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HOW TO DETERMINE THE EIGENVALUES OF G–CIRCULANT MATRICES

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Abstract. For a given nonnegative integer g, a matrix $C_{n,g}$ of size n is called g-circulant if $C_{n,g} = [a_{(r-gs) \bmod n}]_{r,s=0}^{n-1}$. Such matrices arise in wavelet analysis, subdivision algorithms, and more generally when dealing with multigrid/multilevel methods for structured matrices and approximations of boundary value problems. In this paper, we study the eigenvalues of g-circulants. The relationship to the harmonic analysis is explored and based on the new recursive formulas for eigenvalues of such class of matrices are obtained. This result represents an extension of the work due to E. Ngondiep and S. Serra Capizzano in establishing bounds for preconditioners for the linear system of equations determined by the same matrix and it could be seen as a tool for the analysis of the preconditioners. Numerical experiments are presented to illustrate the theoretical result.

1. Introduction

Let g be a nonnegative integer $(g \geqslant 1)$. We consider the problem of the eigenvalues of g-circulant matrices $C_{n,g}$. A g-circulant is a matrix in which each row (except the first) is obtained from the preceding row by shifting the elements cyclically g columns to the right. In other words, the entries of a g-circulant $C_{n,g} = [a_{r,s}]_{r,s=0}^{n-1}$ obey the rule $a_{r,s} = a_{(r-gs) \bmod n}$. Obviously, a g-circulant is uniquely determined by its first row and the shifting parameter $g \in \mathbb{N}$. For a g-circulant $C_{n,g}$, its first row vector, say, $(a_0, a_1, \ldots, a_{n-1})$, can be recorded in

$$\theta_{C_{n,g}}(x) = \sum_{r=0}^{n-1} a_r x^r,$$

which is called the Hall polynomial of $C_{n,g}$. In particular, the Hall polynomial of a g-circulant $C_{n,g}$ can be written as $\theta_{C_{n,g}}(x) = \sum_{j=0}^{r-1} x^{\alpha^j}$, where $\alpha_0 < \alpha_1 < \ldots < \alpha_{r-1} \leqslant n-1$, and $r = \theta_{C_{n,g}}(1)$. For the algebraic properties of such matrices we refer to Section 5.1 of the classical book by Davis [7], while additional new results can be found in [22] and references therein. On the other hand, such kind of matrices arises in important applications such as wavelet analysis [6, 24, 1, 5, 4, 25], in subdivision algorithm and more generally when dealing with multigrid/multilevel methods for structured matrices

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and approximation of boundary value problems [10, 17, 7, 2, 24, 4], and of course in the numerical approximation of one-dimensional PDEs with constant coefficients [3]. Furthermore, it is instructive to recall that Gilbert Strang [20] has shown rich connections between dilation equations in the wavelet context and multigrid algorithm [12, 23], when constructing the restriction/prolongation operators [2] with various boundary conditions. It is worth noticing that the use of different boundary conditions is quite natural when dealing with signal/image restoration problems or differential equations [15, 12, 11].

In some recent works [9, 16, 13], we addressed the problems of characterizing of singular values of g-circulant matrices and we provided an asymptotic analysis of the singular value/eigenvalues distribution for the g-Toeplitz sequences. Furthermore, we presented a brief analysis for problems of regularizing preconditioning of g-Toeplitz sequences via g-circulants. In [14] the authors established bounds for the preconditioners for the linear system of equations determined by g-circulant matrices. Moreover, let M be a given matrix, then the task in constructing preconditioners is to give a nonsingular matrix P easily invertible such that $P^{-1}M$ is close to the identity matrix. There, a bound on the eigenvalues of $P^{-1}M$ gives an information on the quality of the preconditioner (the tighter the eigenvalues are clustered around 1 the better). The result of this work could be seen as a tool for this analysis. On the other hand, the authors [18] analyzed the g-cirulant matrices and they provided closed expressions of the eigenvalues for such matrices. In this note, we are still interested in the study of g-circulant matrices but we provide the explicit formulas of the eigenvalues. Specifically, the attention is focused on the following three items:

(i1) a detailed study of the eigenvalues of g-circulant matrices together with the explicit formulas of such values;

(i2) a technical approach that computes these eigenvalues in recursive way. This item along with item (i1) are our original contributions and they represent both an improvement and a generalization of the result presented in [18];

(i3) a few numerical examples concerning the eigenvalues of g-circulants structures obtained by technical approach (stated in second item) and the simulation of such values, and regarding the effectiveness of the numerical eigenvalues according to the theoretical indications given in the first two items.

In connection with the singular values of $C_{n,g} = F_n D_n F_n^{\star} Z_{n,g}$ [15, 16], other sparse and structured matrices $M_{n,g}$, can be chosen in appropriate algebras of matrices so that $C_{n,g}$ can be written as

$$C_{n,g} = F_n D_n M_{n,g} F_n^{\star}, \tag{1}$$

where $D_n \in \mathbb{C}^{n\times n}$ is a diagonal matrix, $F_n \in \mathbb{C}^{n\times n}$ is the Fourier matrix and $M_{n,g} \in \mathbb{C}^{n\times n}$. In addition, the matrices given in decomposition (1) are defined by

$$D_n = \mathrm{diag}(\sqrt{n} F_n^{\star} \underline{a}), \tag{2}$$

$$F_n = \frac{1}{\sqrt{n}} \left[e^{-\hat{\imath}\frac{2\pi jk}{n}} \right]_{j,k=0}^{n-1} \text{ Fourier matrix, where } \hat{\imath}^2 = -1, \tag{3}$$

$$\underline{a} = [a_0, a_1, \ldots, a_{n-1}]^T, \text{ first column of the matrix } C_{n,g}, \tag{4}$$

$$M_{n,g} = F_n^{\star} Z_{n,g} F_n, \tag{5}$$

$$Z_{n,g} = [\delta_{r-gs}]_{r,s=0}^{n-1} \text{ where } \delta_k = \begin{cases} 1 \text{ if } k \equiv 0 \bmod n, \\ 0 \text{ otherwise.} \end{cases} \tag{6}$$

Unfortunately, the study of the eigenvalues of g-circulant matrices asks additional difficulties: we refer to the classical circulant cases where the parameter g is assumed equals 1. However, we construct a finite sequence of matrices $\left\{ M^{(k-1)}_{n_{g_{(k-1)}}, g_k} \cdot \Delta^{(k-1)}_{n_{g_{(k-1)}}} \right\}_{k=0}^{s}$ of decreasing size $n_{g_{(k-1)}}$, with initial assumptions: $M^{(-1)}_{n_{g_{(-1)}}, g_0} = M_{n,g}$ and $\Delta^{(-1)}_{n_{g_{(-1)}}} = D_n$. We compute the eigenvalues of $D_n M_{n,g}$ and use equation (1) to provide the spectrum of $C_{n,g}$. Furthermore, we observe that also the case of nonpositive g can be taken into consideration and can be reduced to the case of a nonnegative g. In fact, the role of circulants will be played by (-1)-circulant matrices (called also anti-circulants or skew-circulants) [7, 4]; as for circulants, (-1)-circulants for a commutative algebra simultaneously diagonalized by another unitary transform that can be written as the product of the Fourier matrix and a diagonal unitary matrix. Finally, it is worth noticing to remind that in [21] the author determined close formulae of such values in the case where the positive integers n and g are coprime.

The paper is organized as follows. Section 2 is reserved to some preliminary results. In section 3, we give some preparatory tools. Section 4 deals with the characterizing of the eigenvalues of g-circulant matrices. Some numerical evidences are considered and discussed in section 6. We end the paper by drawing the conclusion in section 7.

2. Some preliminary results

This section includes some lemmas and propositions of which we will make use in our work. We denote by $\mathrm{Eig}(A)$ the spectrum of a matrix A.

LEMMA 2.1. *Let a, b, k be three positive integers, then the following equalities hold*

$$(a \bmod k)(b \bmod k) \bmod k = ab \bmod k, \tag{7}$$

$$(a \bmod k \pm b \bmod k) \bmod k = (a \pm b) \bmod k, \tag{8}$$

$$a \bmod k + b < k \Leftrightarrow a \bmod k + b = (a+b) \bmod k. \tag{9}$$

Proof. The euclidian division of both integers a and b by k yields $a = a_0 + r_1 k$ and $b = b_0 + r_2 k$, where r_1 and r_2, are the remainders of divisions of a and b, respectively. Obviously, $0 \leqslant a_0, b_0 < k$. By straightforward computations, it is easy to see that: $(a \bmod k)(b \bmod k) \bmod k = a_0 b_0 \bmod k$ and $ab \bmod k = a_0 b_0 \bmod k$. Furthermore, $(a \bmod k \pm b \bmod k) \bmod k = (a_0 \pm b_0) \bmod k$ and $(a \pm b) \bmod k = (a_0 \pm$

$b_0)$ mod k. Relation (9) holds thanks to the following equalities: $a \bmod k + b = a_0 + b$ and $(a+b) \bmod k = (a_0+b) \bmod k$. $\square$

We state the well known theorem (Euler-Fermat Theorem) and other useful results which play a crucial role in our analysis.

PROPOSITION 2.1. (Euler-Fermat Theorem) *Let $a,b \in \mathbb{N}^\star$ (with $a<b$) and $\varphi(b)$ be the Euler number of b. If $(a,b)=1$, where (n,p) refers to the greater common divisor of two integers n and p, then*

$$a^{\varphi(b)} \equiv 1 \bmod b.$$

Furthermore,

$$\varphi(b) = b-1$$

if b is coprime.

To obtain a simple form of the matrix sequences stated in Section 1, we should use the mapping given by Lemma 2.2.

