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# HYPONORMALITY OF FINITE RANK PERTURBATIONS OF NORMAL OPERATORS

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*Abstract.* Let $T$ be an arbitrary finite rank perturbation of a normal operator $N$ acting on a separable, infinite dimensional, complex Hilbert space $\mathscr{H}$. It is proved that the hyponormality and normality of $T$ are equivalent. Thus every hyponormal finite rank perturbation of a normal operator has a nontrivial hyperinvariant subspace.

## 1. Introduction and notation

This paper is a continuation of first and second authors' earlier paper [12] in which we discussed the hyponormality of rank-one perturbations of normal operators acting on a separable, infinite dimensional, complex Hilbert space $\mathscr{H}$. The notation and terminology in what follows are taken from [12]. For the convenience of the reader we recall a few pertinent definitions. The algebra of bounded linear operators on $\mathscr{H}$ is denoted by $\mathscr{L}(\mathscr{H})$. For nonzero vectors $u$ and $v$ in $\mathscr{H}$ we write $u \otimes v$ for the rank-one operator in $\mathscr{L}(\mathscr{H})$ by $(u \otimes v)(x) = \langle x, v\rangle u$, $x \in \mathscr{H}$. For $X, Y \in \mathscr{L}(\mathscr{H})$, we denote by $[X,Y] = XY - YX$. An operator $T \in \mathscr{L}(\mathscr{H})$ is *normal* if $[T^*, T] = 0$, and $T \in \mathscr{L}(\mathscr{H})$ is *hyponormal* if $[T^*, T]$ is positive, i.e., $\langle [T^*, T]x, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. An operator $T$ in $\mathscr{L}(\mathscr{H})$ is called a *finite rank perturbation of a normal operator* if there exist nonzero vectors $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ in $\mathscr{H}$ and a normal operator $N \in \mathscr{L}(\mathscr{H})$ such that $T$ is unitarily equivalent to an operator $N + \sum_{j=1}^n u_j \otimes v_j$. In particular, for $n = 1$, such operator $T$ is referred to a *rank-one perturbation of a normal operator*. The rank-one perturbations of normal operators can be applied to some areas in mathematical physics (cf. [3], [13], [16]). And also the finite rank perturbations of a normal operator can be applied to solve the von Neumann invariant subspace problem of bounded operators (cf. [17]). E. Ionascu([11]) detected the structure of rank-one perturbations of diagonal operators. Also, in [14] one discussed some properties of rank-one perturbations of unilateral shifts operators. Moreover, in [4] one considered rank-one perturbations of weighted shifts to examine distinctions among various sorts of weak hyponormalities;

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see [10] for weak hyponormalities. In [12], Jung-Lee proved that if $T$ in $\mathscr{L}(\mathscr{H})$ is a rank-one perturbation of a normal operator, then the hyponormality and normality of $T$ are equivalent. As a continued study, we detect the structure of $[T^*,T]$ and prove that if $T$ is a finite rank perturbation of a normal operator, then hyponormality and normality of $T$ are equivalent in Section 2. This implies obviously that if $T$ in $\mathscr{L}(\mathscr{H})$ is a hyponormal finite rank perturbation of a normal operator, then $T$ has a nontrivial hyperinvariant subspace.

Throughout this note, we write $\mathbb{C}$ for the set of complex numbers. For $A \in \mathscr{L}(\mathscr{H})$, $\operatorname{ran} A$ denotes the range of $A$ as usual. Since $(Au)\otimes v = A(u\otimes v)$, we denote it by $Au\otimes v$. For a subset $X$ of $\mathscr{H}$, $\vee X$ is the subspace of $\mathscr{H}$ spanned by $X$.

## 2. Main theorem

Let $\{u_k\}_{k=1}^n$ and $\{v_k\}_{k=1}^n$ be nonzero vectors in $\mathscr{H}$ and let

$$T := N + \sum_{k=1}^{n} u_k \otimes v_k \tag{2.1}$$

be a finite rank perturbation of a normal operator $N$ in $\mathscr{L}(\mathscr{H})$. We first introduce the main theorem of this note as following.

THEOREM 2.1. *Let $T$ be a finite rank perturbation of a normal operator $N$ in $\mathscr{L}(\mathscr{H})$. Then $T$ is hyponormal if and only if $T$ is normal.*

The proof of Theorem 2.1 will be given after lemma and remark. Let $T$ be a usual finite rank perturbation of a normal operator $N$ in $\mathscr{L}(\mathscr{H})$ as in (2.1). Then a simple computation shows that

$$\begin{aligned} [T^*,T] = \sum_{k=1}^{n} [&N^* u_k \otimes v_k + v_k \otimes N^* u_k - N v_k \otimes u_k - u_k \otimes N v_k \\ &+ \sum_{l=1}^{n} (\langle u_l, u_k\rangle v_k \otimes v_l - \langle v_l, v_k\rangle u_k \otimes u_l)]. \end{aligned} \tag{2.2}$$

For brevity, we denote the subspaces by

$$\mathscr{M} := \vee\{u_k, v_k\}_{k=1}^n$$

and

$$\mathscr{R} := \vee\{u_k, v_k, N^* u_k, N v_k\}_{k=1}^n.$$

By (2.2), we obtain that $\operatorname{ran}([T^*,T]) \subset \mathscr{R}$.

