

HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS ON THE VECTOR VALUED BERGMAN SPACE

PUYU CUI, YUFENG LU AND YANYUE SHI

(Communicated by R. Curto)

Abstract. We arrive at a necessary condition and a sufficient condition for the hyponormal block Toeplitz operator T_{F+G^*} on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$, where F and G are matrix valued analytic polynomials. Both of them are also new for the Bergman space $L_a^2(\mathbb{D})$.

1. Introduction

In the complex plane \mathbb{C} , let \mathbb{T} be the unit circle and \mathbb{D} be the open unit disk with the normalized Lebesgue area measure dA . Denote by $L^\infty(\mathbb{D})$ and $h^\infty(\mathbb{D})$ the space of essential bounded measurable functions and the space of all bounded harmonic functions on \mathbb{D} , respectively. Let $L^2(\mathbb{D})$ be the space of the square integrable functions on \mathbb{D} with respect to dA and $L_a^2(\mathbb{D})$ be the place of all analytic functions in $L^2(\mathbb{D})$. We denote the space of vector valued square integrable functions on \mathbb{D} by $L^2(\mathbb{D}, \mathbb{C}^n) = L^2(\mathbb{D}) \otimes \mathbb{C}^n$ and the vector valued Bergman space on \mathbb{D} by $L_a^2(\mathbb{D}, \mathbb{C}^n) = L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$ respectively, where \otimes means the Hilbert space tensor product.

Let $M_{n \times n}$ be the set of all $n \times n$ complex matrices and $L^\infty(\mathbb{D}) \otimes M_{n \times n}$ be the space of matrix valued essential bounded Lebesgue measurable functions on \mathbb{D} . For $\Phi(z) = [\varphi_{ij}(z)]_{n \times n} \in L^\infty(\mathbb{D}) \otimes M_{n \times n}$, the block Toeplitz operator T_Φ and block Hankel operator H_Φ with matrix symbol $\Phi(z)$ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ are defined by

$$T_\Phi h = P(\Phi h) \text{ and } H_\Phi h = (I - P)(\Phi h), \quad \forall h \in L_a^2(\mathbb{D}, \mathbb{C}^n),$$

where P is the orthogonal projection from $L^2(\mathbb{D}, \mathbb{C}^n)$ onto $L_a^2(\mathbb{D}, \mathbb{C}^n)$. Since $L^\infty(\mathbb{D}, \mathbb{C}^n) = L^\infty(\mathbb{D}) \oplus \dots \oplus L^\infty(\mathbb{D})$ and $L_a^2(\mathbb{D}, \mathbb{C}^n) = L_a^2(\mathbb{D}) \oplus \dots \oplus L_a^2(\mathbb{D})$, the block Toeplitz operator T_Φ and the block Hankel operator H_Φ have the following matrix representations:

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ \vdots & & \vdots \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix} \text{ and } H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ \vdots & & \vdots \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{bmatrix},$$

Mathematics subject classification (2010): 47B35, 47B20.

Keywords and phrases: Hyponormality, block Toeplitz operator, block Hankel operator.

This research is supported by Liaoning Normal University Young Scientist Research Fund (LS2014L001), NSFC (No. 11271059, 11201438, 11501277).

where

$$\Phi(z) = \begin{bmatrix} \varphi_{11}(z) & \cdots & \varphi_{1n}(z) \\ \vdots & & \vdots \\ \varphi_{n1}(z) & \cdots & \varphi_{nn}(z) \end{bmatrix},$$

$T_{\varphi_{ij}}$ is the Toeplitz operator with symbol $\varphi_{ij} \in L^\infty(D)$ and $H_{\varphi_{ij}}$ is the Hankel operator with symbol $\varphi_{ij} \in L^\infty(D)$ ($1 \leq i, j \leq n$) on the Bergman space $L^2_a(\mathbb{D})$, respectively.

A bounded linear operator A on a Hilbert space is called hyponormal if $A^*A - AA^*$ is a positive operator. There is an extensive literature on hyponormal Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ (see [2, 5, 7, 9, 13] for example). Some necessary conditions and some sufficient conditions for the hyponormality of Toeplitz operators with certain symbols on the Bergman space can be found in [1, 6, 8, 11, 12]. Further, the corresponding problem has been studied in many other spaces. Some characterizations of hyponormal Toeplitz operators on the vector valued Hardy space can be seen in [3, 4].

