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JOINTLY HYPONORMAL BLOCK TOEPLITZ PAIRS WITH RATIONAL SYMBOLS

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Abstract. In this paper, we are concerned with joint hyponormality of pairs of block Toeplitz operators acting on the vector-valued Hardy space $H^2_{\mathbb{C}^n}$ of the unit circle. We give a general sufficient condition for the matrix-valued rational symbols of the jointly hyponormal pair to have the same co-analytic inner parts of the coprime factorizations of the symbols and then provide some results under this sufficient condition.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces. Write $\mathscr{B}(\mathscr{H},\mathscr{K})$ for the set of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$ and write $\mathscr{B}(\mathscr{H}) \equiv \mathscr{B}(\mathscr{H},\mathscr{H})$. For $A,B \in \mathscr{B}(\mathscr{H})$, we let $[A,B]$ for the commutators of A and B, i.e., $[A,B] := AB - BA$. An operator $T \in \mathscr{B}(\mathscr{H})$ is called normal if $[T^*,T] = 0$, is called hyponormal if $[T^*,T] \geqslant 0$, and is called subnormal if T has a normal extension, i.e., $T = N|_{\mathscr{H}}$, where N is a normal operator on some Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ such that $\mathscr{H}$ is invariant for N. For an n-tuple $\mathbf{T} \equiv (T_1,\dots,T_n)$ of operators on $\mathscr{H}$, $[\mathbf{T}^*,\mathbf{T}] \in \mathscr{B}(\mathscr{H} \oplus \cdots \oplus \mathscr{H})$ denotes the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*,\mathbf{T}] := \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] & \dots & [T_n^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] & \dots & [T_n^*,T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*,T_n] & [T_2^*,T_n] & \dots & [T_n^*,T_n] \end{pmatrix}.$$

The self-commutator for n-tuples of operators on a Hilbert space was introduced by A. Athavale [2]. By analogy with the case $n=1$, we say that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*,\mathbf{T}]$ is a positive operator on $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$. On the other hand, C. Gu, J. Hendricks and D. Rutherford [10] have considered the hyponormality of block Toeplitz operators and characterized it in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on the $\mathbb{C}^n$-valued Hardy space, then its symbol Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. The hyponormality of the

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Toeplitz operator T_Φ with arbitrary matrix-valued symbol Φ, though solved in principle by the criterion due to Gu, Hendricks and Rutherford [10], is in practice very complicated. Explicit criteria for the hyponormality of block Toeplitz operators T_Φ with matrix-valued trigonometric polynomials or rational functions Φ were established via interpolation problems (cf. [10], [11], [12], [13], [5]).

In this paper, we discuss joint hyponormality of pairs of block Toeplitz operators with matrix-valued rational symbols. In [6], the joint hyponormality of the Toeplitz pair $\mathbf{T} \equiv (T_\varphi, T_\psi)$ was completely characterized when both symbols φ and ψ are trigonometric polynomials. The core of the main result of [6] is that the joint hyponormality of $\mathbf{T} \equiv (T_\varphi, T_\psi)$ (φ and ψ are trigonometric polynomials) forces that the co-analytic parts of φ and ψ necessarily coincide up to a constant multiple, i.e.,

$$\varphi - \beta\psi \in H^2 \text{ for some } \beta \in \mathbb{C}. \tag{1.1}$$

It was shown in [5] that (1.1) is still true for matrix-valued trigonometric polynomials under some invertibility and commutativity assumptions on the Fourier coefficients of the symbols. In this paper, we give a general sufficient condition for the matrix-valued rational symbols of the jointly hyponormal pair to have the same co-analytic inner parts of the coprime factorizations of the symbols and then provide some results under this sufficient condition.

2. Preliminaries

To describe our results, we need to review a few essential facts about (block) Toeplitz operators, and for that we will use [7], [9], [14], and [15]. For an operator $T \in \mathscr{B}(\mathscr{H})$, let $\ker T$ and $\operatorname{ran} T$ denote the kernel and the range of T, respectively. Also, write $\mathbb{T} \equiv \partial\mathbb{D}$ for the unit circle (where $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$). Write $L^2 \equiv L^2(\mathbb{T})$ for the set of square-integrable functions on $\mathbb{T}$ and H^2 for the corresponding Hardy space. Also $L^\infty \equiv L^\infty(\mathbb{T})$ for the set of essentially bounded measurable functions on $\mathbb{T}$. Let $H^\infty := L^\infty \cap H^2$. Given a function $\varphi \in L^\infty$, the *Toeplitz operator* T_φ and the *Hankel operator* H_φ with *symbol* φ on H^2 are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2), \tag{2.1}$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator from L^2 onto L^2 defined by $J(f)(z) = \overline{z}f(\overline{z})$ for $f \in L^2$. A function $\varphi \in L^2$ is said to be of *bounded type* if there are functions $\psi_1,\ \psi_2 \in H^\infty$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