We now discuss matrix structure of the commutator $[T^*,T]$ of $T^*$ and $T$ with $\dim \mathscr{M} = d \leqslant 2n$.

LEMMA 2.2. *Let $T = N + \sum_{k=1}^n u_k \otimes v_k$ be a finite rank perturbation of a normal operator $N$ in $\mathscr{L}(\mathscr{H})$ and suppose that $\dim \mathscr{M} = d \leqslant 2n$. Then there exists an orthonormal system $\{e_i\}_{i=1}^m$ in $\mathscr{H}$ with $m = d + 2n$ such that*

*(i)* $\mathscr{M} = \vee\{e_i\}_{i=1}^d$,
*(ii)* $\mathscr{R} \subset \vee\{e_i\}_{i=1}^m$ $(:= \mathscr{N}_m)$,
*(iii)* $[T^*,T] \cong A_m \oplus 0_{\mathscr{H}\ominus\mathscr{N}_m}$ *relative to* $\mathscr{N}_m \oplus (\mathscr{H}\ominus\mathscr{N}_m)$, *where*

$$A_m \cong \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} & a_{1d+1} & a_{1d+2} & \cdots & a_{1m} \\ \overline{a_{12}} & a_{22} & \cdots & a_{2d} & a_{2d+1} & a_{2d+2} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1d}} & \overline{a_{2d}} & \cdots & a_{dd} & a_{dd+1} & a_{dd+2} & \cdots & a_{dm} \\ \overline{a_{1d+1}} & \overline{a_{2d+1}} & \cdots & \overline{a_{dd+1}} & 0 & 0 & \cdots & 0 \\ \overline{a_{1d+2}} & \overline{a_{2d+2}} & \cdots & \overline{a_{dd+2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1m}} & \overline{a_{2m}} & \cdots & \overline{a_{dm}} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.3a}$$

*with*

$$\begin{aligned} a_{ij} = \sum_{k=1}^n [&\langle e_j, v_k\rangle\langle N^*u_k, e_i\rangle + \langle e_j, N^*u_k\rangle\langle v_k, e_i\rangle \\ &- \langle e_j, u_k\rangle\langle Nv_k, e_i\rangle - \langle e_j, Nv_k\rangle\langle u_k, e_i\rangle \\ &+ \sum_{l=1}^n (\langle u_l, u_k\rangle\langle e_j, v_l\rangle\langle v_k, e_i\rangle - \langle v_l, v_k\rangle\langle e_j, u_l\rangle\langle u_k, e_i\rangle)]. \end{aligned} \tag{2.3b}$$

*Proof.* Suppose that the dimension of $\mathscr{M}$ is $d$. Then, by Gram-Schmidt orthogonal process ([20, Th. 3.5]), we can take an orthonormal system $\{e_i\}_{i=1}^d$ such that

$$\mathscr{M} = \vee\{e_i\}_{i=1}^d. \tag{2.4}$$

Take an extended orthonormal system $\{e_i\}_{i=1}^m$ containing $\{e_i\}_{i=1}^d$ with $m = d+2n$ such that $\mathscr{R} \subset \vee\{e_i\}_{i=1}^m$. We denote by $\mathscr{N}_m := \vee\{e_i\}_{i=1}^m$. It follows from (2.2) that for $h \in \mathscr{H}$,

$$\begin{aligned} [T^*,T]h = \sum_{k=1}^n [&\langle h, v_k\rangle N^*u_k + \langle h, N^*u_k\rangle v_k - \langle h, u_k\rangle Nv_k - \langle h, Nv_k\rangle u_k \\ &+ \sum_{l=1}^n (\langle u_l, u_k\rangle\langle h, v_l\rangle v_k - \langle v_l, v_k\rangle\langle h, u_l\rangle u_k)]. \end{aligned} \tag{2.5}$$

