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ON DIFFERENTIABILITY OF A CLASS OF ORTHOGONALLY INVARIANT FUNCTIONS ON SEVERAL OPERATOR VARIABLES

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Abstract. In this work, we study a connection between two classes of orthogonally invariant functions. Both types of functions are defined on $S^{n_1} \times \ldots \times S^{n_k}$. The functions in the first class take their values in $S^{n_1 \cdots n_k}$, while those in the second take values in $S^{\binom{n}{k}}$, where $n = n_1 + \cdots + n_k$. Here, S^n denotes the set of all $n \times n$ symmetric matrices. Using that connection we establish various smoothness properties of the functions in the first class, using analogous known results about the functions in the second class.

1. Introduction

Let $\mathbb{N}_n := \{1, \ldots, n\}$. Denote by S^n the space of all $n \times n$ symmetric matrices with inner product $\langle A, B \rangle := \operatorname{Tr}(AB)$. Let O^n be the group of $n \times n$ orthogonal matrices. Denote by $\mathbb{R}^n_{\geqslant}$ the convex cone in $\mathbb{R}^n$ of all vectors with non-increasingly ordered coordinates. For any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_{\geqslant}$ be the ordered vector of eigenvalues of A. Let $\operatorname{Diag} x$ be the $n \times n$ matrix with $x \in \mathbb{R}^n$ on the main diagonal.

Fix natural numbers $n_1, \ldots, n_k$ and assume that the k-tuples in $\mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}$ are ordered lexicographically. For any function $f : \mathbb{R}^k \to \mathbb{R}$ define

$$F^H : S^{n_1} \times \ldots \times S^{n_k} \to S^{n_1 \cdots n_k} \text{ by}$$
$$F^H(A_1, \ldots, A_k) := (\otimes_{i=1}^k U_i)\big(\operatorname{Diag}_l f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))\big)(\otimes_{i=1}^k U_i)^\top,$$

where $U_i \in O^{n_i}$ is such that $A_i = U_i(\operatorname{Diag} \lambda(A_i))U_i^\top$ for $i \in \mathbb{N}_k$. Here, $\operatorname{Diag}_l \mathbf{x}_l$ denotes the diagonal matrix with vector $\mathbf{x} \in \mathbb{R}^{n_1 \cdots n_k}$ on the main diagonal and $l \in \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}$.

Several properties of these functions have been studied. For example operator monotonicity and operator convexity are extensively studied in [2], [3], [6], [7], [9], [11], and [12]. In [6], the author shows that, for values $m = 1, 2$, function F^H is C^m at $(A_1, \ldots, A_k)$, if the underlying f is C^p, where $p > m + k/2$, at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for all $l \in \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}$.

To introduce the second class of functions, denote by $\mathbb{N}_{n,k}$ the set of all subsets of $\mathbb{N}_n$ of size k with elements ordered increasingly, where $1 \leqslant k \leqslant n$. The elements of the

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set $\mathbb{N}_{n,k}$ are ordered lexicographically and used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$. Let $f:\mathbb{R}^k \to \mathbb{R}$ be a *symmetric* function, that is invariant under permutations of its arguments. Define $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by

$$\mathbf{f}_\rho(x) := f(x_{\rho_1},\ldots,x_{\rho_k})$$

for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{N}_{n,k}$. Finally, let $U^{(k)}$ be the k*-th multiplicative compound matrix* of an $n\times n$ matrix U. It is known that $U^{(k)}$ is orthogonal, whenever U is, see Section 2 for more details. For any symmetric $f:\mathbb{R}^k\to\mathbb{R}$, define a function $F: S^n \to S^{\binom{n}{k}}$, called *(generated)* k*-isotropic*, by

$$F(A) := U^{(k)}\big(\operatorname{Diag}\mathbf{f}(\lambda(A))\big)\big(U^{(k)}\big)^\top,$$

where $U \in O^n$ is such that $A = U\big(\operatorname{Diag}\lambda(A)\big)U^\top$.

Function F is well-defined and satisfies $F(UAU^\top) = U^{(k)}F(A)\big(U^{(k)}\big)^\top$ for any $U\in O^n$ and any A in the domain of F, as shown in [10].

A characterization of C^1 (generated) k-isotropic functions was obtained in [1] and that was extended in in [10] to C^m for a larger class, called k*-isotropic functions*. The (generated) k-isotropic function F is C^m at A, if and only if the underlying symmetric function f is C^m at $\lambda_\rho(A)$ for all $\rho\in\mathbb{N}_{n,k}$. That result holds for $m=0,1,\ldots$ Later on, [8] showed that, F is analytic at A, if and only if the underlying symmetric function f is analytic at $\lambda_\rho(A)$ for all $\rho\in\mathbb{N}_{n,k}$.

The main goal in this work is to connect F^H and F, when the underlying function f is symmetric. This allows us to characterize differentiability of F^H in terms of symmetric f, using the corresponding known properties of F. In addition, we characterize the analyticity of F^H in terms of f, not necessarily symmetric.