LEMMA 2.2. *Let n and g be two positive integers such that, $1 \leqslant g < n$. If $(n,g) = 1$, then the mapping*

$$\begin{aligned} l : \{0,1,\ldots,n-1\} &\to \{0,1,\ldots,n-1\}, \\ k &\mapsto l(k) = l_k = gk \bmod n, \end{aligned}$$

is bijective and there exists $\varphi(n) \in \mathbb{N}^\star$ such that, for $j \in \{0,1,\ldots,n-1\}$ fixed, and for each $q \in \{0,1,\ldots,\varphi(n)-1\}$, it holds

$$l^q(j) = jg^q \bmod n \quad \textit{and} \quad l^{\varphi(n)}(j) = j, \tag{10}$$

where $l^q = \underbrace{l \circ l \circ \ldots \circ l}_{q \text{ times}}$.

Proof. First, we assume that $(n,g)=1$. Let $k_1,k_2 \in \{0,1,\ldots,n-1\}$ such that, $l_{k_1} = l_{k_2}$. Using this, it is not hard to see that $gk_1 - l_{k_1} \equiv 0 \bmod n$ and $gk_2 - l_{k_1} \equiv 0 \bmod n$. A combination of both congruences along with equality $l_{k_1} - l_{k_2} = 0$, yields $g(k_1-k_2) \equiv 0 \bmod n$. Since $(n,g)=1$, we have $k_1 - k_2 \equiv 0 \bmod n$, which implies $k_1 = k_2$. So, the map l is nonsingular. The map l is bijective thanks to the cardinality of the set $\{0,1,\ldots,n-1\}$. Furthermore, for $j \in \{0,1,\ldots,n-1\}$ fixed, and for every $q \in \{0,1,\ldots,\varphi(n)-1\}$, utilizing the requirement $(n,g)=1$ together with equation (7) and Proposition 2.1 result in

$$l^q(j) = jg^q \bmod n \ \text{ and } \ l^{\varphi(n)}(j) = j. \tag{11}$$

$\square$

Now, using the mapping l given in Lemma 2.2, it is easy to construct the g-matrix $M_{n,g}$.

LEMMA 2.3. *There exists a g-matrix $M_{n,g}$ such that,*

$$Z_{n,g} = F_n M_{n,g} F_n^\star, \tag{12}$$

where $M_{n,g}$, is given by

$$M_{n,g} = [\delta_{gi-j}^{(n)}]_{i,j=0}^{n-1}, \quad \textit{with} \quad \delta_k^{(n)} = \begin{cases} 1 & \textit{if } k \equiv 0 \bmod n, \\ 0 & \textit{otherwise.} \end{cases} \tag{13}$$

Proof. For $j,k = 0,1,\ldots,n-1$, there exists a unique $(q_k, l_k) \in \mathbb{Z}^2$, with $0 \leqslant l_k < n$, such that

$$l_k - gk = q_k n. \tag{14}$$

A simple calculation gives

$$(F_n^\star Z_{n,g})_{jk} = \sum_{l=0}^{n-1} (F_n^\star)_{jl} (Z_{n,g})_{lk} = \sum_{l=0}^{n-1} \delta_{l-gk} (F_n^\star)_{jl} = (F_n^\star)_{jl_k}. \tag{15}$$

Combining the Fourier matrix F_n together with equations (14) and (15) provide

$$(M_{n,g})_{ij} = (F_n^\star Z_{n,g} F_n)_{ij} = \sum_{k=0}^{n-1} (F_n^\star Z_{n,g})_{ik} (F_n)_{kj} \underset{(15)}{=} \sum_{k=0}^{n-1} (F_n^\star)_{il_k} (F_n)_{kj} = \frac{1}{n} \sum_{k=0}^{n-1} \exp\left(\frac{\hat{\imath}2\pi i l_k}{n}\right)$$

$$\begin{aligned} \exp\left(\frac{-\hat{\imath}2\pi kj}{n}\right) &= \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\frac{\hat{\imath}2\pi(il_k - kj)}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\frac{\hat{\imath}2\pi[i(gk+nq_k)-kj]}{n}\right) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\hat{\imath}2\pi i q_k\right) \exp\left(\frac{\hat{\imath}2\pi k(gi-j)}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\frac{\hat{\imath}2\pi k(gi-j)}{n}\right) \\ &= \begin{cases} 1 \text{ if } gi - j \equiv 0 \bmod n, \\ 0 \text{ otherwise,} \end{cases} \end{aligned}$$

where the second equality in the second line comes from relation (14). So, $M_{n,g} = [\delta_{gi-j}^{(n)}]_{i,j=0}^{n-1}$. □

REMARK 2.1. As a straightforward consequence, it comes from relation (1) that

$$\mathrm{Eig}(C_{n,g}) = \mathrm{Eig}(D_n M_{n,g}). \tag{16}$$

The following section provides the main tool that we shall use to describe the spectrum of $C_{n,g}$. The idea is based on a matrix product of order n_g, where $n_g < n$.

3. Some preparatory tools

In the following we denote by $\delta^0 = (n, g)$ the greater common divisor of both nonnegative integers n and g, and by $\widetilde{Z}_{n,g} \in \mathbb{C}^{n\times n_g}$ the matrix $Z_{n,g}$ defined in (6) by considering only the $n_g = \frac{n}{\delta^0}$ first columns. We prove that zero is an eigenvalue of $C_{n,g}$, and we determine its algebraic and geometrical multiplicity. Furthermore, we give a simplified formula of the spectrum of $C_{n,g}$.

LEMMA 3.1. *Let n and g be two integers such that $1 \leqslant g < n$ and $M_{n,g}$ be the g-matrix defined in* (13). *Then*

$$\mathrm{Eig}(C_{n,g}) = \mathrm{Eig}\left(\sum_{j=0}^{\delta^0-1} M_j\Delta_j\right) \cup \{0\},$$

where 0 is an eigenvalue of multiplicity $n - n_g$, $M_j \in \mathscr{M}_{n_g}(\mathbb{C})$ and $\Delta_j \in \mathscr{M}_{n_g}(\mathbb{C})$ is a diagonal matrix.

Proof. Combining equations (6) and (13), straightforward computations yield $M_{n,g} = Z^{'}_{n,g}$. In addition, since $\tilde{Z}^{'}_{n,g} \in \mathscr{M}_{n_g\times n}(\mathbb{C})$ and $n = n_g \times \delta^0$, it holds

$$Z^{'}_{n,g} = \left[\tilde{Z}^{'}_{n,g}\; \tilde{Z}^{'}_{n,g}\; \ldots\; \ldots\; \tilde{Z}^{'}_{n,g}\right]^{'} \text{ and } \tilde{Z}^{'}_{n,g} = [M_0|M_1|\ldots|M_{\delta^0-1}],$$

where "$'$" refers to the transpose of a matrix. So, the matrix $M_{n,g}$ can be written as $M_{n,g} = K \otimes \tilde{Z}^{'}_{n,g}$, where $K \in \mathscr{M}_{\delta^0\times 1}(\mathbb{C})$ is a unit-column matrix whose coefficients are equal 1 and the symbol $\otimes$ denotes the tensor product, that is, $[a_{ij}]\otimes X = [a_{ij}X]$, where $[a_{ij}]$ and X designate two matrices.

In way similar, the product of matrices $M_{n,g}$ and D_n gives

$$M_{n,g}D_n = (K \otimes \tilde{Z}^{'}_{n,g})D_n = K \otimes Q,$$

where $Q = [M_0\Delta_0|M_1\Delta_1|\ldots|M_{\delta^0-1}\Delta_{\delta^0-1}]$. Here, the diagonal matrix D_n is decomposed as $D_n = \mathrm{diag}(\Delta_j,\, j = 0,1,\ldots,\delta^0-1)$, where Δ_j is also a diagonal matrix of order δ^0.

By straightforward calculation the determinant of the matrix $M_{n,g}D_n - \lambda I_n$ gives

$$\left|M_{n,g}D_n - \lambda I_n\right| = \begin{vmatrix} M_0\Delta_0 - \lambda I_{n_g} & M_1\Delta_1 & \ldots & M_{\delta^0-1}\Delta_{\delta^0-1} \\ M_0\Delta_0 & M_1\Delta_1 - \lambda I_{n_g} & M_2\Delta_2\ \ldots & M_{\delta^0-1}\Delta_{\delta^0-1} \\ \vdots & \vdots & \vdots & \vdots \\ M_0\Delta_0 & M_1\Delta_1 & \ldots & M_{\delta^0-1}\Delta_{\delta^0-1} - \lambda I_{n_g} \end{vmatrix}.$$

Adding all the columns and putting the result in the first column results in

$$\left|M_{n,g}D_n - \lambda I_n\right| = \begin{vmatrix} -\lambda I_{n_g} + \sum\limits_{j=0}^{\delta^0-1} M_j\Delta_j & M_1\Delta_1 & \dots & M_{\delta^0-1}\Delta_{\delta^0-1} \\ -\lambda I_{n_g} + \sum\limits_{j=0}^{\delta^0-1} M_j\Delta_j & -\lambda I_{n_g} + M_1\Delta_1 & M_2\Delta_2 \dots & M_{\delta^0-1}\Delta_{\delta^0-1} \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda I_{n_g} + \sum\limits_{j=0}^{\delta^0-1} M_j\Delta_j & M_1\Delta_1 & \dots & -\lambda I_{n_g} + M_{\delta^0-1}\Delta_{\delta^0-1} \end{vmatrix}.$$

Subtracting from the first row the other rows gives the determinant of an upper triangular matrix, multiplying the elements of the diagonal yields the following equality

$$\left|M_{n,g}D_n - \lambda I_n\right| = (-\lambda)^{n-n_g} \left| -\lambda I_{n_g} + \sum_{j=0}^{\delta^0-1} M_j\Delta_j \right|,$$

where I_{n_g} is the identity matrix of dimension $n_g \times n_g$, and $|A|$ denotes the determinant of a square matrix A. So, the spectrum of the matrix product $M_{n,g}D_n$ is given by

$$\mathrm{Eig}(M_{n,g}D_n) = \mathrm{Eig}\left(\sum_{j=0}^{\delta^0-1} M_j\Delta_j\right) \cup \{0\}. \tag{17}$$

A combination of relations (16)–(17) provides

$$\mathrm{Eig}(C_{n,g}) = \mathrm{Eig}(D_nM_{n,g}) = \mathrm{Eig}(M_{n,g}D_n) = \mathrm{Eig}\left(\sum_{j=0}^{\delta^0-1} M_j\Delta_j\right) \cup \{0\},$$

where 0 is of multiplicity $n - n_g$. $\square$

The following Lemma gives a simplified form of the sum $\sum\limits_{j=0}^{\delta^0-1} M_j\Delta_j$.

