$$

is such that the polynomials are in $L_2(\mathbb{R}, \sigma)$ and they are dense in this space. One can also write

$$-m(z)^{-1} = z - q_1 + \left(\sum_{k=1}^{l-1} + \sum_{k=l}^{\infty} \right) \frac{\eta_k}{\alpha_k - z}. \tag{3.10}$$

Note that the measure

$$\widetilde{\rho} := \sum_{k \geqslant l} \eta_k \delta_{\alpha_k}$$

has also the property that all the polynomials form a dense linear subset of $L_2(\mathbb{R}, \widetilde{\rho})$.

Indeed, on one hand the fact that all the polynomials are in $L_2(\mathbb{R}, \sigma)$ implies the same occurs for $L_2(\mathbb{R}, \widetilde{\rho})$. On the other hand, if there is $h \in L_2(\mathbb{R}, \widetilde{\rho})$, such that $\langle h, t^m \rangle_{L_2(\mathbb{R}, \widetilde{\rho})} = 0$ for all $m \in \mathbb{N} \cup \{0\}$, then

$$\sum_{k=l}^{\infty} \alpha_k^m h(\alpha_k) \eta_k = 0 \quad \text{for all } m \in \mathbb{N} \cup \{0\}.$$

Thus, by considering the function

$$\widetilde{h}(\alpha_k) = \begin{cases} h(\alpha_k) & k \geqslant l \\ 0 & k < l, \end{cases}$$

one obtains that

$$\sum_{k=1}^{\infty} \alpha_k^m \widetilde{h}(\alpha_k) \eta_k = 0 \quad \text{for all } m \in \mathbb{N} \cup \{0\}.$$

By the density of the polynomials in $L_2(\mathbb{R}, \sigma)$, one concludes that the norm in $L_2(\mathbb{R}, \sigma)$ of $\widetilde{h}$ vanishes, which in turn implies that $\|h\|_{L_2(\mathbb{R}, \widetilde{\rho})} = 0$.

For completing the proof, set

$$\widetilde{q}_l := q_1, \quad \widetilde{b}_{l-1}^2 \widetilde{m}_l^- := \sum_{k=1}^{l-1} \frac{\eta_k}{\alpha_k - z}, \quad \widetilde{b}_l^2 \widetilde{m}_l^+ := \sum_{k=l}^{\infty} \frac{\eta_k}{\alpha_k - z},$$

and substitute these expressions into (3.10) to obtain

$$-m(z)^{-1} = z - \widetilde{q}_l + \widetilde{b}_{l-1}^2 \widetilde{m}_l^- + \widetilde{b}_l^2 \widetilde{m}_l^+ \,.$$

Finally, note that the r. h. s of the last equation is the l-th Green function of some Jacobi operator by Proposition 3.6.   □

4. Index of determinacy

We begin this section by introducing the following notation. For a nonnegative Borel measurable function h and a Borel measure ν, we denote by $h\nu$ the measure which associates to any Borel set Δ the value

$$\int_\Delta h d\nu \,.$$

Thus $h\nu$ is the measure with density h with respect to ν.

The fact that a measure μ in $\mathcal{M}$, the set of Borel measures on $\mathbb{R}$ with infinite support and all their moments finite (see Introduction), is indeterminate N-extremal or determinate may be changed by adding or substracting the mass at only one point.

PROPOSITION 4.1. *Let* $\mu \in \mathcal{M}$ *be indeterminate* N-*extremal.*

(a) If $\lambda \notin \mathrm{supp}\,\mu$, *then* $\mu + a\delta_\lambda$ *(a > 0) is not* N-*extremal.*

(b) If $\lambda \in \mathrm{supp}\,\mu$, *then* $\mu - \mu(\{\lambda\})\delta_\lambda$ *is determinate.*

$\mathrm{supp}\,\mu$ *is the minimal closed set whose complement has* μ-*zero measure.*

Proof. (a) (Communicated by A. Durán) Let $\widetilde{\mu}$ be an N-extremal measure having the same moments as μ and such that

$$\widetilde{\mu}(\{\lambda\}) > 0. \tag{4.1}$$

The existence of such a $\widetilde{\mu}$ is guaranteed by [2, Thm. 3.41] and [31, Thm. 5]. Thus, the measures $\mu + a\delta_\lambda$ and $\widetilde{\mu} + a\delta_\lambda$ have the same moments, but

$$\mu(\{\lambda\}) + a < \widetilde{\mu}(\{\lambda\}) + a$$

as a consequence of (4.1) and the fact that $\mu(\{\lambda\}) = 0$. The last inequality shows that $\mu + a\delta_\lambda$ is not N-extremal since, by [2, Thm. 3.41] and [31, Thm. 5], if an N-extremal measure gives weight to a point, then no other solution of the moment problem can give more weight to that point.

(b) ([12, Thm. 7]) We give an alternative proof based on [2, Thm. 3.4]. Define

$$\widetilde{\mu} := \mu - \mu(\{\lambda\})\delta_\lambda \,.$$

Note that $\widetilde{\mu} \in \mathcal{M}$ and $\widetilde{\mu}(\{\lambda\}) = 0$. If $\widetilde{\mu}$ is indeterminate, then there exists a solution of the moment problem γ such that $\gamma(\{\lambda\}) > 0$ due to [2, Thm. 3.41] (see also [31, Thm. 5]). Now

$$\widetilde{\gamma} := \gamma + \mu(\{\lambda\})\delta_\lambda$$

is a solution of the moment problem associated with μ and gives more weight to λ than μ which is a contradiction $\quad\square$

REMARK 4.2. Since the polynomials are dense in $L_2(\mathbb{R}, \mu)$ if and only if μ is N-extremal, part (a) of Proposition 4.1 shows that the density can be destroyed by adding just one point mass to the measure.