2. Main definition

2.1. Tensor products

Denote by $\otimes_{i=1}^k \mathbb{R}^{n_i}$ the tensor product of $\mathbb{R}^{n_i}$, $i\in\mathbb{N}_k$. This is a linear space of dimension $n_1\cdots n_k$ consisting of formal finite linear combinations of $\{x_1\otimes\ldots\otimes x_k : x_i\in\mathbb{R}^{n_i}, i\in\mathbb{N}_k\}$, with all necessary identifications made so that the product is multi-linear. The inner product between $u_1\otimes\ldots\otimes u_k$ and $v_1\otimes\ldots\otimes v_k$ in $\otimes_{i=1}^k\mathbb{R}^{n_i}$ is $\langle u_1,v_1\rangle\cdots\langle u_k,v_k\rangle$. The tensor product $A_1\otimes\ldots\otimes A_k$, between operators A_i on $\mathbb{R}^{n_i}$, $i\in\mathbb{N}_k$, is a linear operator on $\otimes_{i=1}^k\mathbb{R}^{n_i}$ defined by

$$(A_1\otimes\ldots\otimes A_k)(x_1\otimes\ldots\otimes x_k) := (A_1x_1)\otimes\ldots\otimes(A_kx_k)$$

and extended by linearity. For short introduction to tensor product and its properties, see [4, Chapter I].

Denote by $\{e_i^1,\ldots,e_i^{n_i}\}$ the standard orthonormal basis in $\mathbb{R}^{n_i}$ for $i\in\mathbb{N}_k$. Let $\{\mathbf{e}^l : l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}\}$ denote the standard orthonormal basis in $\mathbb{R}^{n_1\cdots n_k}$. An isometry $\mathscr{T}:\mathbb{R}^{n_1\cdots n_k}\to\mathbb{R}^{n_1}\otimes\ldots\otimes\mathbb{R}^{n_k}$ is defined by

$$\mathscr{T}(\mathbf{e}^l) := e_1^{l_1}\otimes\ldots\otimes e_k^{l_k} \quad \text{for all } l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}.$$

Given any $n_i \times n_i$ matrix A_i for all $i \in \mathbb{N}_k$ and any $\mathbf{x} \in \mathbb{R}^{n_1 \cdots n_k}$, we have

$$\mathscr{T}((\otimes_{i=1}^k A_i)\mathbf{x}) = (\otimes_{i=1}^k A_i)(\mathscr{T}\mathbf{x}),$$

where on the right-hand side, A_i is viewed as an operator on $\mathbb{R}^{n_i}$ with respect to the standard basis for all $i \in \mathbb{N}_k$.

For any $A_i \in S^{n_i}$ for all $i \in \mathbb{N}_k$, the self-adjoint operator corresponding to the symmetric matrix $F^H(A_1, \ldots, A_k)$ is

$$\begin{aligned}\mathscr{F}^H(A_1, \ldots, A_k) &:= \mathscr{T} \circ F^H(A_1, \ldots, A_k) \circ \mathscr{T}^{-1} \\ &= \sum_{l \in \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))(\otimes_{i=1}^k u_i^{l_i}) \otimes (\otimes_{i=1}^k u_i^{l_i}),\end{aligned}$$

where $U_i \in O^{n_i}$ is such that $A_i = U_i\big(\operatorname{Diag}\lambda(A_i)\big)U_i^\top$ and $u_i^{l_i}$ denotes the l_i-th column of U_i for all $i \in \mathbb{N}_k$.

2.2. Anti-symmetric tensor products

The k-tuples in $\mathbb{N}_{n,k}$ are ordered lexicographically and used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$ and matrices of dimension $\binom{n}{k} \times \binom{n}{k}$. For example, $\mathbf{x}_\rho$ is the ρ-th coordinate of a vector $\mathbf{x}$ in $\mathbb{R}^{\binom{n}{k}}$ and $\mathbf{A}_{\rho,\tau}$ is the (ρ,τ)-th element of an $\binom{n}{k} \times \binom{n}{k}$ matrix $\mathbf{A}$. But if $x \in \mathbb{R}^n$, then let $x_\rho := (x_{\rho_1}, \ldots, x_{\rho_k}) \in \mathbb{R}^k$ for any $\rho \in \mathbb{N}_{n,k}$ and if A is an $n \times n$ matrix, let $A_{\rho\tau}$ (without a comma) be the $k \times k$ minor of an A with elements at the intersections of rows $\rho_1, \ldots, \rho_k$ and columns $\tau_1, \ldots, \tau_k$ for any $\rho, \tau \in \mathbb{N}_{n,k}$.

The *k-th multiplicative compound matrix* of $n \times n$ matrix A is an $\binom{n}{k} \times \binom{n}{k}$ matrix, denoted by $A^{(k)}$, such that $A^{(k)}_{\rho,\tau} := \det(A_{\rho\tau})$ for any $\rho, \tau \in \mathbb{N}_{n,k}$. For properties of k-th multiplicative compound matrix, see for example [5].