We recall [1, Lemma 3] that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}. \tag{2.2}$$

If $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp \varphi} \in zH^2.$$

If φ and $\overline{\varphi}$ are of bounded type, then by the Beurling's Theorem, we may write

$$\varphi_- = \theta_0 \overline{b} \quad \text{and} \quad \varphi_+ = \theta_1 \overline{a} \ \ (a, b \in H^2;\ \theta_0, \theta_1 \text{ are inner}). \tag{2.3}$$

By Kronecker's Lemma [14, p. 183], if $f \in H^\infty$, then

$$\overline{f} \text{ is rational} \iff f = \theta \overline{b} \text{ with a finite Blaschke product } \theta. \tag{2.4}$$

Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n \equiv M_{n\times n}$. For $\mathscr{X}$ a Hilbert space, let $L^2_{\mathscr{X}} \equiv L^2_{\mathscr{X}}(\mathbb{T})$ be the Hilbert space of $\mathscr{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $H^2_{\mathscr{X}} \equiv H^2_{\mathscr{X}}(\mathbb{T})$ be the corresponding Hardy space. We also let $L^\infty_{\mathscr{X}} \equiv L^\infty_{\mathscr{X}}(\mathbb{T})$ be the Hilbert space of $\mathscr{X}$-valued bounded measurable functions on $\mathbb{T}$ and let $H^\infty_{\mathscr{X}} \equiv H^\infty_{\mathscr{X}}(\mathbb{T}) = L^\infty_{\mathscr{X}} \cap H^2_{\mathscr{X}}$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$, then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes *block Toeplitz operator* with *symbol* Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A *block Hankel operator* with *symbol* $\Phi \in L^\infty_{M_n}$ is an operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $P_n^\perp$ is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $(H^2_{\mathbb{C}^n})^\perp$ and J_n denotes the unitary operator from $L^2_{\mathbb{C}^n}$ onto $L^2_{\mathbb{C}^n}$ given by $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$, with I_n the $n \times n$ identity matrix. For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$. For a matrix-valued function $\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function. A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \ (A_j \in M_n),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. It was known ([4, Lemma 2.1]) that if $\Theta_i = \theta_i I_n$ for an inner function θ_i $(i \in J)$, then

$$\text{left-g.c.d.}\{\Theta_i : i \in J\} = \text{right g.c.d.}\{\Theta_i : i \in J\} = \theta_d I_n, \text{ where } \theta_d = \text{g.c.d.}\{\theta_i : i \in J\}$$
$$\text{left-l.c.m.}\{\Theta_i : i \in J\} = \text{right l.c.l.}\{\Theta_i : i \in J\} = \theta_d I_n, \text{ where } \theta_d = \text{l.c.m.}\{\theta_i : i \in J\}:$$

they are both *diagonal-constant* inner functions, i.e., diagonal inner functions, constant along the diagonal. If there is no confusion, we write δ for δI_n for $\delta \in L^\infty$.

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left[P_n^\perp(\Phi)\right]^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. Suppose $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if $\theta \equiv \text{l.c.m.}\{\theta_{ij} : i,j = 1,2,\cdots,n\}$, then we can write

$$\Phi = [\varphi_{ij}] = [\theta_{ij}\overline{b}_{ij}] = [\theta\overline{a}_{ij}] \equiv \theta A^* \quad (A \equiv [a_{ji}] \in H^\infty_{M_n}). \tag{2.5}$$

In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products, as we observed in (2.4). By contrast with scalar-valued functions, in (2.5) θI_n and A need not be (right) coprime. If $\Omega = \text{left-g.c.d.}\{A, \theta I_n\}$ in the representation (2.5):

$$\Phi = \theta A^*,$$

then $\theta I_n = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell \in H^2_{M_n}$ because $\det \theta I_n$ is not identically zero) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = {A_\ell}^*\Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{2.6}$$

$A_\ell^*\Omega_\ell$ is called the *left coprime factorization* of Φ; similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{2.7}$$

In this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ. As a consequence of the Beurling-Lax-Halmos Theorem, we can see that ([10, Corollary 2.5]; [4, Remark 2.2])

$$\Phi = \Omega_r A_r^* \text{ (right coprime factorization)} \iff \ker H_{\Phi^*} = \Omega_r H^2_{\mathbb{C}^n}. \tag{2.8}$$

Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then, in view of (2.7), we may write

$$\Phi = \Theta_+ A^*, \ \ \Phi^* = \Theta_0 B^* \quad \text{(right coprime factorization)},$$

where $\Theta_+, \Theta_0 \in H^\infty_{M_n}$.