4.1. Characterization of the index of determinacy

DEFINITION 4.3. For a determinate measure μ, Berg and Durán introduce in [13] the index of determinacy as follows.

$$\mathrm{ind}_z \, \mu = \sup\{k \in \mathbb{N} \cup \{0\} : |t - z|^{2k} \, \mu \text{ is determinate}\},$$

where $z \in \mathbb{C}$. Since the index of determinacy happens to be constant [13, Lem. 3.5] at complex numbers outside the support of μ, one can define

$$\mathrm{ind} \, \mu := \mathrm{ind}_z \, \mu \qquad z \notin \mathrm{supp} \, \mu.$$

In [14, Lem. 2.1], the index of determinacy of a measure is characterized when the measure is multiplied by an arbitrary polynomial. The next assertion, which follows directly from results due to C. Berg and A. Durán, describes the general situation.

PROPOSITION 4.4. *Let r be a polynomial with simple zeros, $\mu \in \mathcal{M}$ and*

$$l := \#\{\text{zeros of } r \text{ outside } \mathrm{supp} \, \mu\}.$$

Then

(a) μ *is determinate and* $\mathrm{ind} \, \mu = l - 1$ *if and only if* $|r|^2 \mu$ *is indeterminate and N-extremal.*

(b) μ *is determinate and* $\mathrm{ind} \, \mu = k \geqslant l$ *if and only if* $|r|^2 \mu$ *is determinate and $k = \mathrm{ind} |r|^2 \mu + l$.*

(c) μ *is indeterminate or μ is determinate and* $\mathrm{ind} \, \mu < l - 1$ *if and only if* $|r|^2 \mu$ *is indeterminate and not N-extremal.*

Proof. (a) ($\Rightarrow$) Let $a \notin \mathrm{supp} \, \mu$ be a zero of r. Write $r = (t - a)\hat{p}$. Since $\mathrm{ind} \, \mu = l - 1$, we get $\mathrm{ind} |\hat{p}|^2 \mu = 0$ by [14, Lem. 2.1(ii)]. Thus, $|t - a|^2 |\hat{p}|^2 \mu$ is indeterminate by Definition 4.3. Due to [13, Lem. A(1)] (cf. [29]), $|t - a|^2 |\hat{p}|^2 \mu$ is N-extremal. ($\Leftarrow$) Now, assume that $|r|^2 \mu$ is indeterminate N-extremal and let a and $\hat{p}$ be as before.

Using the contrapositive of [13, Prop. 3.2], one has $|\hat{p}|^2\mu$ is determinate. $|\hat{p}|^2\mu$ has zero index of determinacy since, otherwise $|r|^2\mu$ would be determinate. Applying again [14, Lem. 2.1(ii)] to $|\hat{p}|^2\mu$, one proves the assertion.

(b) ($\Rightarrow$) This is [14, Lem. 2.1(ii)]. ($\Leftarrow$) $|r|^2\mu$ determinate implies μ is determinate by [13, Prop. 3.2(i)]. We must have $\operatorname{ind}\mu \geqslant l$ since $\operatorname{ind}\mu < l$ implies $|r|^2\mu$ is indeterminate by [14, Lem. 2.1(i)]. From [14, Lem. 2.1(ii)] follows $k = \operatorname{ind}\mu$

(c) ($\Rightarrow$) If μ is indeterminate apply [13, Prop. 3.2(i)]. If μ is determinate then by [14, Lem. 2.1(i)] $|r|^2\mu$ is an indeterminate measure and by (a) above it cannot be N-extremal. ($\Leftarrow$) If μ is determinate then $\operatorname{ind}\mu < l-1$ since otherwise we are in cases (a) or (b) above. $\quad\square$

COROLLARY 4.5. *Let ρ be the spectral measure of a Jacobi operator J. For the n-th Green function $G(z,n)$ of J to be the l-th Green function of some other Jacobi operator for any $l \in \mathbb{N}$ it is necessary and sufficient that*

$$\operatorname{ind}\rho \geqslant \#\{\text{zeros of } \pi_n \text{ outside } \operatorname{supp}\rho\} - 1. \tag{4.2}$$

Proof. Suppose that (4.2) holds. Then, by Proposition 4.4, the polynomials are dense in $L_2(\mathbb{R}, \pi_n^2\rho)$. One direction of the assertion then follows from Theorem 3.7. If one assumes that

$$\operatorname{ind}\rho < \#\{\text{zeros of } \pi_n \text{ outside } \operatorname{supp}\rho\} - 1,$$

then the polynomials are not dense in $L_2(\mathbb{R}, \pi_n^2\rho)$ by Proposition 4.4 . Therefore $\pi_n^2\rho$ cannot be the spectral measure of a Jacobi operator due to Theorem 2.4 and then, by (2.10) and (3.1), $G(z,n)$ is not the Weyl m-function of a Jacobi operator. $\quad\square$

LEMMA 4.6. *Let μ be a determinate measure. If a measure ν is such that $\nu(\mathscr{A}) \leqslant \mu(\mathscr{A})$, for any Borel set $\mathscr{A}$, then ν is determinate.*

Proof. (Communicated by C. Berg) Suppose that there is a measure σ different from ν having the same moments as ν. Then $\sigma + \tau$ and $\nu + \tau$ are two measures with the same moments, as long as τ has finite moments. If one takes $\tau = \mu - \nu$, then $\mu = \nu + \tau$ has the same moments as $\sigma + \tau$, which is a contradiction because μ is determinate. $\quad\square$

With the help of Definition 4.3, one can give more general and precise statements regarding what happens when mass points are added or removed from a measure in $\mathscr{M}$. The next statement is essentially a reformulation of results by C. Berg and A. Durán.