In this paper, we consider the hyponormality of the block Toeplitz operator T_{F+G^*} on the vector valued Bergman space $L^2_a(\mathbb{D}, \mathbb{C}^n)$, where $F(z) = \sum_{k=0}^N A_k z^{m+k}$ and $G(z) = \sum_{k=0}^N B_k z^{m+k}$ with $A_k, B_k \in M_{n \times n}$, $m, N \geq 1$. Recall that G^* is the adjoint of the matrix G . Denote by $M_j = \sum_{k=0}^{N-j} (m+k)(m+k+j)(A_{k+j}A_k^* - B_{k+j}B_k^*)$ for $1 \leq j \leq N$. Inspired by the work in [6, 8, 11], we use the properties of the matrices $\{M_j\}_{j=1}^N$ to study the hyponormality of Block Toeplitz operators T_{F+G^*} on $L^2_a(\mathbb{D}, \mathbb{C}^n)$.

2. A necessary condition

Our goal in this section is to show a necessary condition for the Toeplitz operators to be hyponormal with matrix valued harmonic polynomial symbols on $L^2_a(\mathbb{D}, \mathbb{C}^n)$. Firstly, we recall some useful results.

THEOREM 1. ([10]) *Let $\Phi(z) \in h^\infty(\mathbb{D}) \otimes M_{n \times n}$. If T_Φ on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ is hyponormal, then $\Phi^*(z)\Phi(z) = \Phi(z)\Phi^*(z)$ on \mathbb{D} .*

THEOREM 2. ([10]) *Suppose that $F(z)$ and $G(z)$ are vector valued analytic polynomials. If T_{F+G^*} on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ is hyponormal, then $F'(z) [F'(z)]^* - G'(z)[G'(z)]^*$ is a positive semi-definite matrix for all $z \in \mathbb{T}$.*

THEOREM 3. *Suppose that $\Phi(z) = F(z) + G^*(z) \in h^\infty(\mathbb{D}) \otimes M_{n \times n}$, where F and G are vector valued analytic functions. Then T_Φ is hyponormal on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ if and only if $H_{F^*}^*H_{F^*} - H_{G^*}^*H_{G^*} \geq 0$ on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ and $\Phi^*(z)\Phi(z) = \Phi(z)\Phi^*(z)$ on \mathbb{D} .*

Proof. A direct calculation shows that

$$\begin{aligned} T_\Phi^*T_\Phi - T_\Phi T_\Phi^* &= T_\Phi^*T_\Phi - T_\Phi^*\Phi + (T_\Phi\Phi^* - T_\Phi T_\Phi^*) + T_\Phi^*\Phi - \Phi\Phi^* \\ &= -H_\Phi^*H_\Phi + H_\Phi^*H_{\Phi^*} + T_\Phi^*\Phi - \Phi\Phi^* \\ &= H_{F^*}^*H_{F^*} - H_{G^*}^*H_{G^*} + T_\Phi^*\Phi - \Phi\Phi^*. \end{aligned}$$

Combining Theorem 1, we get the desired result. \square

Let $F(z) = \sum_{k=0}^N A_k z^{m+k}$ and $G(z) = \sum_{k=0}^N B_k z^{m+k}$ with $A_k, B_k \in M_{n \times n}$, $m, N \geq 1$. It is clear that there exist polynomials f_{ij} and g_{ij} such that $F(z) = [f_{ij}(z)]_{n \times n}$ and $G(z) = [g_{ij}(z)]_{n \times n}$. Denote by $M_j = \sum_{k=0}^{N-j} (m+k)(m+k+j)(A_{k+j}A_k^* - B_{k+j}B_k^*)$ for $1 \leq j \leq N$. Then

$$\begin{aligned} & F'(z)[F'(z)]^* - G'(z)[G'(z)]^* \\ &= \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) \\ & \quad + 2Re \sum_{j=1}^N \sum_{k=0}^{N-j} (m+k)(m+k+j)(A_k A_{k+j}^* - B_k B_{k+j}^*) \bar{z}^j, \quad \forall z \in \mathbb{T}. \end{aligned}$$

By Theorem 2, if T_{F+G^*} is hyponormal, then

$$\left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} + 2Re \sum_{j=1}^N \langle M_j e, e \rangle_{\mathbb{C}^n} \bar{z}^j \geq 0$$

for $e \in \mathbb{C}^n$ and $z \in \mathbb{T}$. Integral the above inequality of $z = e^{i\theta}$ with respect to θ from 0 to 2π , it follows that $\sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*)$ is a positive semi-definite matrix. In general, there does not exist complex number $\bar{z} \in \mathbb{T}$ such that

$$\bar{z}^j \langle M_j e, e \rangle_{\mathbb{C}^n} = -|\langle M_j e, e \rangle_{\mathbb{C}^n}|$$

for any $1 \leq j \leq N$. In the following, we consider the relationship between the above positive semi-definite matrix and the matrix M_j ($1 \leq j \leq N$).