For $\Phi, \Psi \in L^\infty_{M_n}$, let

$$[T_\Phi, T_\Psi]_p := H^*_{\Psi^*} H_\Phi - H^*_{\Phi^*} H_\Psi.$$

Then $[T_\Phi^*, T_\Phi]_p$ is called the *pseudo-selfcommutator* of T_Φ. Also T_Φ is said to be *pseudo-hyponormal* if $[T_\Phi^*, T_\Phi]_p$ is positive semidefinite. Thus if T_Φ is pseudo-hyponormal then since

$$[T_\Phi^*, T_\Phi]_p = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi = H_{\Phi_+^*}^* H_{\Phi_+^*} - H_{\Phi_-}^* H_{\Phi_-},$$

it follows that $||H_{\Phi_+^*} f|| \geqslant ||H_{\Phi_-^*} f||$ for all $f \in H^2_{\mathbb{C}^n}$, and hence

$$\Theta_+ H^2_{\mathbb{C}^n} = \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} = \Theta_0 H^2_{\mathbb{C}^n}. \tag{2.9}$$

Thus by Corollary IX.2.2 of [8], Θ_0 is a left inner divisor of Θ_0, i.e., $\Theta_+ = \Theta_0\Theta_1$ for some inner function $\Theta_1 \in H^\infty_{M_n}$. Thus, if $\Phi \in L^\infty_{M_n}$ is rational function and T_Φ is pseudo-hyponormal, then we can write

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_0 B^* \quad \text{(right coprime factorization).} \tag{2.10}$$

For notational convenience we write

$$H^2_0 := zH^2_{M_n} \quad \text{and} \quad \mathscr{Z}(\theta) := \text{the set of all zeros of an inner function } \theta.$$

On the other hand, we have [3, Lemma 3.3] that if $A \in H^\infty_{M_n}$ and θ be a finite Blaschke product, then $A(\alpha)$ is invertible for each $\alpha \in \mathscr{Z}(\theta)$ if and only if A and θI_n are right (or left) coprime. Thus if θ is a finite Blaschke product then we shall say that $A \in H^\infty_{M_n}$ and θI_n are *coprime* whenever they are right or left coprime. Hence if in the representation (2.10), $\Theta_i = \theta_i I_n$ $(i = 1, 2)$ with a finite Blaschke product θ_i then we shall write

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_0 B^* \quad \text{(coprime),} \tag{2.11}$$

where $\theta_0\theta_1$ and θ_0 are called the *analytic inner part* and the *co-analytic inner part* of the coprime factorizations, respectively. If $\Theta \in H^\infty_{M_n}$ is an inner matrix function, we write

$$\mathscr{H}_\Theta := H^2_{M_n} \ominus \Theta H^2_{M_n};$$
$$\mathscr{K}_\Theta := H^2_{M_n} \ominus H^2_{M_n}\Theta.$$

If $\Theta = \theta I_n$ for an inner function θ then $\mathscr{H}_\Theta = \mathscr{K}_\Theta$. If $\Theta \in H^\infty_{M_n}$ is an inner matrix function and $A \in H^2_{M_n}$, then a straightforward calculation shows that ([5, Lemma 4.4])

$$A \in \mathscr{K}_\Theta \Longleftrightarrow \Theta A^* \in H^2_0. \tag{2.12}$$

NOTATION. For a closed subspace $\mathscr{X}$ of $H^2_{M_n}$, we write $P_\mathscr{X}$ for the orthogonal projection from $H^2_{M_n}$ onto $\mathscr{X}$. If $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ and Δ_1 and Δ_2 are inner functions in $H^\infty_{M_n}$, write

$$\Phi_{\Delta_1,\Delta_2} := P_{H^{2\perp}}(\Phi_-^*\Delta_1) + P_{H^2_0}(\Delta_2^*\Phi_+)$$

and

$$\Phi^{\Delta_1,\Delta_2} := P_{H^{2\perp}}(\Delta_1\Phi_-^*) + P_{H^2_0}(\Phi_+\Delta_2^*),$$

where $H^2 \equiv H^2_{M_n}$ and abbreviate

$$\Phi_\Delta \equiv \Phi_{\Delta,\Delta} \quad \text{and} \quad \Phi^\Delta \equiv \Phi^{\Delta,\Delta}.$$

If $\Delta_i := \delta_i I_n$ for some inner functions δ_i $(i = 1, 2)$, then we have that $\Phi_{\Delta_1,\Delta_2} = \Phi^{\Delta_1,\Delta_2}$.