LEMMA 3.2. *Setting* $g_1 = g \bmod n_g$, $n_{g_{(-1)}} = n$, $n_{g_{(0)}} = n_g$, *and denoting the parameter* $\delta_k^{(n_g)}$ *by*

$$\delta_k^{(n_g)} = \begin{cases} 1 \text{ if } k \equiv 0 \bmod n_g, \\ 0 \text{ otherwise.} \end{cases}$$

Then relation (18) *holds,*

$$\sum_{k=0}^{\delta^0-1} M_k\Delta_k = \Delta_{n_g}^{(0)} M_{n_g,g_1}^{(0)}, \tag{18}$$

where $\Delta_{n_g}^{(0)}$ *is a diagonal matrix of size* n_g, $M_{n_g,g_1}^{(0)}$ *is a matrix of order* n_g. *Both matrices are defined as*

$$\left(\Delta_{n_g}^{(0)}\right)_{jj} = d_{gj \bmod n} \text{ and } \left(M_{n_g,g_1}^{(0)}\right)_{ij} = \delta_{g_1 i-j}^{(n_g)}, \tag{19}$$

for $i,j=0,1,\dots,n_g-1$, *where* $d_k := d_{kk} = (D_n)_{k,k}$. D_n *is the diagonal matrix given in* (2).

Proof. For $k=0,1,\dots,\delta^0-1$, we have that

$$M_k = \left[(M_{n,g})_{kn_g+i,kn_g+j}\right]_{i,j=0}^{n_g-1} \text{ and } \Delta_k = \left[(D_n)_{kn_g+i,kn_g+j}\right]_{i,j=0}^{n_g-1}.$$

In addition, using the definitions M_k and Δ_k together with the coefficients of the product $M_k\Delta_k$ result in

$$\begin{aligned}(M_k\Delta_k)_{i,j} &= \sum_{p=0}^{n_g-1}(M_k)_{ip}(\Delta_k)_{pj} = (M_k)_{ij}(\Delta_k)_{jj}\\ &= \delta^{(n)}_{g(kn_g+i)-kn_g-j}d_{kn_g+j,kn_g+j} = \delta^{(n)}_{gi-(kn_g+j)}d_{kn_g+j},\end{aligned}$$

for $i,j=0,1,\dots,n_g-1$. Now, we define the quantities $\widetilde{g}$ and g_1 as $\widetilde{g} = \frac{g-g \bmod n_g}{n_g}$ and $g_1 = g \bmod n_g$, respectively. More specifically, $g = \widetilde{g}n_g + g_1$. Furthermore, simple calculations yield $\widetilde{g}i = q_i\delta^0 + r_i$, where $0 \leqslant r_i < \delta^0$. Using this, it is not hard to see that $gi = n_g\tilde{g}i + g_1 i = n_g(q_i\delta^0 + r_i) + g_1 i = q_i n + r_i n_g + g_1 i,\ 0 \leqslant r_i < \delta^0$. So, $r_i n_g = (g-g_1)i \bmod n$. Utilizing this, straightforward computations provide

$$\begin{aligned}\left(\sum_{k=0}^{\delta^0-1} M_k\Delta_k\right)_{i,j} &= \sum_{k=0}^{\delta^0-1}\delta^{(n)}_{gi-kn_g-j}d_{kn_g+j} = \sum_{k=0}^{\delta^0-1}\delta^{(n)}_{(r_i-k)n_g+g_1 i-j}d_{kn_g+j} \overset{(a)}{=} \delta^{(n)}_{g_1 i-j}d_{r_i n_g+j}\\ &= \delta^{(n)}_{g_1 i-j}d_{(g-g_1)i \bmod n+j} \overset{(b)}{=} \begin{cases} d_{gi \bmod n} & \text{if } j = g_1 i \bmod n\\ 0 & \text{otherwise}\end{cases}\\ &\overset{(c)}{=} \begin{cases} d_{gi \bmod n} & \text{if } j = g_1 i \bmod n_g\\ 0 & \text{otherwise}\end{cases} = \delta^{(n_g)}_{g_1 i-j}d_{gi \bmod n} \overset{(d)}{=} \left(\Delta^{(0)}_{n_g}M^{(0)}_{n_g,g_1}\right)_{ij},\end{aligned}$$

where, **(a)** holds since there exists a unique $k_i \in \{0,1,\dots,\delta^0-1\}$ such that $k_i = r_i$, **(b)** comes from Lemma 2.1, **(c)** comes from estimate $j < n_g$, and **(d)** follows from the definition of entries of $\Delta^{(0)}_{n_g}M^{(0)}_{n_g,g_1}$. □

REMARK 3.1. If $g_1 \neq 0$ then $(n_g,g) = (n_g,g_1)$.

Armed with above tools we are able to characterize the eigenvalues of g-circulant matrices $C_{n,g}$.

4. Characterization of eigenvalues

This section studies the eigenvalues of the g-circulant matrices. The case where the positive integers n and g are coprime is introduced and some particular cases are presented. This tool provides an idea on a close formula of eigenvalues. The approach used for characterizing such values consists in reducing at each step the nonnegative integers n and g, discuss following the greater common divisor of reduced quantities the eigenvalues of $C_{n,g}$ and provide an iterative scheme that computes in recursive way the spectrum of $C_{n,g}$. For the sake of simplicity, we use the notation $d_s := d_{ss} = (D_n)_{s,s}$, where D_n is the diagonal matrix defined in (2).

LEMMA 4.1. *Let $l : \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\}$ be the mapping defined in Lemma 2.2 which satisfies $l_j^q = l^q(j)$, for $0 \leqslant q \leqslant \varphi(n)-1$ and $0 \leqslant j \leqslant n-1$ ($\varphi(n)$ is the Euler indicator), where $l^0 = l^{\varphi(n)}$ is the identity. If $(n,g)=1$, it holds*

$$(M_{n,g}D_n)^{\varphi(n)} = \operatorname{diag}\left(\prod_{p=0}^{\varphi(n)-1} d_{l_j^p l_j^p} : j = 0,1,\ldots,n-1\right). \tag{20}$$

Proof. We prove this result by mathematical induction. Since $(n,g)=1$, using the Euler-Fermat theorem, we have that $g^{\varphi(n)} \equiv 1 \bmod n$. Using Lemma 2.2, simple computations give

$$(M_{n,g}D_n)_{ik} = \sum_{l=0}^{n-1}(M_{n,g})_{il}(D_n)_{lk} = \sum_{l=0}^{n-1} d_{lk}\delta_{gi-l}^{(n)} = d_{l_ik}, \text{ for } i,k = 0,1,\ldots,n-1. \tag{21}$$

In way similar, we have $(M_{n,g}D_nM_{n,g}D_n)_{ik} = \sum_{p=0}^{n-1}(M_{n,g}D_n)_{ip}(M_{n,g}D_n)_{pk} = \sum_{p=0}^{n-1} d_{l_ip}d_{l_pk} = d_{l_i^2k}d_{l_il_i}$. The third equality holds thank to relation (21) and the last one comes from Lemma 2.2. Let us assume that for every $q \in \{2,3,\ldots,\varphi(n)-1\}$,

$$(M_{n,g}D_n)_{ik}^q = d_{l_i^qk}d_{l_il_i}\ldots d_{l_i^{q-1}l_i^{q-1}}. \tag{22}$$

Plugging relations (22), (2) and Lemma 2.2, it is easy to see that

$$\begin{aligned}(M_{n,g}D_n)_{ik}^{\varphi(n)} &= \sum_{p=0}^{n-1}(M_{n,g}D_n)_{ip}^{\varphi(n)-1}(M_{n,g}D_n)_{pk} = \sum_{p=0}^{n-1} d_{l_i^{\varphi(n)-1}p}d_{l_il_i}\ldots d_{l_i^{\varphi(n)-2}l_i^{\varphi(n)-2}}d_{l_pk} \\ &= d_{l_i^{\varphi(n)-1}l_i^{\varphi(n)-1}}d_{l_il_i}\ldots d_{l_i^{\varphi(n)-2}l_i^{\varphi(n)-2}}d_{l_i^{\varphi(n)}k} = d_{ik}d_{l_il_i}d_{l_i^2l_i^2}\ldots d_{l_i^{\varphi(n)-1}l_i^{\varphi(n)-1}},\end{aligned}$$

which ends the proof. □

LEMMA 4.2. *Let $l : \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\}$ be the mapping defined in Lemma 2.2 that satisfies $l_j^q = l^q(j)$, for $0 \leqslant q \leqslant \varphi(n)-1$ and $0 \leqslant j \leqslant n-1$, where $l^0 = l^{\varphi(n)}$ is the identity. Assume that $(n,g)=1$, the spectrum of $C_{n,g}$ is given by*

$$\operatorname{Eig}(C_{n,g}) = \operatorname{Eig}(M_{n,g}D_n) = \left\{\exp\left(\hat{\imath}\frac{2k(j)\pi}{\varphi(n)}\right)\prod_{m=0}^{\varphi(n)-1} f_{l_{p(j)}^m};\ j = 0,1,\ldots,n-1\right\},$$

where $l_{p(j)}^m = g^m p(j) \bmod n$, $f_{l_{p(j)}^m} = L_{l_{p(j)}^m}^{\frac{1}{\varphi(n)}} \exp\left(\hat{\imath}\frac{\theta_{l_{p(j)}^m}}{\varphi(n)}\right)$, with $d_k = (D_n)_{k,k} = L_k \exp(\hat{\imath}\theta_k)$. D_n is the diagonal matrix defined in (2).