THEOREM 4. *Let $F(z) = \sum_{k=0}^N A_k z^{m+k}$ and $G(z) = \sum_{k=0}^N B_k z^{m+k}$ with $A_k, B_k \in M_{n \times n}$, $m, N \geq 1$. If T_{F+G^*} is hyponormal on $L_a^2(\mathbb{D}, \mathbb{C}^n)$, then the following statements hold for any $e \in \mathbb{C}^n$.*

(1) $\left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} \geq \frac{2}{N+1} \sum_{j=1}^N |\langle M_j e, e \rangle_{\mathbb{C}^n}|$, if $N \geq 3$;

(2) $\left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} \geq \frac{2}{N} \sum_{j=1}^N |\langle M_j e, e \rangle_{\mathbb{C}^n}|$, if $1 \leq N \leq 2$.

Proof. For every $e = (e_1, \dots, e_n)^T \in \mathbb{C}^n$, let

$$c_j = \begin{cases} -\frac{\langle M_j e, e \rangle_{\mathbb{C}^n}}{|\langle M_j e, e \rangle_{\mathbb{C}^n}|}, & \text{if } \langle M_j e, e \rangle_{\mathbb{C}^n} \neq 0 \\ 1, & \text{if } \langle M_j e, e \rangle_{\mathbb{C}^n} = 0 \end{cases}.$$

For $0 < a < 1$, u_l is defined as follows:

$$u_l = \begin{cases} 1, & \text{if } l = 0 \\ \frac{\prod_{s=1}^i c_s c_{(N+1)-s}}{c_{(N+1)-i}} a^{2i-1}, & \text{if } l = (i-1)(N+1) + i, 1 \leq i \leq \lfloor \frac{N+1}{2} \rfloor \\ \left(\prod_{s=1}^i c_s c_{(N+1)-s} \right) a^{2i}, & \text{if } l = i(N+1), 1 \leq i \leq \lfloor \frac{N}{2} \rfloor \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

- (i) $u_0 \bar{u}_1 = \bar{c}_1 a$;
- (ii) $u_{(i-1)(N+1)} \bar{u}_{(i-1)(N+1)+i} = \bar{c}_i a^{4i-3}$, when $2 \leq i \leq \lfloor \frac{N+1}{2} \rfloor$;
- (iii) $u_{(i-1)(N+1)+i} \bar{u}_{i(N+1)} = \overline{c_{(N+1)-i}} a^{4i-1}$, when $1 \leq i \leq \lfloor \frac{N}{2} \rfloor$.

Moreover, $u_l \bar{u}_{l+j} \neq 0$ for any $l \in \mathbb{Z}_+$ and $1 \leq j \leq N$ if and only if one of the above cases holds.

For every integer $p \geq 1$, we have $\hat{h}(z) = \sum_{l=0}^\infty u_l z^{l+p} \in L_a^2(\mathbb{D})$. In fact,

$$\|\hat{h}(z)\|^2 \leq \sum_{l=0}^\infty |u_l|^2 = 1 + a^2 + a^4 + \dots + a^{2N} = \frac{1 - a^{2(N+1)}}{1 - a^2} < +\infty.$$

Denote $h = e \cdot \hat{h} = (e_1 \hat{h}, \dots, e_n \hat{h})^T = (h_1, h_2, \dots, h_n)^T$. Here, M^T means the transpose of the M . It is easy to see that $h \in L_a^2(\mathbb{D}, \mathbb{C}^n)$. By Theorem 3, we have