3. Main results

We begin with:

HYPONORMALITY OF BLOCK TOEPLITZ OPERATORS. ([10]) *For each* $\Phi \in L^\infty_{M_n}$, *let*

$$\mathscr{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ ||K||_\infty \leqslant 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Then T_Φ *is hyponormal if and only if* Φ *is normal and* $\mathscr{E}(\Phi)$ *is nonempty.*

We observe that if $\mathbf{T} \equiv (T_\varphi, T_\psi)$, then the self-commutator of $\mathbf{T}$ can be expressed as:

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T^*_\varphi, T_\varphi] & [T^*_\psi, T_\varphi] \\ [T^*_\varphi, T_\psi] & [T^*_\psi, T_\psi] \end{pmatrix} = \begin{pmatrix} H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}} & H^*_{\overline{\varphi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\varphi_-}} \\ H^*_{\overline{\psi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \end{pmatrix}. \tag{3.1}$$

Pairs of block Toeplitz operators will be called *block Toeplitz pairs*. For a block Toeplitz pair $\mathbf{T} \equiv (T_\Phi, T_\Psi)$, the *pseudo-commutator* of $\mathbf{T}$ is defined by

$$[\mathbf{T}^*, \mathbf{T}]_p := \begin{pmatrix} [T^*_\Phi, T_\Phi]_p & [T^*_\Psi, T_\Phi]_p \\ [T^*_\Phi, T_\Psi]_p & [T^*_\Psi, T_\Psi]_p \end{pmatrix} = \begin{pmatrix} H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Phi^*_-} & H^*_{\Phi^*_+} H_{\Psi^*_+} - H^*_{\Psi^*_-} H_{\Phi^*_-} \\ H^*_{\Psi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Psi^*_-} & H^*_{\Psi^*_+} H_{\Psi^*_+} - H^*_{\Psi^*_-} H_{\Psi^*_-} \end{pmatrix}.$$

Then $\mathbf{T} = (T_\Phi, T_\Psi)$ is said to be *pseudo-(jointly) hyponormal* if $[\mathbf{T}^*, \mathbf{T}]_p \geqslant 0$. Observe that if $\Phi \in L^\infty_{M_n}$ then

$$[T^*_\Phi, T_\Phi] = [T^*_\Phi, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

Thus we have

$$T_\Phi \text{ is hyponormal} \iff T_\Phi \text{ is pseudo-hyponormal and } \Phi \text{ is normal;}$$

and (via Theorem 3.3 of [10]) T_Φ is pseudo-hyponormal if and only if $\mathscr{E}(\Phi) \neq \emptyset$.

Let $\Phi, \Psi \in L^\infty_{M_n}$ be matrix-valued rational functions of the form

$$\Phi_+ = \theta_0 \theta_1 A^*, \ \ \Phi_- = \theta_0 B^*, \ \ \Psi_+ = \theta_2 \theta_3 C^*, \ \ \Psi_- = \theta_2 D^* \quad \text{(coprime)}. \tag{3.2}$$

In [5], it was shown that if the pair (T_Φ, T_Ψ) is pseudo-hyponormal and if θ_0 and θ_2 are not coprime then $\theta_0 = \theta_2$. The following question arises at once.

QUESTION A. Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be hyponormal, where Φ and Ψ are given in (3.2). If $\theta_0 = \theta_2$, does it follow that $\theta_1 = \theta_3$?

However, in [5], it was also shown that the answer to Question A is negative even for scalar-valued symbols. In this paper, we give a general sufficient condition for the answer to Question A to be affirmative and then provide some results under the condition that $\theta_0 = \theta_2$.

The following two lemmas are needed for our main results.

LEMMA 3.1. [5, Lemma 9.13] *Let* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a pseudo-hyponormal Toeplitz pair with matrix-valued rational symbols* $\Phi, \Psi \in L^\infty_{M_n}$ *of the form*

$$\Phi_+ = \theta \theta_1 A^*, \ \ \Phi_- = \theta_0 B^*, \ \ \Psi_+ = \theta \theta_3 C^*, \ \ \Psi_- = \theta_2 D^* \quad \text{(coprime)},$$

where $\theta := \text{l.c.m.}(\theta_0, \theta_2)$. *If we let* $\delta := \text{g.c.d.}(\theta_1, \theta_3)$, *then*

$$\mathbf{T}: \textit{ pseudo-hyponormal} \Longleftrightarrow \mathbf{T}_\Delta := (T_{\Phi^{1,\delta}}, T_{\Psi^{1,\delta}}): \textit{ pseudo-hyponormal.}$$