Proof. Putting $\mathrm{Eig}(M_{n,g}D_n)=\mathrm{diag}(\beta_0,\beta_1,\ldots,\beta_{n-1})$, by the Schur theorem [19] there exists a unitary matrix U such that,

$$U^{\star}(M_{n,g}D_n)U=R=\begin{pmatrix}\beta_0 & \star & \ldots & \star\\ & \beta_1 & \star\ldots & \vdots\\ & & \ddots & \star\\ 0 & & & \beta_{n-1}\end{pmatrix}. \tag{23}$$

Using relation (23), a simple calculation gives

$$(U^{\star}(M_{n,g}D_n)U)^{\varphi(n)}=U^{\star}(M_{n,g}D_n)^{\varphi(n)}U=R^{\varphi(n)}=\begin{pmatrix}\beta_0^{\varphi(n)} & \star & \ldots & \star\\ & \beta_1^{\varphi(n)} & \star\ldots & \vdots\\ & & \ddots & \star\\ 0 & & & \beta_{n-1}^{\varphi(n)}\end{pmatrix}. \tag{24}$$

So, it comes from (24) the equality $\mathrm{Eig}\left((M_{n,g}D_n)^{\varphi(n)}\right)=\mathrm{Eig}(R^{\varphi(n)})$. In addition, combining relations (20) and (24) results in

$$\left\{\prod_{p=0}^{\varphi(n)-1}d_{l_j^p}\,:\,j=0,1,\ldots,n-1\right\}=\left\{\beta_j^{\varphi(n)}\,:\,j=0,1,\ldots,n-1\right\}.$$

Hence, for each $j\in\{0,1,\ldots n-1\}$, there exists $i_j\in\{0,1,\ldots,n-1\}$ ($i_j=p(j)$, where p is a mapping from $\{0,1,\ldots,n-1\}$ to $\{0,1,\ldots,n-1\}$) satisfying $\beta_j^{\varphi(n)}=\prod_{m=0}^{\varphi(n)-1}d_{l^m_{p(j)}}$. Setting $\beta_j=a_j\exp\left(\hat{\imath}\alpha_j\right)=[a_j,\alpha_j]$ and $d_t=L_t\exp\left(\hat{\imath}\theta_t\right)=[L_t,\theta_t]$, simple computations yield $[a_j^{\varphi(n)},\varphi(n)\alpha_j]=\left[\prod_{m=0}^{\varphi(n)-1}L_{l^m_{p(j)}},\ \sum_{m=0}^{\varphi(n)-1}\theta_{l^m_{p(j)}}\right]$. Using this, the modulus and the argument of β_j are given by

$$a_j=\left(\prod_{m=0}^{\varphi(n)-1}L_{l^m_{p(j)}}\right)^{\frac{1}{\varphi(n)}}\ \text{and}\ \alpha_j\in\left\{\frac{1}{\varphi(n)}\left(2k\pi+\sum_{m=0}^{\varphi(n)-1}\theta_{l^m_{p(j)}}\right):k=0,1,\ldots,\varphi(n)-1\right\}. \tag{25}$$

Furthermore, let $k_j=k(j)$ be an element of the set $\{0,1,\ldots,\varphi(n)-1\}$, which corresponds to the index of α_j, the explicit formula of α_j is given by

$$\alpha_j=\frac{1}{\varphi(n)}\left(2k(j)\pi+\sum_{m=0}^{\varphi(n)-1}\theta_{l^m_{p(j)}}\right). \tag{26}$$

Combining relations (25) and (26), β_j becomes

$$\beta_j=\exp\left(\hat{\imath}\frac{2k(j)\pi}{\varphi(n)}\right)\prod_{m=0}^{\varphi(n)-1}L_{l^m_{p(j)}}^{\frac{1}{\varphi(n)}}\exp\left(\hat{\imath}\frac{\theta_{l^m_{p(j)}}}{\varphi(n)}\right)=\exp\left(\hat{\imath}\frac{2k(j)\pi}{\varphi(n)}\right)\prod_{m=0}^{\varphi(n)-1}f_{l^m_{p(j)}}, \tag{27}$$

where $l^m_{p(j)} = g^m p(j) \bmod n$ and $f_{l^m_{p(j)}} = L^{\frac{1}{\varphi(n)}}_{l^m_{p(j)}} \exp\left(\hat{i}\frac{\theta_{l^m_{p(j)}}}{\varphi(n)}\right)$. So, the spectrum of the matrix $M_{n,g}D_n$ is given by

$$\mathrm{Eig}(M_{n,g}D_n) = \left\{\exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n)}\right)\prod_{m=0}^{\varphi(n)-1} f_{l^m_{p(j)}};\ j=0,1,\ldots,n-1\right\}. \tag{28}$$

Using the following equation

$$\mathrm{Eig}(M_{n,g}D_n) = \mathrm{Eig}(D_nM_{n,g}),$$

the proof is completed thank to relations (16) and (28). □

In the following Lemma, we analyze the case where the parameter g is a divisor of n, of multiplicity p ($p \geqslant 1$). This is a special case for Multigrid method with different size reduction [8].

LEMMA 4.3. *Consider the mapping l defined in Lemma 2.2. If $n = g^p.n_0$, with $p \geqslant 1$, $n_0 \geqslant 1$ and $(n_0, g) = 1$; we have*

$$\mathrm{Eig}(C_{n,g}) = \left\{\exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_0)}\right)\prod_{q=0}^{\varphi(n_0)-1} f_{l^{p+q}_{v(j)}};\ j=0,1,\ldots,n_0-1\right\}\cup\{0:\ mult. = n-n_0\},$$

where $l^{p+q}_{v(j)} = g^{p+q}v(j) \bmod n$ and $f_{l^{p+q}_{v(j)}} = L^{\frac{1}{\varphi(n_0)}}_{l^{p+q}_{v(j)}}\exp\left(\hat{i}\frac{\theta_{l^{p+q}_{v(j)}}}{\varphi(n_0)}\right)$, with $d_j = L_j\exp\left(\hat{i}\theta_j\right)$, $v(j) \in \{0,1,\ldots,n_0-1\}$ and $k(j) \in \{0,1,\ldots,\varphi(n_0)-1\}$.

Proof. First, we recall that $\delta^0 = (n,g) = g$. A combination of (17)–(18) provides

$$\begin{aligned}\mathrm{Eig}(M_{n,g}D_{n,g}) &= \mathrm{Eig}\left(\sum_{j=0}^{g-1} M_j\Delta_j\right)\cup\{0:\ mult. = n-g^{p-1}n_0\}.\\ &= \mathrm{Eig}\left(M^{(0)}_{g^{p-1}n_0,g}\Delta^{(0)}_{g^{p-1}n_0}\right)\cup\{0:\ mult. = n-g^{p-1}n_0\},\end{aligned}$$

where $(\Delta^{(0)}_{g^{p-1}n_0})_{ii} = d_{gi \bmod n} = d_{gi}$; $(M^{(0)}_{g^{p-1}n_0,g})_{ij} = \delta^{(g^{p-1}n_0)}_{gi-j}$; $i,j = 0,1,\ldots,g^{p-1}n_0-1$. In way similar, for $p>1$, it holds

$$\begin{aligned}\mathrm{Eig}(M^{(0)}_{g^{p-1}n_0,g}\Delta^{(0)}_{g^{p-1}n_0}) &= \mathrm{Eig}\left(\sum_{j=0}^{g-1} M_j\Delta_j\right)\cup\{0:\ mult. = g^{p-1}n_0-g^{p-2}n_0\}\\ &= \mathrm{Eig}\left(M^{(1)}_{g^{p-2}n_0,g}\Delta^{(1)}_{g^{p-2}n_0}\right)\cup\{0:\ mult. = g^{p-1}n_0-g^{p-2}n_0\},\end{aligned}$$

where $(\Delta^{(1)}_{g^{p-2}n_0})_{ii} = d_{g(gi) \bmod n} = d_{g^2i}$, $(M^{(1)}_{g^{p-2}n_0,g})_{ij} = \delta^{(g^{p-2}n_0)}_{gi-j}$, for $i,j = 0,1,\ldots,g^{p-2}n_0-1$. By mathematical induction, one constructs a finite sequence of matrices

$\left\{M^{(k)}_{g^{p-k-1}n_0,g}\Delta^{(k)}_{g^{p-k-1}n_0}\right\}_{k=0}^{p-1}$, of decreasing order $g^{p-k-1}n_0$, satisfying for $k=1,2,\dots,p-1$,

$$\begin{aligned}\mathrm{Eig}\left(M^{(k-1)}_{g^{p-k}n_0,g}\Delta^{(k-1)}_{g^{p-k}n_0}\right) &= \mathrm{Eig}\left(\sum_{j=0}^{g-1}M_j\Delta_j\right)\cup\{0:\ \mathit{mult.}=g^{p-k}n_0-g^{p-k-1}n_0\}\\ &=\mathrm{Eig}\left(M^{(k)}_{g^{p-k-1}n_0,g}\Delta^{(k)}_{g^{p-k-1}n_0}\right)\cup\{0:\ \text{mult.}=(g^{p-k}-g^{p-k-1})n_0\},\end{aligned}\tag{29}$$

where $(\Delta^{(k)}_{g^{p-k-1}n_0})_{ii}=d_{g^{k+1}i}$ and $(M^{(k)}_{g^{p-k-1}n_0,g})_{ij}=\delta^{(g^{p-k-1}n_0)}_{gi-j}$, for $i,j=0,1,\dots,g^{p-k-1}n_0-1$. Combining relations (16) and (29), the spectrum of $C_{n,g}$ becomes

$$\begin{aligned}\mathrm{Eig}(C_{n,g}) &= \mathrm{Eig}(M_{n,g}D_n)\\ &=\mathrm{Eig}\left(M^{(0)}_{g^{p-1}n_0,g}\Delta^{(0)}_{g^{p-1}n_0}\right)\cup\{0:\ \text{mult.}=g^pn_0-g^{p-1}n_0\}\\ &\ \ \vdots\\ &=\mathrm{Eig}\left(M^{(p-1)}_{n_0,g}\Delta^{(p-1)}_{n_0}\right)\cup\{0:\ \text{mult.}=g^pn_0-n_0\}\\ &=\mathrm{Eig}\left(M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}\right)\cup\{0:\ \text{mult.}=g^pn_0-n_0\},\end{aligned}\tag{30}$$

where $g_1=g \bmod n_0$. On the other hand, for $i,j=0,1,\dots,n_0-1$, $(\Delta^{(p-1)}_{n_0})_{jj}=d_{g^pj \bmod n}=d_{g^pj}$ and $(M^{(p-1)}_{n_0,g_1})_{ij}=\delta^{(n_0)}_{g_1i-j}=\begin{cases}1 \text{ if } j\equiv g_1 i \bmod n_0\\ 0 \text{ otherwise}\end{cases}$. To complete the proof, we must compute the eigenvalues of the matrix $M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}$.