$$\begin{aligned} 0 &\leq \langle (H_{F^*}^* H_{F^*} - H_{G^*}^* H_{G^*}) h, h \rangle_{L_a^2(\mathbb{D}, \mathbb{C}^n)} \\ &= \sum_{k=0}^N \langle [(A_k H_{\bar{z}^{m+k}}^*) (A_k^* H_{z^{m+k}}) - (B_k H_{\bar{z}^{m+k}}^*) (B_k^* H_{z^{m+k}})] h, h \rangle_{L_a^2(\mathbb{D}, \mathbb{C}^n)} \\ &\quad + 2\text{Re} \sum_{j=1}^N \sum_{k=0}^{N-j} \langle [(A_{k+j} H_{\bar{z}^{m+k+j}}^*) (A_k^* H_{z^{m+k}}) - (B_{k+j} H_{\bar{z}^{m+k+j}}^*) (B_k^* H_{z^{m+k}})] h, h \rangle_{L_a^2(\mathbb{D}, \mathbb{C}^n)} \\ &= \sum_{k=0}^N \sum_{l=0}^\infty \sum_{q=0}^\infty \langle (A_k A_k^* - B_k B_k^*) e, e \rangle_{\mathbb{C}^n} \langle H_{\bar{z}^{m+k}} (u_l z^{l+p}), H_{z^{m+k}} (u_q z^{q+p}) \rangle_{L_a^2(\mathbb{D})} \\ &\quad + 2\text{Re} \sum_{j=1}^N \sum_{k=0}^{N-j} \sum_{l=0}^\infty \sum_{q=0}^\infty \langle (A_{k+j} A_k^* - B_{k+j} B_k^*) e, e \rangle_{\mathbb{C}^n} \\ &\quad \times \langle H_{\bar{z}^{m+k}} (u_l z^{l+p}), H_{\bar{z}^{m+k+j}} (u_q z^{q+p}) \rangle_{L_a^2(\mathbb{D})} \\ &= \sum_{k=0}^N \sum_{l=0}^\infty \langle (A_k A_k^* - B_k B_k^*) e, e \rangle_{\mathbb{C}^n} u_l \bar{u}_l \langle H_{\bar{z}^{m+k}} z^{p+l}, H_{z^{m+k}} z^{p+l} \rangle_{L_a^2(\mathbb{D})} \\ &\quad + 2\text{Re} \sum_{j=1}^N \sum_{k=0}^{N-j} [\langle (A_{k+j} A_k^* - B_{k+j} B_k^*) e, e \rangle_{\mathbb{C}^n} \\ &\quad \times \sum_{l=0}^\infty (u_l \bar{u}_{l+j} \langle H_{\bar{z}^{m+k}} z^{p+l}, H_{\bar{z}^{m+k+j}} z^{p+l+j} \rangle_{L_a^2(\mathbb{D})})]. \end{aligned}$$

The last equality comes from the fact that $\{H_{\mathbb{Z}^m}(z^k)\}$ and $\{H_{\mathbb{Z}^s}(z^l)\}$ are orthogonal in $L^2(\mathbb{D})$ for $k - m \neq l - s$, and $\langle H_{\mathbb{Z}^{m_2}}^* H_{\mathbb{Z}^{m_1}} z^{k+m_1-m_2}, z^k \rangle_{L^2_a(\mathbb{D})} > 0, \forall k \geq \max\{0, m_2 - m_1\}$.

Divide the both sides of the above inequality by $\|H_{\mathbb{Z}^m} z^p\|_{L^2_a(\mathbb{D})}^2$ and let $p \rightarrow +\infty$. Note that

$$\lim_{p \rightarrow +\infty} \frac{\langle H_{\mathbb{Z}^{m+k}} z^{p+l}, H_{\mathbb{Z}^{m+k}} z^{p+l} \rangle_{L^2_a(\mathbb{D})}}{\|H_{\mathbb{Z}^m} z^p\|_{L^2_a(\mathbb{D})}^2} = \frac{(m+k)^2}{m^2}$$

and

$$\lim_{p \rightarrow +\infty} \frac{\langle H_{\mathbb{Z}^{m+k}} z^{p+l}, H_{\mathbb{Z}^{m+k+j}} z^{p+l+j} \rangle_{L^2_a(\mathbb{D})}}{\|H_{\mathbb{Z}^m} z^p\|_{L^2_a(\mathbb{D})}^2} = \frac{(m+k)(m+k+j)}{m^2}.$$

The details can be found in [11]. Therefore,

$$\begin{aligned} & \left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} \left(\sum_{l=0}^{\infty} |u_l|^2 \right) \\ & \geq -2\text{Re} \sum_{j=1}^N \sum_{l=0}^{\infty} u_l \overline{u_{l+j}} \langle M_j e, e \rangle_{\mathbb{C}^n}. \end{aligned}$$

Combining conditions (ii) and (iii) of u_l , it is easy to check that

$$\begin{aligned} & \left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} \\ & \geq 2 \sum_{j=1}^{\lfloor \frac{N+1}{2} \rfloor} a^{4(j-1)+1} \frac{1-a^2}{1-a^{2(N+1)}} |\langle M_j e, e \rangle_{\mathbb{C}^n}| \\ & \quad + 2 \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} a^{4j-1} \frac{1-a^2}{1-a^{2(N+1)}} |\langle M_{(N+1)-j} e, e \rangle_{\mathbb{C}^n}|. \end{aligned}$$

Letting $a \rightarrow 1$, we get the desired conclusion.