PROPOSITION 4.7. *Let $\mathscr{F} \subset \mathbb{R}$ be a finite set and*

$$\beta := \sum_{\xi \in \mathscr{F}} \beta_\xi \delta_\xi, \qquad \beta_\xi > 0, \tag{4.3}$$

$\mu \in \mathscr{M}$, and $l := \#\{\xi \in \mathscr{F} \text{ outside } \operatorname{supp}\mu\}$.

(a) μ is determinate and $\operatorname{ind}\mu = l - 1$ if and only if $\mu + \beta$ is indeterminate N-extremal.

(b) μ is determinate and $\operatorname{ind}\mu = k \geqslant l$ if and only if $\mu + \beta$ is determinate and $k = \operatorname{ind}(\mu + \beta) + l$.

(c) μ is indeterminate or μ is determinate with $\operatorname{ind}\mu < l - 1$ if and only if $\mu + \beta$ is indeterminate and not N-extremal.

Proof. (a) One direction is [13, Thm. 3.6] and the converse is [13, Lem. 3.7, Thm. 3.9].

(b) Let

$$\widetilde{\beta} = \beta + \sum_{i=1}^{k+1-l} a_i \delta_{\xi_i}, \qquad a_i > 0$$

where $\xi_i \notin \operatorname{supp}(\mu + \beta)$ for $i \in \{1,\dots,k+1-l\}$. Then $\widetilde{\beta}$ is a measure such that $\#\{\xi \in \operatorname{supp}\widetilde{\beta} \text{ outside } \operatorname{supp}\mu\} = k+1$. Applying (a) above we get $\operatorname{ind}\mu = k$ if and only if $\mu + \widetilde{\beta} = \mu + \beta + \sum_{i=1}^{k+1-l} a_i \delta_{\xi_i}$ is indeterminate N-extremal and this happens if and only if $\operatorname{ind}(\mu + \beta) = k - l$ by (a) again since $\xi_i \notin \operatorname{supp}(\mu + \beta)$.

(c) ($\Rightarrow$) Let $\mathscr{C} \subset \{\xi \in \mathscr{F} \text{ outside } \operatorname{supp}\mu\}$ be such that $\#\mathscr{C} = \operatorname{ind}\mu + 1 < l$. Define

$$\widetilde{\gamma} := \mu + \sum_{\lambda \in \mathscr{C}} \beta_\lambda \delta_\lambda .$$

By item (a), $\widetilde{\gamma}$ is indeterminate N-extremal. By Proposition 4.1 and Lemma 4.6, $\mu + \beta$ is indeterminate not N-extremal. ($\Leftarrow$) If μ is determinate then $\operatorname{ind}\mu < l - 1$ since otherwise we are in cases (a) or (b) above. $\square$

REMARK 4.8. A measure of finite index of determinacy is discrete (cf. [13, Cor. 3.4]). In view of Proposition 4.7(a), this is a consequence of the fact that an indeterminate N-extremal measure is discrete [2, Chap. 3 Sec. 2 Pag. 101].

REMARK 4.9. There are measures with infinite index of determinacy being discrete. Indeed, take an indeterminate N-extremal measure and remove the mass at an infinite set of points. By Lemma 4.6 and Proposition 4.7(a), the index of determinacy of the modified measure is not finite.

The following assertion is related to [27, Rem. p. 231, Thm. 5] (see also [13] Lemma B and the comment before Lemma D)

LEMMA 4.10. *Let $\mathscr{I} \subset \mathbb{R}$ be an infinite discrete set and $\widetilde{\mathscr{F}}$ a finite set in $\mathbb{R}$ such that $\mathscr{I} \cap \widetilde{\mathscr{F}} = \emptyset$. Consider a sequence $\{\beta_\xi\}_{\xi \in \mathscr{I} \cup \widetilde{\mathscr{F}}}$ of positive numbers. Define*

$$\mu := \sum_{\xi \in \mathscr{I}} \beta_\xi \delta_\xi \quad and \quad \widetilde{\mu} = \mu - \sum_{\xi \in \mathscr{F}} \beta_\xi \delta_\xi + \sum_{\xi \in \widetilde{\mathscr{F}}} \beta_\xi \delta_\xi ,$$

where $\mathscr{F}$ is a finite subset of $\mathscr{I}$. Suppose that $\mu \in \mathscr{M}$ is either indeterminate N-extremal or determinate with finite index of determinacy. For $\#\mathscr{F} = \#\widetilde{\mathscr{F}}$ to hold, it is necessary and sufficient that either $\operatorname{ind}\mu = \operatorname{ind}\widetilde{\mu}$ or μ and $\widetilde{\mu}$ are simultaneously indeterminate N-extremal.

Proof. ($\Rightarrow$)

i) For the case when μ is indeterminate N-extremal, the proof is essentially given in [12, Thm. 8].

ii) If $0 \leqslant \operatorname{ind}\mu = k < +\infty$, choose a set $\mathscr{A} \subset \mathbb{R} \setminus (\mathscr{I} \cup \widetilde{\mathscr{F}})$ such that $\#\mathscr{A} = k+1$ and consider the measure

$$\mu + \sum_{\xi \in \mathscr{A}} a_\xi \delta_\xi,$$

where $a_\xi > 0$. By Proposition 4.7 (a), this measure is indeterminate N-extremal. It then follows from i) that the measure

$$\widetilde{\mu} + \sum_{\xi \in \mathscr{A}} a_\xi \delta_\xi$$

is indeterminate N-extremal too. Using again Proposition 4.7 (a) we get that $\operatorname{ind}\widetilde{\mu} = k = \operatorname{ind}\mu$.