Thus, by (2.5), $[T^*,T]\mathscr{N}_m \subset \mathscr{R} \subset \mathscr{N}_m$, and so $\mathscr{N}_m$ is a reducing subspace for $[T^*,T]$. Considering some orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\mathscr{H}$ containing $\{e_i\}_{i=1}^m$, we get $[T^*,T]e_i = 0,\ i \geqslant m+1$. Hence we have a decomposition

$$[T^*,T] \cong A_m \oplus 0_{\mathscr{H}\ominus\mathscr{N}_m}$$

relative to $\mathscr{N}_m \oplus (\mathscr{H}\ominus\mathscr{N}_m)$, where $A_m$ is unitarily equivalent to an $m\times m$ complex matrix $(a_{ij})_{1\leqslant i,j\leqslant m}$. Substituting $e_j$ for $h$ in (2.5), we obtain that

$$\begin{aligned} a_{ij} &= \langle [T^*,T]e_j,e_i\rangle \\ &= \sum_{k=1}^{n}[\langle e_j,v_k\rangle\langle N^*u_k,e_i\rangle + \langle e_j,N^*u_k\rangle\langle v_k,e_i\rangle \\ &\quad - \langle e_j,u_k\rangle\langle Nv_k,e_i\rangle - \langle e_j,Nv_k\rangle\langle u_k,e_i\rangle \\ &\quad + \sum_{l=1}^{n}(\langle u_l,u_k\rangle\langle e_j,v_l\rangle\langle v_k,e_i\rangle - \langle v_l,v_k\rangle\langle e_j,u_l\rangle\langle u_k,e_i\rangle)]. \end{aligned} \tag{2.6}$$

Using (2.6), we can obtain (2.3a) and $a_{ji}=\overline{a_{ij}}$. Hence the proof is complete. □

Let $\mathscr{H}_1,\mathscr{H}_2$ be Hilbert spaces and let $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ be the Banach space of all bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. We recall a well-known result in operator theory below.

REMARK 2.3. Suppose $A\in\mathscr{L}(\mathscr{H}_1)$, $B\in\mathscr{L}(\mathscr{H}_2,\mathscr{H}_1)$ and $C\in\mathscr{L}(\mathscr{H}_2)$, and let

$$S := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

relative to some decomposition. Then it follows from [18] that $S\geqslant 0$ if and only if $A\geqslant 0, C\geqslant 0$ and $B=\sqrt{A}E\sqrt{C}$, for some contraction $E\in\mathscr{L}(\mathscr{H}_2,\mathscr{H}_1)$. Hence if every diagonal entry of the positive matrix $S$ is zero, then $S=0$.

Now we are ready to give the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Since every normal operator is hyponormal, we prove only the sufficiency. So we suppose that $T$ is hyponormal and put $d=\dim\mathscr{M}$. Then it follows from Lemma 2.2 that there exists an orthonormal system $\{e_i\}_{i=1}^m$ in $\mathscr{H}$ with $m=d+2n$ such that $[T^*,T]\cong A_m\oplus 0_{\mathscr{H}\ominus\mathscr{N}_m}$, where $A_m$ and $\mathscr{N}_m$ are as in Lemma 2.2. It is obvious that $A_m\geqslant 0$. Hence $a_{ii}\geqslant 0$ for all $1\leqslant i\leqslant d$ and by Remark 2.3, $a_{ij}=0$, $d+1\leqslant j\leqslant m$. Now it is sufficient to see that $a_{ii}=0$ for all $1\leqslant i\leqslant d$. Recall from (2.4) and (2.3b) that

$$u_i = \sum_{k=1}^{d}\langle u_i,e_k\rangle e_k, \text{ for } 1\leqslant i\leqslant n, \tag{2.7a}$$

$$v_i = \sum_{k=1}^{d}\langle v_i,e_k\rangle e_k, \text{ for } 1\leqslant i\leqslant n \tag{2.7b}$$

and

$$\begin{aligned} a_{ii} = 2\mathrm{Re}\Bigg(&\sum_{k=1}^{n}(\langle e_i,v_k\rangle\langle N^*u_k,e_i\rangle - \langle e_i,u_k\rangle\langle Nv_k,e_i\rangle) \\ &+ \sum_{1\leqslant k<l\leqslant n}(\langle u_l,u_k\rangle\langle e_i,v_l\rangle\langle v_k,e_i\rangle - \langle v_l,v_k\rangle\langle e_i,u_l\rangle\langle u_k,e_i\rangle)\Bigg) \\ &+ \sum_{k=1}^{n}(\|u_k\|^2|\langle e_i,v_k\rangle|^2 - \|v_k\|^2|\langle e_i,u_k\rangle|^2). \end{aligned} \tag{2.7c}$$