Now we prove (2). Let

$$v_l = \begin{cases} (c_1 c_N)^k a^{2k}, & l = k(N+1) \\ c_1^{k+1} c_N^k a^{2k+1}, & l = k(N+1) + 1 \\ 0, & \text{otherwise} \end{cases}$$

with $0 < a < 1$ and $k = 0, 1, \dots$. Direct calculations imply that if $v_l \overline{v_{l+j}} \neq 0$, then $v_l \overline{v_{l+j}} = \overline{c_j} a^{2kN+(2j-1)}$ for $1 \leq j \leq N$. Replacing u_l by v_l , we set $\tilde{h}(z) = \sum_{l=0}^{\infty} v_l z^{l+p} \in L^2_a(\mathbb{D})$ for $p \geq 1$. Then $\|\tilde{h}(z)\|^2 \leq \sum_{l=0}^{\infty} |u_l|^2 = \frac{1+a^2}{1-a^4} < +\infty$. As in the proof of (1), we

get

$$\begin{aligned} & \left\langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e, e \right\rangle_{\mathbb{C}^n} \sum_{l=0}^{\infty} |u_l|^2 \\ & \geq 2 \sum_{j=1}^N \sum_{k=0}^{+\infty} a^{2kN+(2j-1)} a^{2kN+2j-1} |\langle M_j e, e \rangle_{\mathbb{C}^n}| \\ & = 2 \sum_{j=1}^N a^{2j-1} \frac{1-a^2}{1-a^{2N}} |\langle M_j e, e \rangle_{\mathbb{C}^n}|. \end{aligned}$$

The rest of the proof is similar to the proof of (1). \square

In the case of $n = 1$, we have the following Corollary.

COROLLARY 1. Given $f(z) = \sum_{k=0}^N a_k z^{m+k}$ and $g(z) = \sum_{k=0}^N b_k z^{m+k}$ with $m, N \geq$

1. Let $M_j = \sum_{p=0}^{N-j} (m+p)(m+p+j)(a_{p+j} \bar{a}_p - b_{p+j} \bar{b}_p)$. If $T_{f+\bar{g}}$ is hyponormal on $L_a^2(\mathbb{D})$, then

- (1) $\sum_{i=0}^N (m+i)^2 (|a_i|^2 - |b_i|^2) \geq \frac{2}{N+1} \sum_{j=1}^N |M_j|$, if $N \geq 3$;
- (2) $\sum_{i=0}^N (m+i)^2 (|a_i|^2 - |b_i|^2) \geq \frac{2}{N} \sum_{j=1}^N |M_j|$, if $1 \leq N \leq 2$.

3. A sufficient condition

The following Theorem is our main result in this section. As a corollary, we also get a sufficient condition for the hyponormal Toeplitz operators with polynomial symbols on $L_a^2(\mathbb{D})$.

THEOREM 5. Suppose that $\Phi(z) = F(z) + G^*(z) \in h^\infty(\mathbb{D}) \otimes M_{n \times n}$, that $F(z) = \sum_{k=0}^N A_k z^{m+k}$ and $G(z) = \sum_{k=0}^N B_k z^{m+k}$ with $A_k, B_k \in M_{n \times n}$, $m \geq 1$ and $N \geq 1$. If the following conditions hold, then T_{F+G^*} is hyponormal on $L_a^2(\mathbb{D}, \mathbb{C}^n)$.

- (1) $\Phi^*(z)\Phi(z) = \Phi(z)\Phi^*(z)$ on \mathbb{D} ;
- (2) $B_j B_j^* - A_j A_j^*$ is positive semi-definite;
- (3) $|\langle M_j e_1, e_1 \rangle_{\mathbb{C}^n}| |\langle M_j e_2, e_2 \rangle_{\mathbb{C}^n}| \geq |\langle M_j e_1, e_2 \rangle_{\mathbb{C}^n}|^2$;
- (4) $\text{Re} \left[\langle (A_{k+j} A_k^* - B_{k+j} B_k^*) e_1, e_2 \rangle_{\mathbb{C}^n} \overline{\langle (A_{k+j+l} A_{k+l}^* - B_{k+j+l} B_{k+l}^*) e_1, e_2 \rangle_{\mathbb{C}^n}} \right] \geq 0$;
- (5) $\langle \sum_{i=0}^N (m+i)^2 (A_i A_i^* - B_i B_i^*) e, e \rangle_{\mathbb{C}^n} \geq 2 \sum_{j=1}^N |\langle M_j e, e \rangle_{\mathbb{C}^n}|$,

for all $e_1, e_2, e \in \mathbb{C}^n$, $1 \leq j \leq N$, $0 \leq k \leq N - j$ and $0 \leq l \leq N - (k + j)$.