($\Leftarrow$)

Assume without loss of generality that $\#\mathscr{F} < \#\widetilde{\mathscr{F}}$ and let $\mathscr{G} \subset \widetilde{\mathscr{F}}$ be such that $\#\mathscr{F} = \#\mathscr{G}$. Then

$$\widetilde{\mu} = \nu + \sum_{\xi \in \widetilde{\mathscr{F}} \setminus \mathscr{G}} \beta_\xi \delta_\xi,$$

where

$$\nu = \mu - \sum_{\xi \in \mathscr{F}} \beta_\xi \delta_\xi + \sum_{\xi \in \mathscr{G}} \beta_\xi \delta_\xi.$$

By what was proven in i) and ii) above, either $\operatorname{ind}\nu = \operatorname{ind}\mu$ or μ and ν are simultaneously indeterminate N-extremal. By Proposition 4.7 neither $\operatorname{ind}\widetilde{\mu} = \operatorname{ind}\mu$ nor μ and $\widetilde{\mu}$ are simultaneously indeterminate N-extremal since $\widetilde{\mathscr{F}} \setminus \mathscr{G}$ is not in the support of ν. $\square$

A consequence of the previous lemma is the following result,

LEMMA 4.11. *Let $\mathscr{I} \subset \mathbb{R}$ be an infinite discrete set and $\{\beta_\xi\}_{\xi \in \mathscr{I}}$ be a sequence of positive numbers. Assume that $\mathscr{F}_1, \mathscr{F}_2 \subset \mathscr{I}$ are finite sets and $\sum_{\xi \in \mathscr{I} \setminus \mathscr{F}_1} \beta_\xi \delta_\xi$ is N-extremal not having infinite index of determinacy.*

$$\#\mathscr{F}_1 = \#\mathscr{F}_2$$

if and only if either

$$\operatorname{ind} \sum_{\xi \in \mathscr{I} \setminus \mathscr{F}_1} \beta_\xi \delta_\xi = \operatorname{ind} \sum_{\xi \in \mathscr{I} \setminus \mathscr{F}_2} \beta_\xi \delta_\xi$$

or the measures $\sum_{\xi \in \mathscr{I} \setminus \mathscr{F}_1} \beta_\xi \delta_\xi$ and $\sum_{\xi \in \mathscr{I} \setminus \mathscr{F}_2} \beta_\xi \delta_\xi$ are simultaneously indeterminate N-extremal.

Proof. Observe that

$$\sum_{\xi\in\mathscr{I}\setminus\mathscr{F}_1}\beta_\xi\,\delta_\xi = \sum_{\xi\in\mathscr{I}\setminus\mathscr{F}_2}\beta_\xi\,\delta_\xi - \sum_{\xi\in(\mathscr{I}\setminus\mathscr{F}_2)\cap\mathscr{F}_1}\beta_\xi\,\delta_\xi + \sum_{\xi\in(\mathscr{I}\setminus\mathscr{F}_1)\cap\mathscr{F}_2}\beta_\xi\,\delta_\xi$$

and apply lemma 4.10, noting that $\#(\mathscr{I}\setminus\mathscr{F}_2)\cap\mathscr{F}_1 = \#(\mathscr{I}\setminus\mathscr{F}_1)\cap\mathscr{F}_2$ if and only if $\#\mathscr{F}_1 = \#\mathscr{F}_2$. □

THEOREM 4.12. *Let J be a Jacobi operator (see Definition 2.3) and ρ its spectral measure. Assume that ρ is a determinate measure and r is a polynomial with simple zeros. Then*

$$\operatorname{ind}\rho \geqslant \#(\{\text{zeros of } r\}\setminus\sigma(J)) - 1 \tag{4.4}$$

if and only if $r(J)e_1$ is a cyclic vector for J.

Proof. ($\Leftarrow$) Let $u = r(J)e_1$ and assume that u is a cyclic vector for J, i.e.,

$$\overline{\operatorname{span}_{k\in\mathbb{N}\cup\{0\}}\{J^k u\}} = l_2(\mathbb{N}). \tag{4.5}$$

It follows from [4, Sec. 69, Thm. 2] that there is a unitary map $\Phi : L_2(\mathbb{R},\mu) \to l_2(\mathbb{N})$, where $\mu(\Delta) := \langle u, E(\Delta)u\rangle$ for any Borel set Δ of $\mathbb{R}$ (see Section 2), such that $\Phi J\Phi^{-1}$ is the operator of multiplication by the independent variable. Thus, (4.5) is equivalent to

$$\overline{\operatorname{span}_{k\in\mathbb{N}\cup\{0\}}\{t^k\}} = L_2(\mathbb{R},\mu).$$

For finishing this part of the proof, it only remains to note that

$$\mu = |r|^2\rho$$

and recur to Proposition 4.4 recalling that N-extremality is equivalent to density of polynomials (see Introduction) and that $\sigma(J) = \operatorname{supp}\rho$.

($\Rightarrow$) First note that $r(J)e_1$ is in $\operatorname{dom}(J^k)$ for all $k\in\mathbb{N}\cup\{0\}$ since $r(J)e_1$ is a finite vector, that is, the corresponding sequence has a finite number of elements different from zero. By Proposition 4.4, (4.4) implies

$$\overline{\operatorname{span}_{k\in\mathbb{N}\cup\{0\}}\{t^k\}} = L_2(\mathbb{R},|r|^2\rho) \tag{4.6}$$

and let $w\in l_2(\mathbb{N})$ be such that

$$\left\langle J^k r(J)e_1, w\right\rangle = 0 \quad \text{for all } k = 0,1,2,\ldots$$

This means that

$$\int_{\mathbb{R}} h(t)t^k\overline{r(t)}d\rho = 0 \quad \text{for all } k = 0,1,2,\ldots, \tag{4.7}$$

where $w = \Phi h$ (Φ here corresponds to ρ, see Section 2). If one writes $h = \widetilde{h}r$, then $\widetilde{h}\in L_2(\mathbb{R},|r|^2\rho)$ since

$$+\infty > \int_{\mathbb{R}} |h|^2\,d\rho = \int_{\mathbb{R}} \left|\widetilde{h}\right|^2 |r|^2\,d\rho.$$