For any vectors $x_1, \ldots, x_k \in \mathbb{R}^n$, their *$k$-th anti-symmetric tensor product* (*wedge product*) is defined by

$$x_1 \wedge \ldots \wedge x_k := \frac{1}{\sqrt{k!}} \sum_{\sigma:\mathbb{N}_k \to \mathbb{N}_k} \varepsilon_\sigma x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)},$$

where the summation is over all permutations σ on $\mathbb{N}_k$ and ε_σ is defined to be $+1$, if σ is even and to be -1, if σ is odd. The wedge product is multi-linear and anti-commutative. Denote by $\wedge^k\mathbb{R}^n$ the $\binom{n}{k}$-dimensional subspace of $\otimes^k\mathbb{R}^n$ spanned by all k-th anti-symmetric tensor products with inherited inner product

$$\langle u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k\rangle = \det\big(\langle u_i, v_j\rangle_{i,j=1}^k\big).$$

If A is an operator on $\mathbb{R}^n$, then $\otimes^k A$ keeps the subspace $\wedge^k\mathbb{R}^n$ invariant. Denote by $\wedge^k A$ the restriction of $\otimes^k A$ onto $\wedge^k\mathbb{R}^n$. It is called the *k-th anti-symmetric tensor power (wedge power) of A* and satisfies

$$(\wedge^k A)(x_1 \wedge \ldots \wedge x_k) = (Ax_1) \wedge \ldots \wedge (Ax_k).$$

For properties of k-th wedge power of A, see [5].

Denote by $\{\mathbf{e}^\rho : \rho \in \mathbb{N}_{n,k}\}$ the standard orthonormal basis in $\mathbb{R}^{\binom{n}{k}}$. An isometry $\mathscr{W} : \mathbb{R}^{\binom{n}{k}} \to \wedge^k \mathbb{R}^n$ is defined by

$$\mathscr{W}(\mathbf{e}^\rho) := e^{\rho_1} \wedge \ldots \wedge e^{\rho_k} \text{ for all } \rho \in \mathbb{N}_{n,k}$$

and extended by linearity. The relationship between $A^{(k)}$ and $\wedge^k A$ is:

$$\mathscr{W}(A^{(k)}\mathbf{x}) = (\wedge^k A)(\mathscr{W}\mathbf{x}) \quad \text{for any } \mathbf{x} \in \mathbb{R}^{\binom{n}{k}},$$

where A is viewed as an operator and a matrix with respect to the standard basis.

For future reference, the self-adjoint operator on $\wedge^k \mathbb{R}^n$, corresponding to the symmetric matrix $F(A)$, is

$$\mathscr{F}(A) := \mathscr{W} \circ F(A) \circ \mathscr{W}^{-1} = \sum_{\rho \in \mathbb{N}_{n,k}} f(\lambda_\rho(A))(u_{\rho_1} \wedge \ldots \wedge u_{\rho_k}) \otimes (u_{\rho_1} \wedge \ldots \wedge u_{\rho_k}),$$

where $\{u_1, \ldots, u_n\}$ are the columns of $U \in O^n$, such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

2.3. Operator functions on k variables

We are ready to introduce a class of operator functions on several variables by restricting (generated) k-isotropic functions to block-diagonal matrices. Henceforth, we assume that $n = n_1 + \ldots + n_k$.

Let $F^* : S^{n_1} \times \ldots \times S^{n_k} \to S^{\binom{n}{k}}$ be defined by

$$F^*(A_1, \ldots, A_k) := F(A_1 \oplus \ldots \oplus A_k),$$

where $A_1 \oplus \ldots \oplus A_k$ denotes the block-diagonal matrix with blocks A_i, $i \in \mathbb{N}_k$. The corresponding self-adjoint operator is

$$\mathscr{F}^*(A_1, \ldots, A_k) := \mathscr{W} \circ F(A_1 \oplus \ldots \oplus A_k) \circ \mathscr{W}^{-1}.$$

2.4. Note about domains

We assume that the domain of the symmetric function $f : \mathbb{R}^k \to \mathbb{R}$, denoted by $\operatorname{dom} f \subseteq \mathbb{R}^k$, is a symmetric and open set. Then, it is easy to see that the set $\operatorname{dom}_n f := \{x \in \mathbb{R}^n : x_\rho \in \operatorname{dom} f \text{ for all } \rho \in \mathbb{N}_{n,k}\}$ is also symmetric and open. Then, the domain of a (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$ corresponding to $f : \mathbb{R}^k \to \mathbb{R}$ is

$$\operatorname{dom} F := \{A \in S^n : \lambda(A) \in \operatorname{dom}_n f\}.$$

It is not too difficult to see that for any $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$, there is a $\rho \in \mathbb{N}_{n,k}$, such that $\lambda_\rho(A_1 \oplus \ldots \oplus A_k)$ is a permutation of $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$. Since the set $\operatorname{dom} f$ is symmetric, we see that $A_1 \oplus \ldots \oplus A_k \in \operatorname{dom} F$ implies that $(A_1, \ldots, A_k) \in \operatorname{dom} F^H$. Hence, the domain of F^* is the set of all k-tuples $(A_1, \ldots, A_k)$ from $S^{n_1} \times \ldots \times S^{n_k}$ that satisfy $A_1 \oplus \ldots \oplus A_k \in \operatorname{dom} F$, and this set is sufficient for our needs.