Proof. Let $h(z) = \sum_{p=0}^{\infty} e_p z^p \in L_a^2(\mathbb{D}, \mathbb{C}^n)$ with $\sum_{p=0}^{\infty} \frac{\langle e_p, e_p \rangle_{\mathbb{C}^n}}{p+1} < \infty$. It is easy to check that

$$\begin{aligned} & \langle (H_F^* H_{F^*} - H_G^* H_{G^*})h, h \rangle_{L_a^2(\mathbb{D}, \mathbb{C}^n)} \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*)e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L_a^2(\mathbb{D})}^2 \\ & \quad + 2\operatorname{Re} \sum_{p=0}^{\infty} \sum_{j=1}^N \sum_{k=0}^{N-j} \langle (A_{k+j} A_k^* - B_{k+j} B_k^*)e_p, e_{p+j} \rangle_{\mathbb{C}^n} \langle H_{\bar{z}^{m+k}} z^p, H_{\bar{z}^{m+k+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*)e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L_a^2(\mathbb{D})}^2 \\ & \quad + 2 \sum_{p=0}^{\infty} \sum_{j=1}^N \operatorname{Re} \left[\sum_{k=0}^{N-j} \langle (A_{k+j} A_k^* - B_{k+j} B_k^*)e_p, e_{p+j} \rangle_{\mathbb{C}^n} \langle H_{\bar{z}^{m+k}} z^p, H_{\bar{z}^{m+k+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \right] \\ &\geq \sum_{p=0}^{\infty} \sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*)e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L_a^2(\mathbb{D})}^2 \\ & \quad - 2 \sum_{p=0}^{\infty} \sum_{j=1}^N \left| \sum_{k=0}^{N-j} \langle (A_{k+j} A_k^* - B_{k+j} B_k^*)e_p, e_{p+j} \rangle_{\mathbb{C}^n} \langle H_{\bar{z}^{m+k}} z^p, H_{\bar{z}^{m+k+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \right|. \end{aligned}$$

For non-negative integers m, k, j , we have

$$\begin{aligned} & \langle H_{\bar{z}^{m+j+k}}^* H_{\bar{z}^{m+k}} H_{\bar{z}^{m+j}} z^p, z^{p+j} \rangle_{L_a^2(\mathbb{D})} \\ &= \begin{cases} \frac{(m+k)(m+j+k)(p+m+j+1)}{m(m+j)(k+m+j+p+1)} \langle H_{\bar{z}^{m+j}}^* H_{\bar{z}^m} z^p, z^{p+j} \rangle_{L_a^2(\mathbb{D})}, & \text{if } p \geq m+k \\ \frac{(p+1)(p+j+1)(m+p+j+1)}{m(m+j)(k+m+j+p+1)} \langle H_{\bar{z}^{m+j}}^* H_{\bar{z}^m} z^p, z^{p+j} \rangle_{L_a^2(\mathbb{D})}, & \text{if } m \leq p < m+k \\ \frac{(p+m+j+1)}{(p+m+j+k+1)} \langle H_{\bar{z}^{m+j}}^* H_{\bar{z}^m} z^p, z^{p+j} \rangle_{L_a^2(\mathbb{D})}, & \text{if } 0 \leq p < m \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} & \langle H_{\bar{z}^{m+k}} z^p, H_{\bar{z}^{m+k+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \\ & \leq \frac{(m+k+j)(m+k)}{m(m+j)} \langle H_{\bar{z}^m} z^p, H_{\bar{z}^{m+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \\ & \leq \frac{(m+k+j)(m+k)}{m^2} \|H_{\bar{z}^m} z^p\|_{L_a^2(\mathbb{D})} \|H_{\bar{z}^m} z^{p+j}\|_{L_a^2(\mathbb{D})}, \end{aligned}$$

for $m, k, j = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Recall $M_j = \sum_{k=0}^{N-j} (m+k)(m+k+j)(A_{k+j} A_k^* - B_{k+j} B_k^*)$ for $1 \leq j \leq N$. Condition (4) implies that,

$$\begin{aligned} & \left| \sum_{k=0}^{N-j} \langle (A_{k+j} A_k^* - B_{k+j} B_k^*)e_p, e_{p+j} \rangle_{\mathbb{C}^n} \langle H_{\bar{z}^{m+k}} z^p, H_{\bar{z}^{m+k+j}} z^{p+j} \rangle_{L_a^2(\mathbb{D})} \right| \\ & \leq \frac{1}{m^2} |\langle M_j e_p, e_{p+j} \rangle_{\mathbb{C}^n}| \|H_{\bar{z}^m} z^p\|_{L_a^2(\mathbb{D})} \|H_{\bar{z}^m} z^{p+j}\|_{L_a^2(\mathbb{D})}. \end{aligned}$$

To finish the proof, we only need to prove that

$$\begin{aligned} & \sum_{p=0}^{\infty} \sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*) e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L^2_{\bar{a}}(\mathbb{D})}^2 \\ & \geq \frac{2}{m^2} \sum_{p=0}^{\infty} \sum_{j=1}^N |\langle M_j e_p, e_{p+j} \rangle_{\mathbb{C}^n}| \|H_{\bar{z}^m} z^p\|_{L^2_{\bar{a}}(\mathbb{D})} \|H_{\bar{z}^m} z^{p+j}\|_{L^2_{\bar{a}}(\mathbb{D})}. \end{aligned}$$