Hence, taking into account (4.7), one has, for any $k \in \mathbb{N} \cup \{0\}$,

$$0 = \int_{\mathbb{R}} h(t)t^k \overline{r(t)} d\rho = \int_{\mathbb{R}} \widetilde{h}(t)t^k |r(t)|^2 d\rho = \left\langle t^k, \widetilde{h} \right\rangle_{L_2(\mathbb{R},|r|^2\rho)}.$$

Due to (4.6), this implies that $\widetilde{h} = 0$, viz.,

$$0 = \left\| \widetilde{h} \right\|_{L_2(\mathbb{R},|r|^2\rho)}^2 = \int_{\mathbb{R}} \left| \widetilde{h} \right|^2 |r|^2 d\rho = \int_{\mathbb{R}} |h|^2 d\rho.$$

Whence $\|h\|_{L_2(\mathbb{R},\rho)} = 0$. Thus, the vector w must vanish which means that $r(J)e_1$ is a cyclic vector. $\square$

In fact, as shown below, $\mathrm{ind}\,\rho$ is the only natural number satisfying the assertion of Theorem 4.12.

COROLLARY 4.13. *Let J and ρ and r be as in Theorem 4.12. If $k \in \mathbb{N} \cup \{0\}$ is such that $r(J)e_1$ is a cyclic vector for J whenever*

$$\#(\{zeros\ of\ r\} \setminus \sigma(J)) \leqslant k+1 \tag{4.8}$$

and it is not a cyclic vector for J whenever

$$\#(\{zeros\ of\ r\} \setminus \sigma(J)) > k+1, \tag{4.9}$$

then $k = \mathrm{ind}\,\rho$.

Proof. Suppose that $\mathrm{ind}\,\rho < k$. Choose a polynomial r such that $k = \#(\{zeros\ of\ r\} \setminus \sigma(J)) - 1$. It follows from (4.8) that $r(J)e_1$ is a cyclic vector. But $\mathrm{ind}\,\rho < \#(\{zeros\ of\ r\} \setminus \sigma(J)) - 1$ implies that $r(J)e_1$ is not cyclic by Theorem 4.12. So, assuming $\mathrm{ind}\,\rho < k$ leads to a contradiction. Therefore $\mathrm{ind}\,\rho \geqslant k$. Let $\mathrm{ind}\,\rho > k$. If r is such that $\mathrm{ind}\,\rho = \#(\{zeros\ of\ r\} \setminus \sigma(J)) - 1$, then Theorem 4.12 implies that $r(J)e_1$ is cyclic vector. But in this case $k < \#(\{zeros\ of\ r\} \setminus \sigma(J)) - 1$ and (4.9) implies that $r(J)e_1$ is a not cyclic vector. We get again a contradiction. Therefore $k = \mathrm{ind}\,\rho$. $\square$

4.2. Stability of the index of determinacy

Let us study the stability of the index of determinacy and the N-extremality for measures. First we deal with the case when the support of the measure does not change.

PROPOSITION 4.14. *Changing the weights of a measure can change its index of determinacy.*

Proof. Consider the following criterion for a measure to be determinate [21, Thm. 5.2, pag. 84]: If there is $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{\varepsilon|t|} d\mu < \infty, \tag{4.10}$$

then μ is determinate. Thus, an indeterminate, N-extremal measure ν, can be transformed into μ by changing the weights so that (4.10) holds. Now consider a measure σ of index n obtained from ν by removing the mass at $n+1$ points. The measure $\tilde{\sigma}$ obtained by removing from μ the mass at the same $n+1$ points has index of determinacy greater than n. $\square$

PROPOSITION 4.15. *By changing a finite number of weights the index of determinacy is preserved.*

Proof. From Proposition 4.7(a), a measure has an infinite index of determinacy if and only if, after adding any finite number of mass points, it remains determinate. Thus, changing a finite number of weights do not alter the infinite index of determinacy. Suppose that ind $\mu < \infty$, then, by Proposition 4.7(a) (see also [14, Pag. 129]), μ is obtained by removing from an indeterminate N-extremal measure μ_0 the mass at a finite set of points. According to [27, Thm. 5(b)] the measure $\tilde{\mu}_0$ obtained by modifying the weight of μ_0 at one mass point is indeterminate N-extremal. Adding to $\tilde{\mu}_0$ the same masses at the same points that were substracted from μ_0 to obtain μ yields a measure η with the same index of determinacy as μ. Note that η is equal to μ with one weight modified. $\square$

THEOREM 4.16. *Let J and $\widehat{J}$ be Jacobi operators as given in Definition 2.3 with spectral measures ρ and $\widehat{\rho}$, respectively. Suppose that, for some $n \in \mathbb{N} \setminus \{1\}$,*

$$\#(\sigma(J_n^-) \cap \sigma(J)) = \#(\sigma(\widehat{J}_n^-) \cap \sigma(\widehat{J})), \tag{4.11}$$

where J_n^- and $\widehat{J}_n^-$ are given in Definition 3.2. Consider the measure ρ_n given in (3.8) and the corresponding measure $\widehat{\rho}_n$ for $\widehat{J}$. If ind $\rho_n =$ ind $\widehat{\rho}_n$ or ρ_n and $\widehat{\rho}_n$ are simultaneously indeterminate N-extremal, then ind $\rho =$ ind $\widehat{\rho}$. Conversely, if

$$\text{ind}\,\rho = \text{ind}\,\widehat{\rho} \geqslant n - (\#(\sigma(J_n^-) \cap \sigma(J)) + 1),$$

then, ind $\rho_n =$ ind $\widehat{\rho}_n$ or ρ_n and $\widehat{\rho}_n$ are simultaneously indeterminate N-extremal.