3. Connecting F^H to F^*

3.1. Introducing the linear map Π

We introduce a linear map Π that links the operator functions $\mathscr{F}^H$ and $\mathscr{F}^*$, whenever both of them are defined in terms of the same symmetric function $f:\mathbb{R}^k \to \mathbb{R}$.

Let the linear map $\Pi_i : \mathbb{R}^{n_i} \to \oplus_{j=1}^k \mathbb{R}^{n_j}$ for $i \in \mathbb{N}_k$, be the embedding of $\mathbb{R}^{n_i}$ in $\oplus_{j=1}^k \mathbb{R}^{n_j}$:

$$\Pi_i(u) := 0 \oplus \ldots \oplus u \oplus \ldots \oplus 0 \quad \text{for any } u \in \mathbb{R}^{n_i},$$

where u appears in the i-th place of the direct sum.

Let $\Pi : \otimes_{j=1}^k \mathbb{R}^{n_j} \to \wedge^k(\oplus_{j=1}^k \mathbb{R}^{n_j})$ be a linear map defined by

$$\Pi(e_1^{l_1} \otimes \ldots \otimes e_k^{l_k}) := \Pi_1(e_1^{l_1}) \wedge \ldots \wedge \Pi_k(e_k^{l_k}) = (e_1^{l_1} \oplus \ldots \oplus 0) \wedge \ldots \wedge (0 \oplus \ldots \oplus e_k^{l_k}).$$

It can be extended by linearity to any vector in $u_1 \otimes \ldots \otimes u_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$:

$$\Pi(u_1 \otimes \ldots \otimes u_k) = \Pi_1(u_1) \wedge \ldots \wedge \Pi_k(u_k). \tag{3.1}$$

Next, we show that Π preserves the inner product.

LEMMA 3.1. *For any $s_1, \ldots, s_k \in \mathbb{N}_k$ with $s_1 \leqslant \ldots \leqslant s_k$, let $u_j \in \mathbb{R}^{n_j}$ and $v_j \in \mathbb{R}^{n_{s_j}}$ for $j \in \mathbb{N}_k$. Define $\mathbf{u} := \Pi_1(u_1) \wedge \ldots \wedge \Pi_k(u_k)$ and $\mathbf{v} := \Pi_{s_1}(v_1) \wedge \ldots \wedge \Pi_{s_k}(v_k)$. The inner product between $\mathbf{u}$ and $\mathbf{v}$ is given by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} \prod_{j=1}^k \langle u_j, v_j \rangle & \text{if } s_1, \ldots, s_k \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Π preserves the inner product.

Proof. *Case I.* If all $s_1, \ldots, s_k$ are distinct, then $s_i = i$ for $i \in \mathbb{N}_k$ and we have

$$\mathbf{v} = \Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)$$

with $v_i \in \mathbb{R}^{n_i}$ for $i \in \mathbb{N}_k$. Calculate $\langle \mathbf{u}, \mathbf{v} \rangle$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \det\left(\langle \Pi_i(u_i), \Pi_j(v_j) \rangle_{i,j=1}^k\right) = \prod_{j=1}^k \langle u_j, v_j \rangle,$$

since $\langle \Pi_i(u_i), \Pi_j(v_j) \rangle = 0$, whenever $i \neq j$.

Case II. Suppose now that $s_1, \ldots, s_k$ are not distinct. Without loss of generality, assume that $s_{k-1} = s_k$, other cases being analogous. Then, we calculate the inner product between $\mathbf{u}$ and $\mathbf{v}$ by

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \langle \Pi_1(u_1) \wedge \ldots \wedge \Pi_{k-1}(u_{k-1}) \wedge \Pi_k(u_k), \Pi_{s_1}(v_1) \wedge \ldots \wedge \Pi_{s_{k-1}}(v_{k-1}) \wedge \Pi_{s_k}(v_k) \rangle \\ &= \langle (u_1 \oplus \ldots \oplus 0) \wedge \ldots \wedge (0 \oplus \ldots \oplus u_{k-1} \oplus 0) \wedge (0 \oplus \ldots \oplus u_k), \\ &\qquad \Pi_{s_1}(v_1) \wedge \ldots \wedge \Pi_{s_{k-1}}(v_{k-1}) \wedge \Pi_{s_k}(v_k) \rangle \\ &= \det(\langle \Pi_i(u_i), \Pi_{s_j}(v_j) \rangle_{i,j=1}^k) = 0, \end{aligned}$$

since either the $(k-1)$-th row or the k-th row of the determinant is zero.

For any $u := u_1 \otimes \ldots \otimes u_k, v := v_1 \otimes \ldots \otimes v_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$, we have

$$\langle u, v\rangle = \langle u_1 \otimes \ldots \otimes u_k, v_1 \otimes \ldots \otimes v_k\rangle = \prod_{i=1}^{k} \langle u_i, v_i\rangle = \langle \Pi(u), \Pi(v)\rangle,$$

hence, the linear map Π preserves the inner product. □

The linear map $\Pi : \otimes_{i=1}^k \mathbb{R}^{n_i} \to \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i})$ is an injection, so we can consider the inverse map Π^{-1}, defined on the range of Π. That is, $\Pi^{-1} \circ \Pi(u) = u$ for any $u \in \otimes_{i=1}^k \mathbb{R}^{n_i}$.