LEMMA 3.2. [5, Corollary 9.21] (Hyponormality of Rational Block Toeplitz Pairs) *Let* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a block Toeplitz pair, where* $\Phi, \Psi \in L^\infty_{M_n}$ *are matrix-valued rational functions of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}. \tag{3.3}$$

Assume that θ_0 *and* θ_2 *are not coprime. Assume also that* $\Lambda := B(\gamma_0)D(\gamma_0)^{-1}$ *is a normal matrix commuting with* Φ_- *and* Ψ_- *for some* $\gamma_0 \in \mathscr{Z}(\theta_0)$. *Then the pair* $\mathbf{T}$ *is hyponormal if and only if*

(i) Φ *and* Ψ *are normal and* $\Phi\Psi = \Psi\Phi$;

(ii) $\Phi_- = \Lambda^*\Psi_-$;

(iii) $T_{\Psi^{1,\Omega}}$ *is pseudo-hyponormal with* $\Omega := \theta_0\theta_1\theta_3\overline{\theta}\Delta^*$,

where $\theta := \text{g.c.d.}\{\theta_1, \theta_3\}$ *and* $\Delta := \text{left-g.c.d.}\{\theta_0\theta I_n,\ \overline{\theta}(\theta_3 A - \theta_1 C\Lambda^*_{\gamma_0})\}$.

Proof. This follows from a slight variation of the proof of Corollary 9.21 of [5], in which $B(\gamma_0)$ and $D(\gamma_0)$ are diagonal-constant for some $\gamma_0 \in \mathscr{Z}(\theta_0)$. $\square$

If the symbols are matrix-valued trigonometric polynomials then the answer to Question A is indeed affirmative under an assumption on the outer coefficients.

THEOREM 3.3. *Let* $\Phi, \Psi \in L^\infty_{M_n}$ *be matrix-valued trigonometric polynomials of the form*

$$\Phi(z) := \sum_{j=-m}^{N} A_j z^j \quad \textit{and} \quad \Psi(z) := \sum_{j=-\ell}^{M} B_j z^j \tag{3.4}$$

satisfying

(i) the outer coefficients $A_{-m}, A_N, B_{-\ell}$ *and* B_M *are invertible;*

(ii) $\Lambda := A_{-m}B_{-\ell}^{-1}$ *is a normal matrix commuting with* Φ_- *and* Ψ_-.

If $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *is pseudo-hyponormal then* $N = M$.

Proof. Suppose that $\mathbf{T}$ is pseudo-hyponormal. Then T_Φ and T_Ψ are pseudo-hyponormal so that, by (2.10), $m \leqslant N$ and $\ell \leqslant m$. Thus it follows from Lemma 3.2 that $m = \ell$, and hence we may write

$$\Phi_+ = z^{m+q}A^*, \quad \Phi_- = z^m B^*, \quad \Psi_+ = z^{m+r}C^*, \quad \Psi_- = z^m D^* \quad \text{(coprime)},$$

where $m \geqslant 1$, $q \geqslant 0$ and $r \geqslant 0$. By Lemma 3.1, we may also assume that $r = 0$. Put

$$\Delta := \text{left-g.c.d.}\{z^m I_n,\ A - z^q C\Lambda^*\},$$

where $\Lambda := A_{-m}B_{-m}^{-1} = B(0)D(0)^{-1}$. Then it follows from Lemma 3.2 that $T_{\Psi^{1,\Omega}}$ is pseudo-hyponormal with $\Omega := z^{m+q}\Delta^*$. We want to show that $q = 0$. Assume to the contrary that $q \neq 0$. Since $(A - z^q C\Lambda^*)(0) = A(0) = A_N^*$ is invertible, it follows that Δ is a constant unitary. Observe that

$$H_{(\Psi^{1,\Omega})_+^*} = H_{[P_{H_0^2}(\Psi_+\Omega^*)]^*} = H_{\Omega\Psi_+^*} = H_{z^q\Delta^* C} = 0. \tag{3.5}$$

It thus follows from (2.9) that

$$z^m H^2_{\mathbb{C}_n} = \ker H_{(\Psi^{1,\Omega})_-^*} \supseteq \ker H_{(\Psi^{1,\Omega})_+^*} = H^2_{\mathbb{C}^n},$$

a contradiction. Therefore we must have $q = 0$. This comletes the proof. □

Even when the analytic inner parts of the coprime factorizations of the symbols are not equal for a pseudo-hyponormal pair with rational symbols having the same co-analytic inner parts, we are interested in finding a general sufficient condition for the rational symbols Φ and Ψ of the pseudo-hyponormal pair $\mathbf{T} := (T_\Phi, T_\Psi)$ to have the same analytic inner parts.