Proof. Due to the fact that $\sigma(J_n^-)$ is simple, (4.11) and Remark 3.4 imply that the number of zeros of π_n that are not in the supp ρ is equal to the number of zeros of $\widehat{\pi}_n$ that are not in the supp $\widehat{\rho}$. The assertion then follows from Proposition 4.4 $\square$

The following statement gives a criterion for two measures with the same support and different weights to have the same index of determinacy.

COROLLARY 4.17. *Let J, $\widehat{J}$, ρ, $\widehat{\rho}$, ρ_n, and $\widehat{\rho}_n$ be as in the previous theorem. Assume that J and $\widehat{J}$ are isospectral Jacobi operators. If, for some $n \in \mathbb{N}$, $\rho_n = \widehat{\rho}_n$ and ρ_n is N-extremal, then ind $\rho =$ ind $\widehat{\rho}$*

Proof. For any Borel set Δ,

$$\int_\Delta \pi_n^2(t)d\rho(t) = \rho_n(\Delta) = \widehat{\rho}_n(\Delta) = \int_\Delta \widehat{\pi}_n^2(t)d\widehat{\rho}(t). \tag{4.12}$$

This implies that the zeros of π_n that are in the $\operatorname{supp}\rho$ coincide with the zeros of $\widehat{\pi}_n$ that are in the $\operatorname{supp}\widehat{\rho}$. Therefore (4.11) holds. It remains to apply Theorem 4.16.   □

If $\rho \in \mathscr{M}$ is determinate, then ρ is the spectral measure of a Jacobi operator J in the sense of (2.6). In this case, J is the unique self-adjoint extension of J_0 (see Definition 2.1), i. e., J_0 is essentially self-adjoint [2, Thm. 2.2], [31, Thm. 2].

The following assertion appears in [15, Thm. 1]. We reproduce it here with a brief proof for the reader's convenience.

PROPOSITION 4.18. *Let ρ be a determinate measure. For the measure ρ to have index of determinacy k, it is necessary and sufficient that J_0^l is essentially self-adjoint for $l = 1, \ldots, k+1$ and J_0^{k+2} is not essentially self-adjoint. The measure ρ has infinite index of determinacy if and only if J_0^l is essentially self-adjoint for all $l \in \mathbb{N}$.*

Proof. Let $\mathbb{P}$ be the set of all polynomials, i. e.,

$$\mathbb{P} = \left\{ \sum_{k=0}^{N} a_k t^k : N \in \mathbb{N} \cup \{0\}, \, t \in \mathbb{R}, \, a_k \in \mathbb{C} \right\}.$$

The unitary map Φ introduced in Section 2 satisfies (2.7) and therefore

$$\Phi\mathbb{P} = \operatorname{dom}(J_0).$$

J_0^l is essentially self-adjoint if and only if $\overline{\operatorname{ran}(J_0^l \pm iI)} = l_2(\mathbb{N})$ [28, Cor. to Thm VIII.3]. This implies, by means of the unitary map Φ, that this happens if and only if

$$\overline{(t^l \pm i)\mathbb{P}} = L_2(\mathbb{R}, \rho). \tag{4.13}$$

By [15, Lemma], (4.13) is equivalent to

$$\overline{\mathbb{P}} = L_2(\mathbb{R}, (1 + t^{2l})\rho). \tag{4.14}$$

It follows from

$$1 \leqslant \frac{(1 + x^2)^l}{1 + x^{2l}} \leqslant 2^{l-1}$$

that the polynomials are dense in $L_2(\mathbb{R}, (1 + t^{2l})\rho)$ if and only if they are dense in $L_2(\mathbb{R}, (1 + t^2)^l\rho)$. Thus, by Definition 4.3, $\operatorname{ind}\rho = k$ if and only if (4.14) is satisfied for $l = 1, \ldots, k+1$ but does not hold for $l = k+2$.   □

As a consequence of the previous proposition, one has the following assertion.

COROLLARY 4.19. *Let J be a Jacobi operator and ρ its spectral measure. If ρ is determinate, then the index of determinacy of ρ coincides with the index of determinacy of σ_n, as defined in (3.6), for any $n \in \mathbb{N}$. The measure ρ is indeterminate N-extremal if and only if σ_n is indeterminate N-extremal.*

Proof. Define $B := \mathbb{O} \oplus J_n^+ \restriction_{\text{dom}(J_0)}$, where $\mathbb{O}$ is the null operator in $\text{span}\{e_k\}_{k=1}^n$ and $\oplus$ indicates that we are considering the orthogonal sum of operators (see [16, Sec. 3.6]). Note that the domain of J_0^l and B^l is the same for all $l \in \mathbb{N}$. Since the matrix corresponding to the operator J_0 is tridiagonal, there exists a finite rank operator C such that $B^l + C = J_0^l$ for any $l \in \mathbb{N}$. Note that the rank of C depends on l and n. By the Kato-Rellich theorem (see [24, Chap. 5, Sec. 4, Thm. 4.4]) B^l is essentially self-adjoint if and only if J_0^l is essentially self-adjoint. Now, since $B^l = \mathbb{O}^l \oplus (J_n^+ \restriction_{\text{dom}(J_0)})^l$, B^l is essentially self-adjoint if and only if $(J_n^+ \restriction_{\text{dom}(J_0)})^l$ is essentially self-adjoint. The result follows from Proposition 4.18 and [2, Addenda and Problems to Chap. 1].   □

COROLLARY 4.20. *Let J and $\widetilde{J}$ be Jacobi operators as defined in Section 2 such that $\widetilde{J} = J + C$, where $\text{rank}(C) < \infty$, and denote by ρ and $\widetilde{\rho}$ the corresponding spectral measures. If ρ is determinate, then $\widetilde{\rho}$ is determinate and $\text{ind}\rho = \text{ind}\widetilde{\rho}$. If ρ is indeterminate N-extremal, then $\widetilde{\rho}$ is indeterminate N-extremal.*