3.2. Connecting F^H to F^*

Let $n := \sum_{i=1}^k n_i$ and let $A := A_1 \oplus \ldots \oplus A_k \in S^n$, where $A_i \in S^{n_i}$ for all $i \in \mathbb{N}_k$. Let $U_i \in O^{n_i}$ be such that $A_i = U_i(\operatorname{Diag}\lambda(A_i))U_i^\top$ for all $i \in \mathbb{N}_k$. Denote by $u_i^{l_i}$ the l_i-th column of U_i and $u_i^{l_i}$ is the eigenvector corresponding to $\lambda_{l_i}(A_i)$ for all $i \in \mathbb{N}_k$. Matrix A is diagonalized by

$$A = (U_1 \oplus \ldots \oplus U_k)\big(\operatorname{Diag}\tilde{\lambda}(A)\big)(U_1 \oplus \ldots \oplus U_k)^\top, \tag{3.2}$$

where $\tilde{\lambda}(A) := (\lambda(A_1), \ldots, \lambda(A_k))$ is not necessarily ordered. For any $t \in \mathbb{N}_n$, the t-th column of $U_1 \oplus \ldots \oplus U_k$ is denoted by $\tilde{u}_t$. For any such t, there exist unique $i \in \mathbb{N}_k$ and $l_i \in \mathbb{N}_{n_i}$, such that $t = \sum_{j=1}^{i-1} n_j + l_i$ and

$$\tilde{u}_t = \Pi_i(u_i^{l_i}). \tag{3.3}$$

This notation allows us to obtain the following representation of $\mathscr{F}^*$.

Recall that any symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ defines a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by $\mathbf{f}_\rho(x) := f(x_\rho)$ for all x in the domain of f and all $\rho \in \mathbb{N}_{n,k}$. Such function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric in the sense of

$$\operatorname{Diag}\mathbf{f}(Px) = P^{(k)}\big(\operatorname{Diag}\mathbf{f}(x)\big)P^{(k)^\top} \tag{3.4}$$

for all $x \in \mathbb{R}^n$ and all $n \times n$ permutation matrix P, see [1].

PROPOSITION 3.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric. For any $A := A_1 \oplus \ldots \oplus A_k$ with $A_i \in S^{n_i}$ for $i \in \mathbb{N}_k$, we have*

$$\mathscr{F}^*(A_1, \ldots, A_k) = \sum_{\rho \in \mathbb{N}_{n,k}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}).$$

Proof. Let P be an $n \times n$ permutation matrix, such that $\lambda(A) = P\tilde{\lambda}(A)$. Using (3.2), we obtain

$$\begin{aligned} A &= (U_1 \oplus \ldots \oplus U_k)\big(\operatorname{Diag}\tilde{\lambda}(A)\big)(U_1 \oplus \ldots \oplus U_k)^\top \\ &= (U_1 \oplus \ldots \oplus U_k)\big(\operatorname{Diag}P^\top\lambda(A)\big)(U_1 \oplus \ldots \oplus U_k)^\top \\ &= (U_1 \oplus \ldots \oplus U_k)P^\top\big(\operatorname{Diag}\lambda(A)\big)P(U_1 \oplus \ldots \oplus U_k)^\top. \end{aligned}$$

Let $U := (U_1 \oplus \ldots \oplus U_k)P^\top$ and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be defined by $\mathbf{f}_\rho(x) := f(x_\rho)$ for all $\rho \in \mathbb{N}_{n,k}$ and all $x \in \mathbb{R}^n$. Such $\mathbf{f}$ is symmetric in the sense of (3.4). Then, we have

$$\begin{aligned} F^*(A_1,\ldots,A_k) = F(A) &= U^{(k)}\big(\mathrm{Diag}\,\mathbf{f}(\lambda(A))\big)U^{(k)\top} \\ &= U^{(k)}\big(\mathrm{Diag}\,\mathbf{f}(P\tilde{\lambda}(A))\big)U^{(k)\top} \\ &= U^{(k)}P^{(k)}\big(\mathrm{Diag}\,\mathbf{f}(\tilde{\lambda}(A))\big)P^{(k)\top}U^{(k)\top} \\ &= (UP)^{(k)}\big(\mathrm{Diag}\,\mathbf{f}(\tilde{\lambda}(A))\big)(UP)^{(k)\top} \\ &= (U_1 \oplus \ldots \oplus U_k)^{(k)}\big(\mathrm{Diag}\,\mathbf{f}(\tilde{\lambda}(A))\big)(U_1 \oplus \ldots \oplus U_k)^{(k)\top}, \end{aligned}$$

where we used (3.4). The rest follows. □

Denote by $\mathscr{M}$ the collection of all $\rho \in \mathbb{N}_{n,k}$ satisfying

$$\rho_i \in \big\{(n_1 + \cdots + n_{i-1}) + 1, \ldots, (n_1 + \cdots + n_{i-1}) + n_i\big\} \quad \text{for all } i \in \mathbb{N}_k,$$

with the understanding that the sums in the parenthesis are zero when $i = 1$. Let

$$\mathscr{M}^c := \mathbb{N}_{n,k} \setminus \mathscr{M}.$$

Define the operator $\mathscr{F}^*_{\mathscr{M}}$ by

$$\mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k) := \sum_{\rho \in \mathscr{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})$$

and let $\mathscr{F}^*_{\mathscr{M}^c}(A_1,\ldots,A_k) := \mathscr{F}^*(A_1,\ldots,A_k) - \mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k)$.