Note that $\zeta_i(k) = \frac{\|H_{\bar{z}^{m+i}} z^k\|_{L^2_{\bar{a}}(\mathbb{D})}^2}{\|H_{\bar{z}^m} z^k\|_{L^2_{\bar{a}}(\mathbb{D})}^2}$ ($i = 1, 2, \dots, N$) is increasing and $\zeta_i(k) \rightarrow \frac{(m+i)^2}{m^2}$

as $k \rightarrow +\infty$. So $\zeta_i(k) \leq \frac{(m+i)^2}{m^2}$. Combining conditions (2) and (5), we have

$$\begin{aligned} & \frac{\sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*) e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L^2_{\bar{a}}(\mathbb{D})}^2}{\|H_{\bar{z}^m} z^p\|_{L^2_{\bar{a}}(\mathbb{D})}^2} \\ & \geq \frac{1}{m^2} \langle \sum_{k=0}^N (m+k)^2 (A_k A_k^* - B_k B_k^*) e_p, e_p \rangle_{\mathbb{C}^n} \\ & \geq \frac{2}{m^2} \sum_{j=1}^N |\langle M_j e_p, e_p \rangle_{\mathbb{C}^n}| \geq 0. \end{aligned}$$

Let $\eta(p) = \left(\sum_{k=0}^N \langle (A_k A_k^* - B_k B_k^*) e_p, e_p \rangle_{\mathbb{C}^n} \|H_{\bar{z}^{m+k}} z^p\|_{L^2_{\bar{a}}(\mathbb{D})}^2 \right)^{1/2}$ for $p \geq 0$. Then

$$\frac{\eta^2(p)}{\|H_{\bar{z}^m} z^p\|_{L^2_{\bar{a}}(\mathbb{D})}^2} \geq \frac{2}{m^2} \sum_{k=1}^N |\langle M_k e_p, e_p \rangle_{\mathbb{C}^n}|$$

and

$$\frac{\eta^2(p+j)}{\|H_{\bar{z}^m} z^{p+j}\|_{L^2_{\bar{a}}(\mathbb{D})}^2} \geq \frac{2}{m^2} \sum_{k=1}^N |\langle M_k e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}|.$$

It follows that

$$\begin{aligned} & \eta(p)\eta(p+j) \tag{1} \\ & \geq \frac{2}{m^2} \left(\sum_{k=1}^N |\langle M_k e_p, e_p \rangle_{\mathbb{C}^n}| \right)^{\frac{1}{2}} \left(\sum_{k=1}^N |\langle M_k e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}| \right)^{\frac{1}{2}} \\ & \quad \times \|H_{\bar{z}^m} z^p\|_{L^2_{\bar{a}}(\mathbb{D})} \|H_{\bar{z}^m} z^{p+j}\|_{L^2_{\bar{a}}(\mathbb{D})}. \end{aligned}$$

Thus

$$\begin{aligned}
 \sum_{p=0}^{\infty} \eta^2(p) &= \frac{1}{2} \eta^2(0) + \frac{1}{2} \sum_{j=1}^{N-1} \left(\frac{\sum_{k=j+1}^N |\langle M_k e_j, e_j \rangle_{\mathbb{C}^n}|}{\sum_{k=1}^N |\langle M_k e_j, e_j \rangle_{\mathbb{C}^n}|} \eta^2(j) \right) \\
 &\quad + \frac{1}{2} \sum_{p=0}^{\infty} \sum_{j=1}^N \left(\frac{|\langle M_j e_p, e_p \rangle_{\mathbb{C}^n}|}{\sum_{k=1}^N |\langle M_k e_p, e_p \rangle_{\mathbb{C}^n}|} \eta^2(p) \right. \\
 &\quad \left. + \frac{|\langle M_j e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}|}{\sum_{k=1}^N |\langle M_k e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}|} \eta^2(p+j) \right) \\
 &\geq \sum_{p=0}^{\infty} \sum_{j=1}^N \frac{|\langle M_j e_p, e_p \rangle_{\mathbb{C}^n}|^{\frac{1}{2}} |\langle M_j e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}|^{\frac{1}{2}} \eta(p) \eta(p+j)}{(\sum_{k=1}^N |\langle M_k e_p, e_p \rangle_{\mathbb{C}^n}|)^{\frac{1}{2}} (\sum_{k=1}^N |\langle M_k e_{p+j}, e_{p+j} \rangle_{\mathbb{C}^n}|)^{\frac{1}{2}}} \\
 &\geq 2 \sum_{p=0}^{\infty} \sum_{j=1}^N \frac{1}{m^2} |\langle M_j e_p, e_{p+j} \rangle_{\mathbb{C}^n}| \|H_{\bar{z}^m} z^p\|_{L_a^2(\mathbb{D})} \|H_{\bar{z}^m} z^{p+j}\|_{L_a^2(\mathbb{D})}.
 \end{aligned}$$

The last inequality follows from the inequality (1) and the condition (3). Hence we finish the proof. \square

In particular, when $n = 1$, we have the following result.