Spectrum of the matrix $M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}$. Let $\tilde{d}_j=d_{g^pj}$ and let introduce the mapping $\tilde{l}$ defined by

$$\begin{aligned}\tilde{l}:\{0,1,\dots,n_0-1\}&\to\{0,1,\dots,n_0-1\}\\ j\quad&\mapsto\tilde{l}(j)=\tilde{l}_j=g_1j \bmod n_0.\end{aligned}$$

This mapping is bijective (the proof is obvious according to Lemma 2.2). In addition, since $(n_0,g_1)=1$, it comes from Proposition 2.1 (Euler-Fermat Theorem) that $g_1^{\varphi(n_0)}\equiv 1 \bmod n_0$. In equation (20), replacing n, g and $\varphi(n)$ with n_0, g_1 and $\varphi(n_0)$, respectively, straightforward calculations give

$$\left(M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}\right)^{\varphi(n_0)}=\mathrm{diag}\left\{\prod_{q=0}^{\varphi(n_0)-1}\tilde{d}_{\tilde{l}^q_j}:\ j=0,1,\dots,n_0-1\right\},\tag{31}$$

where $\tilde{l}^q_j=g_1^qj \bmod n_0=g^qj \bmod n_0$. Furthermore,

$$\tilde{d}_{\tilde{l}^q_j}=\tilde{d}_{g^qj \bmod n_0}:=d_{(g^qj \bmod n_0)g^p}=d_{g^{p+q}j \bmod n}=d_{l^p_{g^qj}}=d_{l^{p+q}_j}.\tag{32}$$

Combining relations (31)–(32), simple calculations yield the eigenvalues β_j of $M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}$, that is,

$$\beta_j = \exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_0)}\right)\prod_{m=0}^{\varphi(n_0)-1} L^{\frac{1}{\varphi(n_0)}}_{l^{p+m}_{v(j)}} \exp\left(\hat{i}\frac{\theta_{l^{p+m}_{v(j)}}}{\varphi(n_0)}\right) = \exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_0)}\right)\prod_{m=0}^{\varphi(n_0)-1} f_{l^{p+m}_{v(j)}}, \tag{33}$$

where $l^{p+m}_{v(j)} = g^{p+m}v(j) \bmod n$, $f_{l^{p+m}_{v(j)}} = L^{\frac{1}{\varphi(n_0)}}_{l^{p+m}_{v(j)}} \exp\left(\hat{i}\frac{\theta_{l^{p+m}_{v(j)}}}{\varphi(n_0)}\right)$, with $d_j = L_j\exp\left(\hat{i}\theta_j\right)$, $v(j)\in\{0,1,\dots,n_0-1\}$ and $k(j)\in\{0,1,\dots,\varphi(n_0)-1\}$. So

$$\operatorname{Eig}\left(M^{(p-1)}_{n_0,g_1}\Delta^{(p-1)}_{n_0}\right) = \left\{\exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_0)}\right)\prod_{m=0}^{\varphi(n_0)-1} f_{l^{p+m}_{v(j)}} : j=0,1,\dots,n_0-1\right\}. \tag{34}$$

An assembling of relations (30) and (34) ends the proof. $\square$

In the rest of this work, we analyze in detail the spectrum of the matrix $C_{n,g}$. To attain this purpose we construct by mathematical induction a finite sequence of matrices $\left\{M^{(k-1)}_{n_{g_{(k-1)}},g_k}\cdot\Delta^{(k-1)}_{n_{g_{(k-1)}}}\right\}_{k=0}^{s}$, of decreasing size $n_{g_{(k-1)}}$, which satisfies the initial assumptions: $M^{(-1)}_{n_{g_{(-1)}},g_0} = M_{n,g}$ and $\Delta^{(-1)}_{n_{g_{(-1)}}} = D_n$. We recall that $n_g = \frac{n}{\delta^{(0)}}$, with $\delta^{(0)} = \delta^0$, $g_1 = g \bmod n_g$, and we start with the case where the integers n_g and g_1 are coprime.

LEMMA 4.4. *Consider the mapping l defined in Lemma 2.2. Suppose that $(n_g,g_1) = 1$, then it holds*

$$\operatorname{Eig}(C_{n,g}) = \left\{\exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_g)}\right)\prod_{p=1}^{\varphi(n_g)} f_{l^p_{v(j)}}; j=0,1,\dots,n_g-1\right\}\cup\{0\},$$

where 0 is of multiplicity $n-n_g$, $l^p_{v(j)} = g^p v(j) \bmod n$, and $f_{l^p_{v(j)}} = L^{\frac{1}{\varphi(n_g)}}_{l^{p+m}_{v(j)}}\exp\left(\hat{i}\frac{\theta_{l^p_{v(j)}}}{\varphi(n_g)}\right)$, with $d_j = (D_n)_{j,j} = L_j\exp\left(\hat{i}\theta_j\right)$, $v(j)\in\{0,1,\dots,n_g-1\}$, and $k(j)\in\{0,1,\dots,\varphi(n_g)-1\}$.

Proof. According to Lemma 3.1 and relation (18), the spectrum of $C_{n,g}$ is given by $\operatorname{Eig}(C_{n,g}) = \operatorname{Eig}(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})\cup\{0: \text{ mult. } = n-n_g\}$. Since $(n_g,g_1)=1$, according to Euler-Fermat theorem (Proposition 2.1) there exists $\varphi(n_g)\in\mathbb{N}^*$, such that $g_1^{\varphi(n_g)} \equiv 1 \bmod n_g$. Furthermore, simple calculations and rearranging terms provide

$$\begin{aligned}(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})^2_{ij} &= \sum_{l=0}^{n_g-1}(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})_{il}(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})_{lj} = \sum_{l=0}^{n_g-1}\delta^{(n_g)}_{g_1 i-l}d_{gi \bmod n}\delta^{(n_g)}_{g_1 l-j}d_{gl \bmod n}\\ &= d_{gi \bmod n}\delta^{(n_g)}_{g(g_1 i \bmod n_g)-j}d_{g(g_1 i \bmod n_g)\bmod n} = \delta^{(n_g)}_{g(g_1 i \bmod n_g)-j}d_{gi \bmod n}d_{g^2 i \bmod n},\end{aligned}$$

the last equality comes from equalities $g(g_1 i \bmod n_g) \bmod n = (g^2 i \bmod \tilde{g}n) \bmod n = g^2 i \bmod n$, where $g = \tilde{g}\delta^0$. Now, we need to give a simplified formula of coefficients of the matrix $\left(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g}\right)^{\varphi(n_g)}$, where $\varphi(n_g)$ designates the Euler indicator of n_g. Setting $\beta^{(h)}(j) = jg_1^h \bmod n_g$, by mathematical induction it is not hard to see that

$$(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})^{\varphi(n_g)}_{ij} = \delta^{(n_g)}_{g\beta^{(\varphi(n_g)-1)}(i)-j} \prod_{p=1}^{\varphi(n_g)} d_{g^p i \bmod n}.$$

Using equations (7) and (8), the application of the Euler-Fermat theorem yields

$$\begin{aligned}
\left(g\beta^{(\varphi(n_g)-1)}(i) - j\right) \bmod n_g &= \left(g\beta^{(\varphi(n_g)-1)}(i) \bmod n_g - j \bmod n_g\right) \bmod n_g \\
&= \left((g_1\beta^{(\varphi(n_g)-1)}(i) \bmod n_g) \bmod n_g - j\right) \bmod n_g \\
&= \left((g_1 \cdot g_1^{\varphi(n_g)-1} i \bmod n_g) \bmod n_g - j\right) \bmod n_g \\
&= \left(g_1^{\varphi(n_g)} i \bmod n_g - j\right) \bmod n_g \\
&= (i-j) \bmod n_g = 0 \Leftrightarrow i = j.
\end{aligned}$$

The last equality comes from $g_1 = g \bmod n_g$ and according to Proposition 2.1, we have $g_1^{\varphi(n_g)} \equiv 1 \bmod n_g$, the equivalence follows from estimate $|i-j| < n_g$. So

$$(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})^{\varphi(n_g)}_{ij} = \begin{cases} \prod_{p=1}^{\varphi(n_g)} d_{g^p i \bmod n} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $d_t = L_t \exp\left(\hat{i}\theta_t\right) = [L_t, \theta_t]$ $(t \in \mathbb{N})$, the trigonometric form of the entry $(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})^{\varphi(n_g)}_{jj}$ is given by

$$(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g})^{\varphi(n_g)}_{jj} = \left[\prod_{p=1}^{\varphi(n_g)} L_{g^p j \bmod n}; \sum_{p=1}^{\varphi(n_g)} \theta_{g^p j \bmod n}\right].$$

Now, let $\mathrm{Eig}\left(M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g}\right) = \{\lambda_j : \ j = 0, 1, \ldots, n_g - 1\}$ be the set of eigenvalues of $M^{(0)}_{n_g,g_1}\Delta^{(0)}_{n_g}$. Similar to case $(n,g) = 1$, a simple calculation gives

$$\begin{aligned}
\lambda_j &= \exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_g)}\right) \prod_{p=1}^{\varphi(n_g)} L^{\frac{1}{\varphi(n_g)}}_{g^p v(j) \bmod n} \exp\left(\hat{i}\frac{\theta_{g^p v(j) \bmod n}}{\varphi(n_g)}\right) \\
&= \exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_g)}\right) \prod_{p=1}^{\varphi(n_g)} f_{g^p v(j) \bmod n},
\end{aligned}$$

where $v(j) \in \{0, 1, \ldots, n_g - 1\}$, $k(j) \in \{0, 1, \ldots, \varphi(n_g) - 1\}$ and $f_{g^p v(j) \bmod n} = L^{\frac{1}{\varphi(n_g)}}_{g^p v(j) \bmod n} \exp\left(\hat{i}\frac{\theta_{g^p v(j) \bmod n}}{\varphi(n_g)}\right)$. This completes the proof. $\square$

Moreover, we have the following fundamental results which play crucial roles in our original contribution (namely Theorem 4.1).