The relationship between $\mathscr{F}^H$ and $\mathscr{F}^*$ is given in the next theorem.

THEOREM 3.2. *Then, for any $A_i \in S^{n_i}$, $i \in \mathbb{N}_k$, the following diagram commutes*

$$\begin{array}{ccc} \otimes_{i=1}^k \mathbb{R}^{n_i} & \xrightarrow{\mathscr{F}^H(A_1,\ldots,A_k)} & \otimes_{i=1}^k \mathbb{R}^{n_i} \\ \downarrow \Pi & & \downarrow \Pi \\ \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i}) & \xrightarrow{\mathscr{F}^*(A_1,\ldots,A_k)} & \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i}) \end{array}$$

Moreover, the operators $\mathscr{F}^*(A_1,\ldots,A_k)$ and $\mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k)$ coincide on the subspace $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

Proof. We need to show that for any $v := v_1 \otimes \ldots \otimes v_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$, we have

$$\Pi \circ \mathscr{F}^H(A_1,\ldots,A_k)(v) = \mathscr{F}^*(A_1,\ldots,A_k) \circ \Pi(v). \tag{3.5}$$

For the right-hand side, we have

$$\mathscr{F}^*(A_1,\ldots,A_k) \circ \Pi(v) = \mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k) \circ \Pi(v) + \mathscr{F}^*_{\mathscr{M}^c}(A_1,\ldots,A_k) \circ \Pi(v). \tag{3.6}$$

Note that the elements ρ of $\mathscr{M}$ have the property that for any $\rho_i \in \rho$, there exists a unique $l_i \in \mathbb{N}_{n_i}$, such that $\rho_i = \sum_{j=1}^{i-1} n_j + l_i$. Thus, using (3.3) and (3.1), we obtain

$$\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k} = \Pi_1(u_1^{l_1}) \wedge \ldots \wedge \Pi_k(u_k^{l_k}) = \Pi(u_1^{l_1} \otimes \ldots \otimes u_k^{l_k}). \tag{3.7}$$

With that we express the first term on the right-hand side of (3.6) as

$$\begin{aligned}
&\mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k) \circ \Pi(v)\\
&= \sum_{\rho\in\mathscr{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})\big(\Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)\big)\\
&= \sum_{\rho\in\mathscr{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})\langle \tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}, \Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)\rangle\\
&= \sum_{\rho\in\mathscr{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})\langle \Pi_1(u_1^{l_1}) \wedge \ldots \wedge \Pi_k(u_k^{l_k}), \Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)\rangle\\
&= \sum_{\rho\in\mathscr{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}) \prod_{j=1}^{k} \langle u_j^{l_j}, u_j\rangle\\
&= \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k)) \prod_{j=1}^{k} \langle u_j^{l_j}, u_j\rangle \Pi(u_1^{l_1} \otimes \ldots \otimes u_k^{l_k}),
\end{aligned}$$

where in the last three equalities we used (3.7) and Lemma 3.1.

We now turn our attention to the second term on the right-hand side of (3.6). Note that the elements ρ of $\mathscr{M}^c$ have the property that for any $\rho_i \in \rho$, there exists unique $s_i \in \mathbb{N}_k$ and $l_i \in \mathbb{N}_{n_{s_i}}$, such that $\rho_i = \sum_{j=1}^{s_i-1} n_j + l_i$. The important observation is that the indexes $s_1,\ldots,s_k$ are not distinct and

$$\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k} = \Pi_{s_1}(u_{s_1}^{l_1}) \wedge \ldots \wedge \Pi_{s_k}(u_{s_k}^{l_k}).$$

With that, we calculate $\mathscr{F}^*_{\mathscr{M}^c}(A_1,\ldots,A_k) \circ \Pi(v)$ by

$$\begin{aligned}
&\mathscr{F}^*_{\mathscr{M}^c}(A_1,\ldots,A_k) \circ \Pi(v)\\
&= \sum_{\rho\in\mathscr{M}^c} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})\big(\Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)\big)\\
&= \sum_{\rho\in\mathscr{M}^c} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \ldots \wedge \tilde{u}_{\rho_k})\langle \Pi_{s_1}(u_{s_1}^{l_1}) \wedge \ldots \wedge \Pi_{s_k}(u_{s_k}^{l_k}), \Pi_1(v_1) \wedge \ldots \wedge \Pi_k(v_k)\rangle\\
&= 0,
\end{aligned}$$

where the last equality is obtained using Lemma 3.1.