Proof. Since $\text{rank}(C) < \infty$, there is $n \in \mathbb{N}$ such that $J_n^+ = \widetilde{J}_n^+$. Therefore, taking into account that, according to [2, Addenda and Problems to Chap. 1], the matrix representations of J and J_n^+ are simultaneously either limit circle case or limit point case, one concludes that ρ and $\widetilde{\rho}$ are simultaneously either determinate or indeterminate N-extremal. If ρ is determinate, then the assertion follows from Corollary 4.19.   □

REMARK 4.21. In the previous proof, one could have used [24, Chap. 5, Sec. 4, Thms. 4.3 and 4.4] (Kato-Rellich theorem) to show that J_0 and $\widetilde{J}_0$ are simultaneously either essentially self-adjoint or not.

The next assertion uses the measures introduced in (3.6), (3.7) and (3.8).

THEOREM 4.22. *For the measure ρ_n to be determinate with index of determinacy k (indeterminate N-extremal) it is necessary and sufficient that $\sigma_n + \mu_n$ is determinate with index of determinacy k (indeterminate N-extremal).*

Proof. Due to Proposition 4.4, the $\text{ind}\rho_n = k$ if and only if

$$\text{ind}\rho = k + \#(\sigma(J_n^-) \setminus \sigma(J)).$$

This is so, because the set of zeros of π_n is the spectrum of J_n^- (see Remark 3.4). Since, according to Corollary 4.19, $\text{ind}\rho = \text{ind}\sigma_n$ one has, using Proposition 4.7,

$$\text{ind}(\sigma_n + \mu_n) = k + \#(\sigma(J_n^-) \setminus \sigma(J)) - \#(\sigma(J_n^-) \setminus \sigma(J_n^+)).$$

In view of Remark 3.5 and Proposition 4.7(b), the last expression yields that $\text{ind}(\sigma_n + \mu_n) = k$ if and only if $\text{ind}\rho_n = k$.

By Proposition 4.4(a), the measure ρ_n is indeterminate N-extremal if and only if

$$\text{ind}\rho = \#\{\text{zeros of } \pi_n \text{ outside } \text{supp}\rho\} - 1.$$

Using Remark 3.4 and Corollary 4.19, one concludes that the last expression is equivalent to

$$\operatorname{ind}\sigma_n = \#(\sigma(J_n^-)\setminus\sigma(J)) - 1 = \#(\sigma(J_n^-)\setminus\sigma(J_n^+)) - 1.$$

This happens if and only if that $\sigma_n + \mu_n$ is indeterminate N-extremal by Proposition 4.7(a).   □

COROLLARY 4.23. *Let ρ be the spectral measure of some Jacobi operator J as in* (2.6). *Define the measure β by* (4.3) *with*

$$\#(\mathscr{F}\setminus\operatorname{supp}\rho) = \#(\sigma(J_n^-)\setminus\sigma(J)).$$

The measure ρ_n has index of determinacy k (is indeterminate N-extremal) if and only if $\rho + \beta$ has index of determinacy k (is indeterminate N-extremal).

Proof. The assertion follows from Theorem 4.22, taking into account Corollary 4.19 and using the same reasoning as in the proof of Proposition 4.7(b).   □

5. Inverse spectral problems

Let J be the Jacobi operator associated with the matrix (2.2) as in Section 2. Fix $n \in \mathbb{N}\setminus\{1\}$ and consider, along with the self-adjoint operator J, the operator

$$
\begin{aligned}
\widetilde{J}(n) = {}& J + [q_n(\theta^2 - 1) + \theta^2 h]\langle e_n, \cdot\rangle e_n \\
& + b_n(\theta - 1)(\langle e_n, \cdot\rangle e_{n+1} + \langle e_{n+1}, \cdot\rangle e_n) \\
& + b_{n-1}(\theta - 1)(\langle e_{n-1}, \cdot\rangle e_n + \langle e_n, \cdot\rangle e_{n-1}), \quad \theta > 0, \quad h \in \mathbb{R},
\end{aligned}
\tag{5.1}
$$

where it has been assumed that $b_0 = 0$. Clearly, $\widetilde{J}(n)$ is a self-adjoint extension of the operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is a Jacobi matrix obtained from (2.2) by modifying the entries b_{n-1}, q_n, b_n. For instance, if $n > 2$, $\widetilde{J}(n)$ is a self-adjoint extension (possibly not proper) of the operator whose matrix representation is

$$
\begin{pmatrix}
q_1 & b_1 & 0 & 0 & 0 & 0 & \cdots \\
b_1 & \ddots & \ddots & 0 & 0 & 0 & \cdots \\
0 & \ddots & q_{n-1} & \theta b_{n-1} & 0 & 0 & \cdots \\
0 & 0 & \theta b_{n-1} & \theta^2(q_n + h) & \theta b_n & 0 & \cdots \\
0 & 0 & 0 & \theta b_n & q_{n+1} & b_{n+1} & \\
0 & 0 & 0 & 0 & b_{n+1} & q_{n+2} & \ddots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \ddots
\end{pmatrix}.
\tag{5.2}
$$

Note that $\widetilde{J}(n)$ is obtained from J by a rank-three perturbation when $n > 1$, and a rank-two perturbation otherwise.

The form of perturbation in (5.1) has a physical meaning in the context of spring-mass linear systems (see [19]).

Define

$$\gamma := \frac{\theta^2 h}{1 - \theta^2}. \tag{5.3}$$

Consider the following inverse problem:

Given two sequences S and $\widetilde{S}$ without finite points of accumulation, $n \in \mathbb{N} \setminus \{1\}$ and $\gamma \in \mathbb{R} \setminus S$, find a Jacobi operator J and parameters θ and h such that $\sigma(J) = S$ and $\sigma(\widetilde{J}(n)) = \widetilde{S}$ and (5.3) holds. We denote this inverse spectral problem by $(S, \widetilde{S}, n, \gamma)$. The operator J is called a solution of the inverse problem $(S, \widetilde{S}, n, \gamma)$.