THEOREM 3.4. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a block Toeplitz pair with matrix-valued rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \ \ \Phi_- = \theta_0 B^*, \ \ \Psi_+ = \theta_0\theta_3 C^*, \ \ \Psi_- = \theta_0 D^* \quad \text{(coprime)}.$$

Suppose $\Lambda \equiv \Lambda_{\gamma_0} := B(\gamma_0)D(\gamma_0)^{-1}$ is a normal matrix commuting with Φ_- and Ψ_- for some $\gamma_0 \in \mathscr{Z}(\theta_0)$. If $\mathbf{T}$ is pseudo-hyponormal and $\delta := \mathrm{GCD}\{\theta_1, \theta_3\}$, then $(\theta_1\overline{\delta})(\theta_3\overline{\delta})$ and θ_0 are coprime.

Proof. Assume that $\mathbf{T}$ is pseudo-hyponormal. By Lemma 3.1, we may assume that θ_1 and θ_3 are coprime. We want to show that $\theta_1\theta_3$ and θ_0 are coprime. Write

$$\theta_0 = \prod_{j=1}^{d_0} b_{\alpha_j}^{p_j}, \ \ \theta_1 = \prod_{j=1}^{d_1} b_{\beta_j}^{n_j}, \ \ \theta_3 = \prod_{j=1}^{d_3} b_{\gamma_j}^{m_j} \qquad (p_j, n_j, m_j \geqslant 1),$$

where $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ $(\lambda \in \mathbb{D})$. Assume to the contrary that $\theta_1\theta_3$ and θ_0 are not coprime. Without loss of generality, we may assume that $\alpha_1 = \beta_1$. Let

$$\omega := \theta_0\theta_1\theta_3 b_{\beta_1}^{-(p_1+n_1)}.$$

Since $\mathbf{T}$ is pseudo-hyponormal, it follows from [5, Lemma 9.8] that $(T_{\Phi_\omega}, T_{\Psi_\omega})$ is pseudo-hyponormal. Write

$$b \equiv b_{\beta_1} \quad \text{and} \quad \delta \equiv \theta_1 b^{-n_1} = \prod_{j=2}^{d_1} b_{\beta_j}^{n_j}.$$

Observe that

$$(\Phi_\omega)_-^* = P_{H^{2\perp}}\left(B\theta_1\theta_3 b^{-(p_1+n_1)}\right) = P_{H^{2\perp}}\left(B\delta\theta_3 b^{-p_1}\right) = b^{-p_1}\left[P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B)\right].$$

We thus have

$$(\Phi_\omega)_- = b^{p_1}\left[P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B)\right]^* \quad \text{(coprime)}.$$

Similarly, we have the following right coprime factorizations:

$$\begin{aligned}(\Phi_\omega)_+ &= b^{(p_1+n_1)}\left[P_{\mathscr{K}_{b^{(p_1+n_1)}}}(\theta_3 A)\right]^*,\\ (\Psi_\omega)_- &= b^{p_1}\left[P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 D)\right]^*,\\ (\Psi_\omega)_+ &= b^{p_1}\left[P_{\mathscr{K}_{b^{p_1}}}(\delta C)\right]^*.\end{aligned}$$

Now we will show that

$$\Lambda \text{ commutes with } (\Phi_\omega)_- \text{ and } (\Psi_\omega)_-.$$

Since Λ is a normal matrix commuting with Φ_- and Ψ_-, it follows from the Fuglede-Putnam Theorem that Λ commutes with B and D and hence Λ commutes with $\delta\theta_3 B$ and $\delta\theta_3 D$. Write

$$B_1 \equiv \delta\theta_3 B - P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B).$$

Then $B_1 \in b^{p_1}H^2_{M_n}$. Thus we can write $B_1 = b^{p_1}B_2$ for some $B_2 \in H^2_{M_n}$. Since Λ commutes with $\delta\theta_3 B$ and $\delta\theta_3 D$, we have that

$$\Lambda P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B) + b^{p_1}\Lambda B_2 = P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B)\Lambda + b^{p_1}B_2\Lambda. \tag{3.6}$$