LEMMA 4.5. *Setting* $\delta^{(0)} = (n_g, g_1) > 1$, *there exists a finite sequence* $\left\{(g_{k+1}, \delta^{(k)}, n_{g_{(k)}})\right\}_k$ *of elements of* $\mathbb{N}^3$ *satisfying:* $g_0 = g$, $n_{g_{(0)}} = n_g$, $g_1 = g \mod n_g$ *and for* $k \in \mathbb{N}$ $(k \neq 0)$, $g_k = g_{k-1} \mod n_{g_{(k-1)}}$, $\delta^{(k-1)} = (n_{g_{(k-1)}}, g_k)$, $n_{g_{(k)}} = \frac{n_{g_{(k-1)}}}{\delta^{(k-1)}}$, *and a matrix sequence* $\left\{M^{(k)}_{n_{g_{(k)}},g_{k+1}} \Delta^{(k)}_{n_{g_{(k)}}}\right\}_k$ *of decreasing order* $n_{g_{(k)}}$ *(this sequence is finite since the matrix* $M_{n,g}D_n$ *is of order* n*) such that,*

a) $M^{(0)}_{n_{g_{(0)}},g_1} \Delta^{(0)}_{n_{g_{(0)}}} = M^{(0)}_{n_g,g_1} \Delta^{(0)}_{n_g}$,

b) $\Delta^{(k)}_{n_{g_{(k)}}} = \mathrm{diag}\left(d_{g^{k+1} j \bmod n)} : j = 0, 1, \ldots, n_{g_{(k)}} - 1\right)$,

c) $M^{(k)}_{n_{g_{(k)}},g_{k+1}} = \left[\delta^{(n_{g_{(k)}})}_{g_{k+1}i-j}\right]_{i,j=0}^{n_{g_{(k)}}-1}$; *where* $\delta^{(n_{g_{(k)}})}_q = \begin{cases} 1 \text{ if } q \equiv 0 \bmod n_{g_{(k)}}, \\ 0 \text{ otherwise.} \end{cases}$

Proof. First, let us set: $g_0 = g$, $n_{g_{(0)}} = n_g$, $g_1 = g \mod n_g$, $\delta^{(0)} = (n_g, g_1)$ and $n_{g_{(1)}} = \frac{n_g}{\delta^{(0)}}$. As proved in Lemmas 3.1-3.2, straightforward calculations yield

$$\mathrm{Eig}\left(M^{(0)}_{n_g,g_1} \Delta^{(0)}_{n_g}\right) = \mathrm{Eig}\left(\sum_{k=0}^{\delta^{(0)}-1} M^{(1)}_k \Delta^{(1)}_k\right) \cup \{0\},$$

where 0 is of multiplicity $n_g - n_{g_{(1)}}$, the entries $\left(M^{(1)}_k\right)_{ij}$ and $\left(\Delta^{(1)}_k\right)_{jj}$, are given by $\left(M^{(1)}_k\right)_{ij} = \left(M^{(0)}_{n_g,g_1}\right)_{kn_{g_{(1)}}+i,kn_{g_{(1)}}+j} = \delta^{(n_g)}_{g_1(kn_{g_{(1)}}+i)-(kn_{g_{(1)}}+j)} = \delta^{(n_g)}_{(g_1 i-j)-kn_{g_{(1)}}}$ (the last equality is a straightforward consequence of relations: $g_1 n_{g_{(1)}} = \frac{g_1}{\delta^{(0)}} \delta^{(0)} n_{g_{(1)}} = \frac{g_1}{\delta^{(0)}} n_g \equiv 0 \bmod n_g$),

$$\left(\Delta^{(1)}_k\right)_{jj} = \left(\Delta^{(0)}_{n_g}\right)_{kn_{g_{(1)}}+j,kn_{g_{(1)}}+j} = d_{g(kn_{g_{(1)}}+j) \bmod n}, \quad k = 0, 1, \ldots, \delta^{(0)} - 1,$$

and $i, j = 0, 1, \ldots, n_{g_{(1)}} - 1$. To end the proof of this result, we should give a simple form of the sum $\sum_{k=0}^{\delta^{(0)}-1} M^{(1)}_k \Delta^{(1)}_k$. Setting $g_2 = g_1 \mod n_{g_{(1)}}$, it holds

$$\begin{aligned}(M^{(1)}_k \Delta^{(1)}_k)_{i,j} &= \sum_{p=0}^{n_{g_{(1)}}-1} (M^{(1)}_k)_{ip} (\Delta^{(1)}_k)_{pj} = (M^{(1)}_k)_{ij} (\Delta^{(1)}_k)_{jj} \\ &= \delta^{(n_g)}_{g_1 i-(kn_{g_{(1)}}+j)} d_{g(kn_{g_{(1)}}+j) \bmod n},\end{aligned}$$

for $i,j=0,1,\ldots,n_{g_{(1)}}-1$. Since $g_2=g_1 \bmod n_{g_{(1)}}$, we can write $g_1=\tilde{g}_1 n_{g_{(1)}}+g_2$, where $\tilde{g}_1\in\mathbb{N}$. Using this, it is not hard to see that $g_1 i=n_{g_{(1)}}\tilde{g}_1 i+g_2 i=n_{g_{(1)}}(q_i\delta^{(0)}+r_i)+g_2 i=q_i n_g+r_i n_{g_{(1)}}+g_2 i$, where $0\leqslant r_i<\delta^{(0)}$. From these equalities, it follows $r_i n_{g_{(1)}}=(g_1-g_2)i \bmod n_g$. Now, straightforward calculations provide

$$
\begin{aligned}
\left(\sum_{k=0}^{\delta^{(0)}-1} M_k^{(1)}\Delta_k^{(1)}\right)_{i,j} &= \sum_{k=0}^{\delta^{(0)}-1} \delta^{(n_g)}_{g_1 i-kn_{g_{(1)}}-j} d_{g(kn_{g_{(1)}}+j) \bmod n}\\
&= \sum_{k=0}^{\delta^{(0)}-1} \delta^{(n_g)}_{(r_i-k)n_{g_{(1)}}+g_2 i-j} d_{g(kn_{g_{(1)}}+j) \bmod n}\\
&= \delta^{(n_g)}_{g_2 i-j} d_{g(r_i n_{g_{(1)}}+j) \bmod n}=\delta^{(n_g)}_{g_2 i-j} d_{g((g_1-g_2)i \bmod n_g+j) \bmod n}\\
&\underset{(f)}{=} \begin{cases} d_{g(g_1 i \bmod n_g) \bmod n} & \text{if } j\equiv g_2 i \mod n_g,\\ 0 & \text{otherwise.}\end{cases}\\
&\underset{(h)}{=} \begin{cases} d_{g^2 i \bmod n} & \text{if } j\equiv g_2 i \mod n_{g_{(1)}},\\ 0 & \text{otherwise.}\end{cases}\\
&= \delta^{(n_{g_{(1)}})}_{g_2 i-j}\, d_{g^2 i \bmod n}=\left(\Delta^{(1)}_{n_{g_{(1)}}} M^{(1)}_{n_{g_{(1)}},g_2}\right)_{ij},
\end{aligned}
$$

where, the third equality holds because there exists a unique $k_i\in\{0,1,\ldots,\delta^{(0)}-1\}$, such that $k_i=r_i$. Equality **(f)** comes from Lemma 2.1. Equality **(h)** comes from estimate $j<n_{g_{(1)}}$. A straightforward calculation of entries of $\Delta^{(1)}_{n_{g_{(1)}}} M^{(1)}_{n_{g_{(1)}},g_2}$ provides the last equality. Here, g_2 can be equal to zero. Finally, for every $k\in\mathbb{N}^\star$, we construct the parameters g_k, $\delta^{(k-1)}$ and $n_{g_{(k)}}$, as follows

$$
g_k=g_{k-1} \bmod n_{g_{(k-1)}},\ \ \delta^{(k-1)}=(n_{g_{(k-1)}},g_k),\ \ n_{g_{(k)}}=\frac{n_{g_{(k-1)}}}{\delta^{(k-1)}}. \tag{35}
$$

Using relation (35) we define the finite sequence $\left\{(g_{k+1},\delta^{(k)},n_{g_{(k)}})\right\}_{k\in\mathbb{N}}$ of elements of $\mathbb{N}^3$ and we construct by mathematical induction the matrix sequence $\left\{M^{(k)}_{n_{g_{(k)}},g_{k+1}}\Delta^{(k)}_{n_{g_{(k)}}}\right\}_{k\in\mathbb{N}}$ of decreasing size $n_{g_{(k)}}$ satisfying

$$
\left(M^{(k)}_{n_{g_{(k)}},g_{k+1}}\right)_{ij}=\delta^{(n_{g_{(k)}})}_{g_{k+1}i-j}, \tag{36}
$$

and

$$
\left(\Delta^{(k)}_{n_{g_{(k)}}}\right)_{ii}=d_{g[g_1(g_2(\ldots(g_{k-1}(g_k i \bmod n_{g_{(k-1)}}) \bmod n_{g_{(k-2)}})\ldots) \bmod n_{g_{(1)}}) \bmod n_{g_0}] \bmod n}\underset{(\beta)}{=} d_{g^{k+1}i \bmod n}, \tag{37}
$$

for $i,j=0,1,\ldots,n_{g_{(k)}}-1$.