Combining the results for the two terms on the right-hand side of (3.6), gives

$$\begin{aligned}
&\mathscr{F}^*(A_1,\ldots,A_k) \circ \Pi(v)\\
&= \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k)) \prod_{j=1}^{k} \langle u_j^{l_j}, u_j\rangle \Pi(u_1^{l_1} \otimes \ldots \otimes u_k^{l_k}).
\end{aligned}$$

This also shows that $\mathscr{F}^*(A_1,\ldots,A_k)$ and $\mathscr{F}^*_{\mathscr{M}}(A_1,\ldots,A_k)$ are the same map when restricted to the subspace $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

For the left-hand side of (3.5), we have

$$\begin{aligned}
&\Pi \circ \mathscr{F}^H(A_1,\ldots,A_k)(v)\\
&= \Pi \circ \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k))(\otimes_{i=1}^k u_i^{l_i})\otimes(\otimes_{i=1}^k u_i^{l_i})(v_1\otimes\ldots\otimes v_k)\\
&= \Pi \circ \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k))\langle\otimes_{i=1}^k u_i^{l_i}, v_1\otimes\ldots\otimes v_k\rangle(\otimes_{i=1}^k u_i^{l_i})\\
&= \Pi \circ \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k))\prod_{i=1}^k\langle u_i^{l_i}, v_j\rangle(u_1^{l_1}\otimes\ldots\otimes u_k^{l_k})\\
&= \sum_{l\in\mathbb{N}_{n_1}\times\ldots\times\mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1),\ldots,\lambda_{l_k}(A_k))\prod_{i=1}^k\langle u_i^{l_i}, v_j\rangle\Pi(u_1^{l_1}\otimes\ldots\otimes u_k^{l_k}).
\end{aligned}$$

This shows that the diagram commutes. $\square$

Theorem 3.2 shows that

$$\begin{aligned}
\mathscr{F}^H(A_1,\ldots,A_k) &= \Pi^{-1}\circ\mathscr{F}^*(A_1,\ldots,A_k)\circ\Pi\\
&= \Pi^{-1}\circ\mathscr{F}(A_1\oplus\ldots\oplus A_k)\circ\Pi \quad \text{and}\\
\mathscr{F}(A_1\oplus\ldots\oplus A_k) &= \Pi\circ\mathscr{F}^H(A_1,\ldots,A_k)\circ\Pi^{-1},
\end{aligned} \tag{3.8}$$

where both sides of the last equality are assumed to be restricted to $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

Thus, one can use (3.8) to infer properties of $\mathscr{F}^H$ from those of $\mathscr{F}$.

4. Differentiability properties of F^H

In this section, we study the differentiability properties of F^H. We start with those associated with a symmetric function $f:\mathbb{R}^k\to\mathbb{R}$. The following is a special case of Theorem 5.1 in [10], which was proven for the more general k-isotropic functions.

THEOREM 4.1. *Let $f:\mathbb{R}^k\to\mathbb{R}$ be symmetric with corresponding (generated) k-isotropic function $F:S^n\to S^{\binom{n}{k}}$. Then, F is C^m at A, if and only if f is C^m at $\lambda_\rho(A)$ for any $\rho\in\mathbb{N}_{n,k}$. Here, $m=0,1,\ldots$*

Theorem 4.1, together with (3.8), allows us to see the following corollary.

COROLLARY 4.1. *Let $f:\mathbb{R}^k\to\mathbb{R}$ be symmetric with corresponding $F^H: S^{n_1}\times\ldots\times S^{n_k}\to S^{n_1\cdots n_k}$. The function F^H is C^m at $(A_1,\ldots,A_k)$, whenever f is C^m at $\lambda_\rho(A_1\oplus\ldots\oplus A_k)$ for any $\rho\in\mathbb{N}_{n,k}$. Here, $m=0,1,\ldots$*

Proof. Suppose that f is C^m at $\lambda_\rho(A_1\oplus\ldots\oplus A_k)$ for any $\rho\in\mathbb{N}_{n,k}$. Using Theorem 4.1, one can obtain that the corresponding (generated) k-isotropic function is C^m at $A_1\oplus\ldots\oplus A_k$. Using (3.8), the corresponding F^H is C^m at $(A_1,\ldots,A_k)$. $\square$

Restricting F^H to diagonal matrices, we get the following converse.

COROLLARY 4.2. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric with corresponding $F^H : S^{n_1} \times \ldots \times S^{n_k} \to S^{n_1 \cdots n_k}$. The function f is C^m at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for any $l \in \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}$, whenever F^H is C^m at $(A_1, \ldots, A_k)$. Here, $m = 0, 1, \ldots$*

An inductive formula for the first and higher-order derivatives of k-isotropic functions was the focus of [10]. A formula for just the first derivative of (generated) k-isotropic functions was obtained in [1]. Thus, at least in theory, one can obtain the formula for the derivatives of F^H using (3.8).

Now, address the analyticity of F^H. Here, the symmetricity of $f : \mathbb{R}^k \to \mathbb{R}$ is not necessary.