But since Λ is a constant matrix, it follows from (2.12) that $\Lambda P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B)$ and $P_{\mathscr{K}_{b^{p_1}}}(\delta\theta_3 B)\Lambda$ are in $\mathscr{K}_{b^{p_1}}$. Thus by (3.6), we have that Λ commutes with $(\Phi_\omega)_-$. Similarly, we also have that Λ commutes with $(\Psi_\omega)_-$. Since Λ is a normal matrix commuting with Φ_- and Ψ_-, it follows from Lemma 3.2 that $\Phi_- = \Lambda^*\Psi_-$ and hence $\Lambda = B(\beta_1)D(\beta_1)^{-1}$. We now apply Lemma 3.2. To do so, let

$$\Delta := \text{left-g.c.d.}\{b^{p_1}I_n,\ P_{\mathscr{K}_{b^{(p_1+n_1)}}}(\theta_3 A) - b^{n_1}P_{\mathscr{K}_{b^{p_1}}}(\delta C)\Lambda^*\}.$$

Put $\Omega := b^{p_1+n_1}\Delta^*$. Since $(T_{\Phi_\omega}, T_{\Psi_\omega})$ is pseudo-hyponormal it follows from Lemma 3.2 that T_Υ is pseudo-hyponormal with $\Upsilon = (\Psi_\omega)^{1,\Omega}$. It thus follows from (2.9) that $n_1 = 0$, a contradiction. This completes the proof. $\square$

We now have:

COROLLARY 3.5. *Let* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a block Toeplitz pair with matrix-valued rational symbols* $\Phi, \Psi \in L^\infty_{M_n}$ *of the form*

$$\Phi_+ = \theta_0\theta_1 A^*,\ \ \Phi_- = \theta_0 B^*,\ \ \Psi_+ = \theta_0\theta_3 C^*,\ \ \Psi_- = \theta_0 D^* \quad \text{(coprime)}.$$

Suppose $\Lambda \equiv \Lambda_{\gamma_0} := B(\gamma_0)D(\gamma_0)^{-1}$ *is a normal matrix commuting with* Φ_- *and* Ψ_- *for some* $\gamma_0 \in \mathscr{Z}(\theta_0)$. *If* $\mathbf{T}$ *is pseudo-hyponormal and* $\mathscr{Z}(\theta_1\theta_3) \subseteq \mathscr{Z}(\theta_0)$, *then* $\theta_1 = \theta_3$.

Proof. If $\theta_1 \neq \theta_3$ and $\mathscr{Z}(\theta_1\theta_3) \subseteq \mathscr{Z}(\theta_0)$, then $\theta_1\theta_3\overline{\text{g.c.d.}\{\theta_1,\theta_3\}}^2$ and θ_0 have a common zero, which is a contradiction by Theorem 3.4. $\square$

If the matrix-valued rational symbols Φ and Ψ have the same co-analytic and analytic inner parts of the coprime factorizations, we get a general necessary condition for the pseudo-hyponormality of the pair $\mathbf{T} := (T_\Phi, T_\Psi)$.

THEOREM 3.6. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a block Toeplitz pair with matrix-valued rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*,\ \ \Phi_- = \theta_0 B^*,\ \ \Psi_+ = \theta_0\theta_1 C^*,\ \ \Psi_- = \theta_0 D^* \quad \text{(coprime)}.$$

Suppose $\Lambda \equiv \Lambda_{\gamma_0} := B(\gamma_0)D(\gamma_0)^{-1}$ is a normal matrix commuting with Φ_- and Ψ_- for some $\gamma_0 \in \mathscr{Z}(\theta_0)$. If $\mathbf{T}$ is pseudo-hyponormal then

$$\Phi - \Lambda\Psi \in \mathscr{K}_{z\theta_1}.$$

Proof. By Lemma 3.1, $\mathbf{T}_{\Theta_1} := (T_{\Phi^{1,\theta_1}}, T_{\Psi^{1,\theta_1}})$ is pseudo-hyponormal. We can write

$$\Phi_+^{1,\theta_1} = \theta_0 A_0^* \quad \text{and} \quad \Psi_+^{1,\theta_1} = \theta_0 C_0^* \quad \text{(coprime)},$$

where $A_0 := P_{\mathscr{K}_{\theta_0}} A$ and $C_0 := P_{\mathscr{K}_{\theta_0}} C$. It follows from Lemma 3.2 that T_Υ is pseudo-hyponormal with

$$\Upsilon := \Psi_-^* + P_{H_0^2}\left(\Psi_+^{1,\theta_1}\Omega^*\right) \quad (\Omega := \theta_0\Delta^*),$$

where $\Delta :=$ left-g.c.d.$\{\theta_0 I_n, A_0 - C_0\Lambda^*\}$. We claim that

$$\Delta = \theta_0 I_n \quad \text{(up to a constant unitary).} \tag{3.7}$$

Observe that

$$\Upsilon_+^* = \left(P_{H_0^2}\left(\Psi_+^{1,\theta_1}\Omega^*\right)\right)^* = P_{H^{2\perp}}(\Delta^* C_0).$$