Let us prove the last equality in (37). Putting $G=g[g_1(\ldots(g_{k-1}(g_k i \bmod n_{g_{(k-1)}}) \bmod n_{g_{(k-2)}})\ldots) \bmod n_{g_{(0)}}] \bmod n$, we can write $g_j=g_{j-1}-m_j.n_{g_{(j-1)}}$, where $m_j\in\mathbb{Z}$, for every $j = 1,\ldots,k$. Hence, simple calculations result in

$$\begin{aligned}
G &:= g[g_1(\ldots(g_{k-1}(g_k i \bmod n_{g_{(k-1)}}) \bmod n_{g_{(k-2)}})\ldots) \bmod n_{g_{(0)}}] \bmod n \\
&= g[g_1(g_2(\ldots(g_{k-1}^2 i \bmod n_{g_{(k-2)}})\ldots) \bmod n_{g_{(1)}}) \bmod n_{g_{(0)}}] \bmod n \\
&= g[g_1(g_2(\ldots(g_{k-2}^3 i \bmod n_{g_{(k-3)}})\ldots) \bmod n_{g_{(1)}}) \bmod n_{g_{(0)}}] \bmod n \\
&= g[g_1(g_2(\ldots(g_{k-3}^4 i \bmod n_{g_{(k-4)}})\ldots) \bmod n_{g_{(1)}}) \bmod n_{g_{(0)}}] \bmod n \\
&\vdots \\
&= g[g_1^k i \bmod n_{g_{(0)}}] \bmod n \\
&= g^{k+1} i \bmod n. \quad \square
\end{aligned}$$

As a straightforward consequence of Lemma 4.5, the following result describes in detailed way the spectrum of $C_{n,g}$.

LEMMA 4.6. *Let $s \in \mathbb{N}$ be the first index associated with the sequences constructed in Lemma 4.5 such that, $\delta^{(s)} \in \{1, n_{g_{(s)}}\}$. Then, we have*

1. *If $\delta^{(s)} = n_{g_{(s)}}$, then $g_{s+1} = 0$. Hence*

$$\mathrm{Eig}\,(C_{n,g}) = \mathrm{Eig}\left(M^{(s)}_{n_{g_{(s)}},0}\Delta^{(s)}_{n_{g_{(s)}}}\right) \cup \{0:\ \textit{mult.} = n - n_{g_{(s)}}\} = \{d_0,\, 0\},$$

where 0 is of multiplicity $n-1$;

2. *For $\delta^{(s)} = 1$, it holds*

$$\begin{aligned}
\mathrm{Eig}\,(C_{n,g}) &= \mathrm{Eig}\left(M^{(s)}_{n_{g_{(s)}},g_{s+1}}\Delta^{(s-1)}_{n_{g_{(s)}}}\right) \cup \{0:\ \textit{mult.} = n - n_{g_{(s)}}\} \\
&= \left\{0,\ \exp\left(\hat{\imath}\frac{2k(j)\pi}{\varphi\left(n_{g_{(s)}}\right)}\right)^{\varphi\left(n_{g_{(s)}}\right)-1} \prod_{h=0} f_{g_{s+1}^h k_s(p(j)) \bmod n};\ \ j = 0,1,\ldots,n_{g_{(s)}}-1\right\},
\end{aligned}$$

where 0 is an eigenvalue of algebraic and geometric multiplicity $n - n_{g_{(s)}}$, $k_s(j) = g^{s+1} j \bmod n$, $p(j) \in \{0,1,\ldots,n_{g_{(s)}}-1\}$, $k(j) \in \left\{0,1,\ldots,\varphi\left(n_{g_{(s)}}\right)-1\right\}$, and

$$f_{g_{s+1}^h k_s(p(j)) \bmod n} = L_{g_{s+1}^h k_s(p(j)) \bmod n}^{\frac{1}{\varphi\left(n_{g_{(s)}}\right)}} \exp\left(\hat{\imath}\frac{\theta_{g_{s+1}^h k_s(p(j)) \bmod n}}{\varphi\left(n_{g_{(s)}}\right)}\right),$$

with $d_j = (D_n)_{j,j} = L_j \exp\left(\hat{\imath}\theta_j\right)$.

Proof. *Case* 1. ($\delta^{(s)} = n_{g_{(s)}}$). Since $\delta^{(s)} = n_{g_{(s)}}$, we have that $g_{s+1} = 0$. So, the matrix $M^{(s)}_{n_{g_{(s)}},0}\Delta^{(s)}_{n_{g_{(s)}}}$ is given by $M^{(s)}_{n_{g_{(s)}},0}\Delta^{(s)}_{n_{g_{(s)}}} = C_1 \otimes d_0$, where C_1 is the matrix of order $n_{g_{(s)}}$ having 1 as entries of the first column, 0 elsewhere and d_0 is the first entry of the diagonal matrix D. A combination of Lemmas 3.1, 3.2 and 4.5, after straightforward computations and rearranging terms gives

$$\begin{aligned}
\mathrm{Eig}\,(C_{n,g}) &= \mathrm{Eig}\left(M^{(0)}_{n_{g_{(0)}},g_1}\Delta^{(0)}_{n_{g_{(0)}}}\right) \cup \{0:\ mult. = n - n_g\} \\
&= \mathrm{Eig}\left(M^{(1)}_{n_{g_{(1)}},g_2}\Delta^{(1)}_{n_{g_{(1)}}}\right) \cup \{0:\ mult. = n_g - n_{g_{(1)}}\} \cup \{0:\ \text{mult.} = n - n_g\} \\
&= \mathrm{Eig}\left(M^{(1)}_{n_{g_{(1)}},g_2}\Delta^{(1)}_{n_{g_{(1)}}}\right) \cup \{0:\ mult. = n - n_{g_{(1)}}\} \\
&\ \vdots \\
&= \mathrm{Eig}\left(M^{(s)}_{n_{g_{(s)}},g_{s+1}}\Delta^{(s)}_{n_{g_{(s)}}}\right) \cup \{0 : mult. : n_{g_{(s-1)}} - n_{g_{(s)}}\} \cup \{0:\ n - n_{g_{(s-1)}}\} \\
&= \{d_0, 0:\ \text{mult.} = n - 1\}.
\end{aligned}$$

Case 2. $\delta^{(s)} = 1$. According to Euler-Fermat theorem (Proposition 2.1), there exists a positive integer $\varphi(n_{g_{(s)}})$, satisfying $g_{s+1}^{\varphi(n_{g_{(s)}})} = 1 \bmod n_{g_{(s)}}$. The entries of $\Delta^{(s)}_{n_{g_{(s)}}} M^{(s)}_{n_{g_{(s)}},g_{s+1}}$, are defined as

$$\begin{aligned}
\left(\Delta^{(s)}_{n_{g_{(s)}}} M^{(s)}_{n_{g_{(s)}},g_{s+1}}\right)_{ij} &= \delta^{(n_{g_{(s)}})}_{g_{s+1}i-j} d_{g[g_1(g_2(\dots(g_{s-1}(g_s i \bmod n_{g_{(s-1)}}) \bmod n_{g_{(s-2)}})\dots) \bmod n_{g_{(1)}}) \bmod n_{g_0}] \bmod n} \\
&= \delta^{(n_{g_{(s)}})}_{g_{s+1}i-j} d_{k_s(i)},
\end{aligned}$$

where $i, j = 0, 1, \dots, n_{g_{(s)}} - 1$, and $k_s(i) = g^{s+1} i \bmod n$. So, the spectrum of $C_{n,g}$ is given by

$$\mathrm{Eig}\,(C_{n,g}) = \mathrm{Eig}\left(M^{(s)}_{n_{g_{(s)}},g_{s+1}}\Delta^{(s)}_{n_{g_{(s)}}}\right) \cup \{0:\ \text{mult.} = n - n_{g_{(s)}}\}.$$

Since the integers $n_{g_{(s)}}$ and g_{s+1} are coprime, using Lemma 4.4, replacing n_g by $n_{g_{(s)}}$, g_1 by g_{s+1}, $p(j)$ by $k_s(p(j))$, and $\varphi(n_g)$ by $\varphi(n_{g_{(s)}})$, we get

$$\begin{aligned}
\mathrm{Eig}(C_{n,g}) = &\left\{ \exp\left(\hat{i}\frac{2k(j)\pi}{\varphi(n_{g_{(s)}})}\right) \prod_{h=0}^{\varphi(n_{g_{(s)}})-1} f_{g^h_{s+1}k_s(p(j)) \bmod n} :\ j = 0, 1, \dots, n_{g_{(s)}} - 1 \right\} \\
&\cup \{0 : \text{mult.} = n - n_{g_{(s)}}\},
\end{aligned}$$

where $f_{g^h_{s+1}k_s(p(j)) \bmod n} = L^{\frac{1}{\varphi(n_{g_{(s)}})}}_{g^h_{s+1}k_s(p(j)) \bmod n} \exp\left(\hat{i}\frac{\theta_{g^h_{s+1}k_s(p(j)) \bmod n}}{\varphi(n_{g_{(s)}})}\right)$, $p(j) \in \{0, 1, \dots, n_{g_{(s)}} - 1\}$, and $k(j) \in \{0, 1, \dots, \varphi(n_{g_{(s)}}) - 1\}$. $\square$

More specifically, the following theorem characterizes the eigenvalues of g-circulant matrices $C_{n,g}$.