COROLLARY 2. *Given $f(z) = \sum_{k=0}^N a_k z^{m+k}$ and $g(z) = \sum_{k=0}^N b_k z^{m+k}$ with $m, N \geq 1$.*

1. *Let $M_j = \sum_{p=0}^{N-j} (m+p)(m+p+j)(a_{p+j} \bar{a}_p - b_{p+j} \bar{b}_p)$. Suppose $|b_k| \geq |a_k|$ for $1 \leq k \leq N$ and $\text{Re}[(a_{k+j} \bar{a}_k - b_{k+j} \bar{b}_k)(\bar{a}_{k+j+l} a_{k+l} - \bar{b}_{k+j+l} b_{k+l})] \geq 0$ with $1 \leq j \leq N-k$ and $0 \leq l \leq N-(k+j)$. Then $T_{f+\bar{g}}$ is hyponormal, if*

$$\sum_{i=0}^N (m+i)^2 (|a_i|^2 - |b_i|^2) \geq 2 \sum_{j=1}^N |M_j|.$$

EXAMPLE 1. Let $F(z) + G^*(z) =$

$$\begin{bmatrix} z + z^2 + z^3 & 4z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{z} + \bar{z}^2 + \bar{z}^3 & 2\bar{z} + \frac{9}{8}\bar{z}^2 + \frac{1}{3}\bar{z}^3 \\ 0 & 0 \end{bmatrix}.$$

Then T_{F+G^*} is hyponormal on $L_a^2(\mathbb{D}, \mathbb{C}^2)$.

EXAMPLE 2. Consider the polynomial $\varphi(z) = \frac{1}{3}\bar{z}^3 + \frac{9}{8}\bar{z}^2 + 2\bar{z} + 4z + \frac{1}{2}z^2 + \frac{1}{6}z^3$. On the Bergman space, Corollary 2 shows that T_φ is a hyponormal operator.

REFERENCES

[1] P. AHERN AND Z. CUCKOVIC, *A mean value in equality with applications to Bergman space operators*, Pacific J. Math. **173** (2) (1996), 295–305.
 [2] C. COWEN, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (3) (1988), 809–812.
 [3] R. E. CURTO, I. S. HWANG AND W. Y. LEE, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230** (4–6) (2012), 2094–2151.

- [4] C. GU, J. HENDRICKS AND D. RUTHERFORD, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (1) (2006), 95–111.
- [5] C. GU AND J. E. SHAPIRO, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319** (3) (2001), 553–572.
- [6] I. S. HWANG, *Hyponormal Toeplitz operators on the Bergman space*, J. Korean Math. Soc. **42** (2) (2005), 387–403.
- [7] I. S. HWANG AND W. Y. LEE, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (6) (2002), 2461–2474.
- [8] I. S. HWANG AND J. LEE, *Hyponormal Toeplitz operators on the Bergman space II*, Bull. Korean Math. Soc. **44** (3) (2007), 517–522.
- [9] D. KANG AND W. Y. LEE, *A criterion on the hyponormality of Toeplitz operators with polynomial symbols via Schur numbers*, Linear Algebra and its Applications **436** (9) (2012), 3608–3617.
- [10] Y. LU, P. CUI AND Y. SHI, *Hyponormal Toeplitz operators on the vector valued Bergman Space*, Bull. Korean Math. Soc. **51** (1) (2014), 237–252.
- [11] Y. LU AND Y. SHI, *Hyponormal Toeplitz Operators on the Weighted Bergman Space*, Integr. equ. oper. theory. **65** (1) (2009), 115–129.
- [12] H. SADRAOUI, *Hyponormality of Toeplitz operators and composition operators*, Thesis, Purdue University, 1992.
- [13] K. ZHU, *Hyponormal Toeplitz operators with polynomial symbols*, Integr. equ. oper. theory **21** (3) (1995), 376–381.

(Received April 11, 2016)

Puyu Cui
School of Mathematics
Liaoning Normal University
Dalian 116029, China
e-mail: cuipuyu1234@163.com

Yufeng Lu
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, China
e-mail: lyfdlut@dlut.edu.cn

Yanyue Shi
School of Mathematical Sciences
Ocean University of China
Qingdao, 266100, China
e-mail: shiyanyue@163.com