When $n > 1$, it was shown in [19, Thms. 5.6] that if there is a solution, then there is an infinite set of solutions. Necessary and sufficient conditions on S and $\widetilde{S}$ for the existence of solutions of the inverse problem are given in [19, Thms. 5.9].

REMARK 5.1. All solutions of this inverse spectral problem have the same Green function at n [19, Prop. 5.3] given by

$$G(z, n) = \frac{\mathfrak{M}_n(z) - \theta^2}{(1 - \theta^2)(\gamma - z)}$$

(see [19, Eq. 4.2]), where the function $\mathfrak{M}_n$ is univocally determined by the sequences S and $\widetilde{S}$ [19, Prop. 4.13]. Moreover, γ and $\mathfrak{M}_n$ uniquely determine θ (see the proof of [19, Prop. 5.4]).

THEOREM 5.2. *Let J and $\widehat{J}$ be Jacobi operators which solve the inverse problem $(S, \widetilde{S}, n, \gamma)$ and ρ and $\widehat{\rho}$ be the corresponding spectral measures. Then, either*

$$\mathrm{ind}\,\rho = \mathrm{ind}\,\widehat{\rho}$$

or ρ and $\widehat{\rho}$ are simultaneously indeterminate N-extremal.

Proof. Due to Remark 5.1, J and $\widehat{J}$ have the same function $G(z, n)$. According to [19, Prop. 3.5], one writes

$$-G(z, n)^{-1} = z - q_n + \sum_{\alpha \in \mathscr{I}} \frac{\eta_\alpha}{\alpha - z},$$

where $\mathscr{I}$ is a discrete subset of $\mathbb{R}$. By Proposition 3.6

$$\sum_{\alpha \in \mathscr{I}} \frac{\eta_\alpha}{\alpha - z} = \begin{cases} b_n^2 m_n^+(z) + b_{n-1}^2 m_n^-(z) \\ b_n^2 \widehat{m}_n^+(z) + b_{n-1}^2 \widehat{m}_n^-(z), \end{cases}$$

where $m_n^\pm$ are given in Definition 3.2 and $\widehat{m}_n^\pm$ are the corresponding functions for $\widehat{J}$. Thus, using the notation introduced in (3.6), one has

$$b_n^2 \sigma_n = \sum_{\alpha \in \mathscr{I} \setminus \mathscr{F}} \eta_\alpha \delta_\alpha \qquad b_n^2 \widehat{\sigma}_n = \sum_{\alpha \in \mathscr{I} \setminus \mathscr{F}} \eta_\alpha \delta_\alpha$$

where $\widehat{\sigma}_n$ is defined as σ_n for the function $\widehat{m}_n^+$ and $\#\mathscr{F} = \#\widehat{\mathscr{F}} = n$. By Lemma 4.11, either

$$\operatorname{ind}\sigma_n = \operatorname{ind}\widehat{\sigma}_n$$

or σ_n and $\widehat{\sigma}_n$ are simultaneously indeterminate N-extremal. Thus, Corollary 4.19 completes the proof. $\square$

THEOREM 5.3. *Let J and J' be solutions of the inverse problems $(S,\widetilde{S},n,\gamma)$ and $(S,\widetilde{S},n',\gamma)$ respectively. Denote by ρ and ρ' the spectral measures corresponding to J and J' and assume that $\operatorname{ind}\rho < +\infty$. Either*

$$\operatorname{ind}\rho = \operatorname{ind}\rho'$$

or ρ and ρ' are simultaneously indeterminate N-extremal if and only if

$$n = n'.$$

Proof. ($\Leftarrow$) This is Theorem 5.2.
($\Rightarrow$) Since S and $\widetilde{S}$ univocally determine $\mathfrak{M}_n$ (see Remark 5.1), one has

$$\mathfrak{M}_n(z) = \mathfrak{M}_{n'}(z) \qquad \text{for all } z \in \mathbb{C} \setminus (S \setminus \widetilde{S}).$$

Again, Remark 5.1 yields

$$G(z,n) = G'(z,n'),$$

where $G'(z,n')$ is the n'-th Green function of J'. Repeating the argumentation of the previous theorem's proof, one arrives at

$$b_n^2 \sigma_n = \sum_{\alpha \in \mathscr{I} \setminus \mathscr{F}} \eta_\alpha \delta_\alpha \qquad (b_{n'}')^2 \sigma_{n'}' = \sum_{\alpha \in \mathscr{I} \setminus \mathscr{F}'} \eta_\alpha \delta_\alpha, \tag{5.4}$$

where

$$n = \#\mathscr{F} \quad \text{and} \quad n' = \#\mathscr{F}' \tag{5.5}$$

The hypothesis and Corollary 4.19 imply that either

$$\operatorname{ind}\sigma_n = \operatorname{ind}\sigma_{n'}'$$

or σ_n and $\sigma_{n'}'$ are simultaneously indeterminate N-extremal. To conclude the proof, one applies Lemma 4.11 to (5.4) and (5.5). $\square$

REMARK 5.4. Under the assumption that $S,\widetilde{S},\gamma$ are fixed, if ρ in the previous theorem is such that $\operatorname{ind}\rho < +\infty$, then the place of the perturbation n is determined uniquely by $\operatorname{ind}\rho$. If $\operatorname{ind}\rho = \infty$, then there are several possible values of n. This happens, in particular, to the inverse problem for finite Jacobi matrices.

REMARK 5.5. The inverse spectral problem for which $\gamma \in S$ is treated analogously.

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