THEOREM 4.2. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a function with corresponding $F^H : S^{n_1} \times \ldots \times S^{n_k} \to S^{n_1 \cdots n_k}$. The function F^H is analytic at $(A_1, \ldots, A_k)$, if and only if f is analytic at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for all $l \in \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_k}$.*

Proof. Suppose $f : \mathbb{R}^k \to \mathbb{R}$ is analytic. Then, the Cauchy integral representation of f holds as follows

$$f(x_1, \ldots, x_k) = \frac{1}{(2\pi i)^k} \oint_{\Gamma_k} \cdots \oint_{\Gamma_1} \frac{f(z_1, \ldots, z_k)}{\prod_{j=1}^k (z_j - x_j)} dz_1 \cdots dz_k,$$

where Γ_j is a positively oriented circle in the complex plane enclosing the points x_j for all $j \in \mathbb{N}_k$. The Dunford-Taylor integral representation of $F^H(A_1, \ldots, A_k)$ for any $A_j \in S^{n_j}$, $j \in \mathbb{N}_k$ is

$$\begin{aligned}
F^H(A_1, \ldots, A_k) &= (\otimes_{i=1}^k U_i)\big(\mathrm{Diag}_l\, f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))\big)(\otimes_{i=1}^k U_i)^\top \\
&= (\otimes_{j=1}^k U_j)\Big(\mathrm{Diag}_l \frac{1}{(2\pi i)^k} \oint_{\Gamma_k} \cdots \oint_{\Gamma_1} \frac{f(z_1, \ldots, z_k)}{\prod_{j=1}^k (z_j - \lambda_{l_j}(A_j))} dz_1 \cdots dz_k\Big)(\otimes_{j=1}^k U_j)^\top \\
&= \frac{1}{(2\pi i)^k} \oint_{\Gamma_k} \cdots \oint_{\Gamma_1} f(z_1, \ldots, z_k)(\otimes_{j=1}^k U_j) \\
&\qquad \Big(\mathrm{Diag}_l \prod_{j=1}^k (z_j - \lambda_{l_j}(A_j))^{-1}\Big)(\otimes_{j=1}^k U_j)^\top dz_1 \cdots dz_k,
\end{aligned}$$

where $U_j \in O^{n_j}$ is such that $A_j = U_j\big(\mathrm{Diag}\,\lambda(A_j)\big)U_j^\top$ and Γ_j is a positively oriented circle in the complex plane enclosing all eigenvalues $\{\lambda_{l_j}(A_j) : l_j \in \mathbb{N}_{n_j}\}$ for all $j \in \mathbb{N}_k$. Notice that

$$(\otimes_{j=1}^k U_j)\Big(\mathrm{Diag}_l \prod_{j=1}^k (z_j - \lambda_{l_j}(A_j))^{-1}\Big)(\otimes_{j=1}^k U_j)^\top = (z_1 I - A_1)^{-1} \otimes \ldots \otimes (z_k I - A_k)^{-1},$$

holds. Thus, we have the integral representation

$$\begin{aligned}
&F^H(A_1, \ldots, A_k) \\
&= \frac{1}{(2\pi i)^k} \oint_{\Gamma_k} \cdots \oint_{\Gamma_1} f(z_1, \ldots, z_k)\big((z_1 I - A_1)^{-1} \otimes \ldots \otimes (z_k I - A_k)^{-1}\big) dz_1 \cdots dz_k.
\end{aligned}$$

Since the eigenvalue map $A_j \mapsto \lambda(A_j)$ is a continuous function, the circle Γ_j encloses the eigenvalues of all matrices in a small neighbourhood of A_j for all $j \in \mathbb{N}_k$. It is easy to see then, that for each fixed $(z_1,\dots,z_k)$, the integrand is analytic in $(A_1,\dots,A_k)$, and so is F^H.

For the other direction, restrict the function F^H to diagonal matrices. $\square$

REFERENCES

[1] B. AMES AND H. SENDOV, *Derivatives of compound matrix valued functions*, J. Math. Anal. Appl. **433** (2016), pp. 1459–1485.

[2] H. ARAKI AND F. HANSEN, *Jensen's operator inequality for functions of several variables*, Proc. Amer. Math. Soc. **128** (2000), pp. 2075–2084.

[3] J. AUJLA, *Matrix convexity of functions of two variables*, Linear Algebra Appl. **194** (1993), pp. 149–160.

[4] R. BHATIA, *Matrix analysis*, vol. 169, Springer Science & Business Media, 2013.

[5] M. FIEDLER, *Additive compound matrices and an inequality for eigenvalues of symmetric stochastic matrices*, Czechoslovak Math. J. **24** (1974), pp. 392–402.

[6] F. HANSEN, *Operator convex functions of several variables*, Publ. Res. Inst. Math. Sci. **33** (1997), pp. 443–463.

[7] F. HANSEN, *Operator monotone functions of several variables*, Math. Inequal. Appl. **6** (2003), pp. 1–7.

[8] T. JIANG, M. MOUSAVI, AND S. SENDOV, *On the analyticity of k-isotropic functions*, Submitted, (2017).

[9] A. KORÁNYI, *On some classes of analytic functions of several variables*, Trans. Amer. Math. Soc. (1961), pp. 520–554.

[10] S. MOUSAVI AND H. SENDOV, *A unified approach to spectral and isotropic functions*, Submitted, (2016).

[11] M. SINGH AND H. VASUDEVA, *Monotone matrix functions of two variables*, Linear Algebra Appl. **328** (2001), pp. 131–152.

[12] Z. ZHANG, *Some operator convex functions of several variables*, Linear Algebra Appl. **463** (2014), pp. 1–9.

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