Thus, by (2.7a-c), we have

$$\begin{aligned}\sum_{i=1}^{d} a_{ii} = 2\mathrm{Re}\Bigg( &\sum_{k=1}^{n}\Big(\sum_{i=1}^{d}\langle e_i, v_k\rangle\langle N^* u_k, e_i\rangle - \sum_{i=1}^{d}\langle e_i, u_k\rangle\langle N v_k, e_i\rangle\Big) \\ &+ \sum_{1\leqslant k<l\leqslant n}\Big(\sum_{i=1}^{d}\langle u_l, u_k\rangle\langle e_i, v_l\rangle\langle v_k, e_i\rangle - \sum_{i=1}^{d}\langle v_l, v_k\rangle\langle e_i, u_l\rangle\langle u_k, e_i\rangle\Big)\Bigg) \\ &+ \sum_{k=1}^{n}\Big(\|u_k\|^2\sum_{i=1}^{d}|\langle e_i, v_k\rangle|^2 - \|v_k\|^2\sum_{i=1}^{d}|\langle e_i, u_k\rangle|^2\Big) \\ = 2\mathrm{Re}\Bigg( &\sum_{k=1}^{n}\Big(\langle N^* u_k, \sum_{i=1}^{d}\langle v_k, e_i\rangle e_i\rangle - \langle N v_k, \sum_{i=1}^{d}\langle u_k, e_i\rangle e_i\rangle\Big) \\ &+ \sum_{1\leqslant k<l\leqslant n}\Big(\langle u_l, u_k\rangle\langle v_k, \sum_{i=1}^{d}\langle v_l, e_i\rangle e_i\rangle - \langle v_l, v_k\rangle\langle u_k, \sum_{i=1}^{d}\langle u_l, e_i\rangle e_i\rangle\Big)\Bigg) \\ &+ \sum_{k=1}^{n}\Big(\|u_k\|^2\|v_k\|^2 - \|v_k\|^2\|u_k\|^2\Big).\end{aligned}$$

By using (2.7a,b) again, we obtain

$$\begin{aligned}\sum_{i=1}^{d} a_{ii} = 2\mathrm{Re}\Bigg( &\sum_{k=1}^{n}\big(\langle N^* u_k, v_k\rangle - \langle N v_k, u_k\rangle\big) \\ &+ \sum_{1\leqslant k<l\leqslant n}\big(\langle u_l, u_k\rangle\langle v_k, v_l\rangle - \langle v_l, v_k\rangle\langle u_k, u_l\rangle\big)\Bigg) = 0.\end{aligned}$$

Thus $a_{ii} = 0$ for all $1 \leqslant i \leqslant d$. Hence the proof is complete. □

## 3. Remark on invariant subspaces

Recall that $\mathscr{M}$ is a *nontrivial invariant* [*hyperinvariant*] *subspace* for $T \in \mathscr{L}(\mathscr{H})$ if $T\mathscr{M} \subset \mathscr{M}$ [$X\mathscr{M} \subset \mathscr{M}$ for $X \in \{T\}' = \{X \in \mathscr{L}(\mathscr{H}) : XT = TX\}$] with $(0) \neq \mathscr{M} \neq \mathscr{H}$. In 1930's, J. von Neumann introduced the invariant subspace problem: does every operator in $\mathscr{L}(\mathscr{H})$ have a nontrivial invariant subspace? Although many operator theorists tried to solve this problem until now, it remains still as an open problem (cf. [17]). An operator $T$ in $\mathscr{L}(\mathscr{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace. In 1978, S. Brown ([1]) proved that every subnormal operator has a nontrivial invariant subspace. The question of whether subnormal operators in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}1_{\mathscr{H}}$ has a nontrivial hyperinvariant subspace is still open (cf. [6], [19]). Note that every subnormal operator is hyponormal. And also the question whether every hyponormal operator has a nontrivial invariant subspace is still open (cf. [2]). We recall the following problem:

(P1) *Does every operator $T$ of the form $T = N + K$, where $N$ is normal operator and $K$ is compact operator, have a nontrivial invariant subspace?*

The theorem of Berger-Shaw reduces the invariant subspace problem for hyponormal operators to a very special case of the following result ([15, Corollary 8.5]):

(P2) *If every operator* $T$ *of the form* $T = N + K$, *where* $N$ *is normal operator and* $K$ *is compact operator, has a nontrivial invariant subspace, every hyponormal operator has a nontrivial invariant subspace.*

As one of effective studies concerning (P2), the following problem was suggested in [15, Problem K].

(P3) *Suppose* $N$ *is a diagonal normal operator whose eigenvalues constitute a dense subset of the unit disc* $\mathbb{D}$. *Does every operator of the form* $N + F$ *have a nontrivial invariant subspace, where* $F$ *is an operator of rank one?*

Despite the fact that Problem (P3) is about forty years old, it has remained stubbornly intractable, although some operator theoriests obtained some partial solutions (cf. [5], [7], [8], [9]). From this point of view, the following corollary which comes immediately from Theorem 2.1 is interesting.

COROLLARY 3.1. *Let* $T$ *be a finite rank perturbation of a normal operator* $N$ *in* $\mathscr{L}(\mathscr{H})$. *If* $T$ *is hyponormal, then* $T$ *has a nontrivial hyperinvariant subspace.*

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