Since T_Υ is pseudo-hyponormal, it follows from (2.9) that

$$\theta_0 H^2_{\mathbb{C}_n} = \ker H_{\Psi_-^*} = \ker H_{\Upsilon_-^*} \supseteq \ker H_{\Upsilon_+^*} = \ker H_{\Delta^* C_0} = \Delta H^2_{\mathbb{C}^n},$$

which implies (3.7). Also since Δ is a left inner divisor of $A_0 - C_0\Lambda^*$, it follows that

$$A_0 - C_0\Lambda^* \in \theta_0 H^2_{M_n}.$$

But since Λ is a constant matrix and $C_0 \in \mathscr{K}_{\theta_0}$ it follows from (2.12) that $C_0\Lambda^* \in \mathscr{K}_{\theta_0}$, and hence $A_0 - C_0\Lambda^* \in \mathscr{K}_{\theta_0}$. Therefore,

$$A_0 - C_0\Lambda^* \in \theta_0 H^2_{M_n} \bigcap \mathscr{K}_{\theta_0 I_n} = \{0\},$$

which implies $A_0 = C_0\Lambda^*$. Put $A_1 := A - A_0$, and $C_1 := C - C_0$. Then $A_1, C_1 \in \theta_0 H^2_{M_n}$. Thus $A_1 = \theta_0 A_2$ and $C_1 = \theta_0 C_2$ for some $A_2, C_2 \in H^2_{M_n}$. Then

$$\begin{aligned}\Phi_+ - \Lambda\Psi_+ &= \theta_0\theta_1(A_0^* + A_1^*) - \theta_0\theta_1\Lambda(C_0^* + C_1^*)\\ &= \theta_0\theta_1(A_1^* - \Lambda C_1^*) \quad (\text{since } A_0^* = \Lambda C_0^*)\\ &= \theta_0\theta_1(\overline{\theta_0}A_2^* - \overline{\theta_0}\Lambda C_2^*)\\ &= \theta_1(A_2 - C_2\Lambda^*)^*.\end{aligned}$$

We thus have

$$z\theta_1(\Phi_+ - \Lambda\Psi_+)^* = z(A_2 - C_2\Lambda^*) \in H^2_0,$$

which implies, by (2.12), $\Phi_+ - \Lambda\Psi_+ \in \mathcal{K}_{z\theta_1}$. Since Λ is a normal amtrix commuting with Φ_- and Ψ_-, it follows from Lemma 3.2 that

$$\Phi - \Lambda\Psi = \Phi_-^* - \Lambda\Psi_-^* + \Phi_+ - \Lambda\Psi_+ = \Phi_+ - \Lambda\Psi_+ \in \mathcal{K}_{z\theta_1},$$

which proves the theorem. $\square$

As we will see in the next result, if the analytic and co-analytic inner parts of the coprime factorizations of the rational symbols are equal then two symbols coincide up to a constant matrix under the assumption of pseudo-hyponormality.

COROLLARY 3.7. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a block Toeplitz pair with matrix-valued rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta A^*, \;\; \Phi_- = \theta B^*, \;\; \Psi_+ = \theta C^*, \;\; \Psi_- = \theta D^* \quad \text{(coprime factorizations)},$$

where θ is a finite Blaschke product. Suppose $\Lambda := B(\gamma_0)D(\gamma_0)^{-1}$ is a normal matrix commuting with Φ_- and Ψ_- for some $\gamma_0 \in \mathcal{Z}(\theta)$. If $\mathbf{T}$ is pseudo-hyponormal then

$$\Phi - \Lambda\Psi \in M_n.$$

Proof. Immediate from Theorem 3.6. $\square$

COROLLARY 3.8. *Let $\Phi, \Psi \in L^\infty_{M_n}$ be matrix-valued trigonometric polynomials of the form*

$$\Phi(z) := \sum_{j=-m}^{N} A_j z^j \quad \textit{and} \quad \Psi(z) := \sum_{j=-\ell}^{M} B_j z^j \tag{3.8}$$

satisfying

(i) the outer coefficients $A_{-m}, A_N, B_{-\ell}$ and B_M are invertible;

(ii) $\Lambda := A_{-m}B_{-\ell}^{-1}$ is a normal matrix commuting with Φ_- and Ψ_-.

If $\mathbf{T} := (T_\Phi, T_\Psi)$ is pseudo-hyponormal then

$$\Phi - \Lambda\Psi \in \mathcal{K}_{z^{N-m+1}}.$$

Proof. By Lemma 3.2 and Theorem 3.3, we have $N = M$ and $m = \ell$. Thus the result follows from Theorem 3.6. □

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