THEOREM 4.1. *Let n and g be two positive integers, then there exists a nonnegative integer s such that, the spectrum of $C_{n,g}$ is given by*

$$\operatorname{Eig}(C_{n,g}) = \{0\} \cup \left\{ \exp\left\{ \hat{\imath}\frac{2k(j)\pi}{\varphi\left(n_{g_{(s)}}\right)} \prod_{h=0}^{\varphi\left(n_{g_{(s)}}\right)-1} f_{g_{s+1}^{h} k_s(p(j)) \bmod n} \right\} ; j = 0,1,\ldots,n_{g_{(s)}}-1 \right\},$$

where 0 *is of multiplicity* $n-n_{g_{(s)}}$, $n_{g_{(s)}}$ *is given by relation* (35), $k_s(j) = g^{s+1} j \bmod n$, $p(j) \in \{0,1,\ldots,n_{g_{(s)}}-1\}$, $k(j) \in \left\{0,1,\ldots,\varphi\left(n_{g_{(s)}}\right)-1\right\}$ *and* $f_{g_{s+1}^{h} k_s(p(j)) \bmod n} = L_{g_{s+1}^{h} k_s(p(j)) \bmod n}^{\frac{1}{\varphi\left(n_{g_{(s)}}\right)}} \exp\left(\hat{\imath}\frac{\theta_{g_{s+1}^{h} k_s(p(j)) \bmod n}}{\varphi\left(n_{g_{(s)}}\right)}\right)$, *with* $d_j = (D_n)_{j,j} = L_j \exp\left(\hat{\imath}\theta_j\right)$.

Proof. The proof comes from Lemmas 4.5 and 4.6. □

REMARK 4.1. we obtain a simplified recursive procedure that computes the eigenvalues of $C_{n,g}$.

5. A recursive procedure

Given a positive integer g, the Fourier matrix F_n and the matrix $Z_{n,g} = [\delta_{r-gs}]_{r,s=0}^{n-1}$, determine the matrix $M_{n,g} = F_n Z_{n,g} F_n^{\star} := [\delta_{gr-s}]_{r,s=0}^{n-1}$. Set $n_{g_{(-1)}} := n$; $\delta^{(-1)} := (n,g)$; $g_{(0)} = g_0 := g$; $n_g := n_{g_{(0)}} := \frac{n}{\delta^{(-1)}}$; $M_{n_{g_{(-1)}},g_0}^{(-1)} := M_{n,g}$; $\Delta_{n_{g_{(-1)}}}^{(-1)} := D_n = \operatorname{diag}(d_j : j = 0,1,2,\ldots,n-1)$; $\delta_s^{(n)} := \delta_s$. Put $k := 0$;

(1) if $\delta^{(k-1)} = 1$;

i. compute using Lemma 4.1 the matrix

$$\left(M_{n_{g_{(k-1)}},g_k}^{(k-1)} \cdot \Delta_{n_{g_{(k-1)}}}^{(k-1)}\right)^{\varphi\left(n_{g_{(k-1)}}\right)} := \operatorname{diag}\left(\prod_{p=0}^{\varphi\left(n_{g_{(k-1)}}\right)-1} d_{l_j^p},\ j = 0,1,\ldots,n_{g_{(k-1)}}-1\right),$$

where $l_j^p := g^p j \bmod n_{g_{(k-1)}}$;

ii. for a fixed $j \in \{0,1,\ldots,n_{g_{(k-1)}}-1\}$, solve the equation

$$z^{\varphi\left(n_{g_{(k-1)}}\right)} = \prod_{p=0}^{\varphi\left(n_{g_{(k-1)}}\right)-1} d_{l_j^p};$$

iii. then the spectrum of $C_{n,g}$ is

$$\operatorname{Eig}(C_{n,g}) = \operatorname{Eig}\left(M_{n_{g_{(k-1)}},g_k}^{(k-1)} \cdot \Delta_{n_{g_{(k-1)}}}^{(k-1)}\right) \cup \{0\} =$$

$$\left\{0,\ \exp\left(\hat{\imath}\frac{2k(j)\pi}{\varphi(n_{g_{(k-1)}})}\right)\prod_{h=0}^{\varphi(n_{g_{(k-1)}})-1} f_{g_k^h R_{k-1}(p(j)) \bmod n_{g_{(k-1)}}}\ :\ j=0,1,\ldots,n_{g_{(k-1)}}-1\right\},$$

0 is of multiplicity $n-n_{g_{(k-1)}}$; $R_{k-1}(j):=g^k j \bmod n$; $d_j:=L_j\exp\left(\hat{\imath}\theta_j\right)$, where $|d_j|=L_j$;

$$f_{g_k^h R_{k-1}(p(j)) \bmod n_{g_{(k-1)}}} := L_{g_k^h R_{k-1}(p(j)) \bmod n_{g_{(k-1)}}}^{\frac{1}{\varphi\left(n_{g_{(k-1)}}\right)}} \exp\left(\hat{\imath}\frac{\theta_{g_k^h R_{k-1}(p(j)) \bmod n_{g_{(k-1)}}}}{\varphi\left(n_{g_{(k-1)}}\right)}\right),$$

where $\varphi(a)$ denotes the Euler indicator associated with the positive integer a, $p(j)\in\{0,1,\ldots,n_{g_{(k-1)}}-1\}$, and $k(j)\in\left\{0,1,\ldots,\varphi\left(n_{g_{(k-1)}}\right)-1\right\}$;
stop.

otherwise,

(2) if $\delta^{(k-1)}:=n_{g_{(k-1)}}$,

i. compute:

$g_k:=0$;

$$\mathrm{Eig}\left(M^{(k-1)}_{n_{g_{(k-1)}},0}\cdot\Delta^{(k-1)}_{n_{g_{(k-1)}}}\right):=\{d_0,0:mult.=n_{g_{(k-1)}}-1\};$$

ii. hence the spectrum of $C_{n,g}$ is given by

$$\begin{aligned}\mathrm{Eig}(C_{n,g}) &:= \mathrm{Eig}\left(M^{(k-1)}_{n_{g_{(k-1)}},0}\cdot\Delta^{(k-1)}_{n_{g_{(k-1)}}}\right)\cup\{0:mult.=n-n_{g_{(k-1)}}\}\\ &:=\{d_0,0:mult.=n-1\};\end{aligned}$$

stop;

otherwise;

(3) put $k:=k+1$, compute:

$$n_{g_{(k-1)}}:=\frac{n_{g_{(k-2)}}}{\delta^{(k-2)}};\ \ g_k:=g_{k-1} \bmod n_{g_{(k-1)}};\ \ \delta^{(k-1)}:=gcd(n_{g_{(k-1)}},g_k);$$

and

$$\Delta^{(k-1)}_{n_{g_{(k-1)}}}:=\mathrm{diag}\left(d_{g^k j \bmod n}\ :\ j=0,1,\ldots,n_{g_{(k-1)}}-1\right);$$

$$M^{(k-1)}_{n_{g_{(k-1)}},g_k}:=\left[\delta^{(n_{g_{(k-1)}})}_{g_k r-s}\right]_{r,s=0}^{n_{g_{(k-1)}}-1};\ \ \delta^{(n_{g_{(k-1)}})}_q:=\begin{cases}1 \text{ if } q\equiv 0 \bmod n_{g_{(k-1)}},\\ 0 \text{ otherwise;}\end{cases}$$

The above scheme determines in recursive way the eigenvalues of $C_{n,g}$. It can break off in step (1) if the positive integers n and g are coprime. In that case the obtained values are like those determined by William F. Trench in [21]. In addition, the scheme can stop in step (2) if $g=0$.

In the following we present some numerical experiments which confirm the theoretical results.

6. Some numerical experiments

Aimed of providing numerical evidences to the theoretical results of the previous section, we now analyze in detail the eigenvalues of g-circulant matrices $C_{n,g}$ in both cases

(i) the integers n and g are coprime,

(ii) the integers n and g are not coprime,

and where the generating function f is chosen in different classes of integrable functions defined over $(-\pi,\pi)$, such as: polynomials (for instance, $f(x)=1+x^3$), trigonometric polynomials (i.e., $f(x)=(1-\cos(x))^3$) and neither polynomials nor trigonometric polynomials (for example, $f(x)=\frac{x-2}{x^2+1}$).

In these numerical experiments, we consider six test cases and we report for each considered case the eigenvalues of $C_{n,g}$. We observe that the symbol f is nonnegative in the case of trigonometric polynomials, and only has a real root in both others cases. However, when the parameter g is strictly greater that 1, and is not coprime with the integer n, the g-circulant matrix, $C_{n,g}$, has at least one eigenvalue equals zero of multiplicity greater than two. The numerical tests have been developed with MatLab $R2009a$, and the eigenvalues have been computed by the built-in MatLab function eig().

• **Test 1:** $n=80$, $g=50$, and the generating function f is an integrable function defined over $(-\pi,\pi)$, by $f(x)=\frac{x-2}{x^2+1}$.

Table 1. *Eigenvalues of* $C_{80,50}$.

$$-125.78-27.55i \quad -10^{-2} \quad 10^{-2} \quad -10^{-2}i \quad 10^{-2}i \quad 0$$

where i is the complex number satisfying $i^2=-1$. When neglecting the four numbers $\pm10^{-2}$ and $\pm10^{-2}i$ with respect to $-125.78-27.55i$, it follows that "0" is an eigenvalue of algebraic and geometrical multiplicity equals $80-1=79$. In that case, we meet the first item of Lemma 4.6. In addition, one also could compute the value of d_0 and compare it to the number $-125.78-27.55i$ to meet the theoretical result given above.

• **Test 2:** $n=11$ and $g=7$ (the integers n and g are coprime), and the generating function f is given in Test 1.

Table 2. *Eigenvalues of* $C_{11,7}$.

$-15.4190-3.7759i$	$-8.2470+0.1980i$	$-6.5556+5.0077i$	$-2.3602+7.9046i$
$6.7883+4.6873i$	$8.2470-0.1980i$	$-6.7883-4.6873i$	$6.5556-5.0077i$
$2.3602-7.9046i$	$2.7368+7.7822i$	$-2.7368-7.7822i$	

The above values are all complex numbers and zero is not on this list, so zero is not an eigenvalue of $C_{11,7}$, which implies that the matrix $C_{11,7}$ is nonsingular. Furthermore, except the complex number, $-15.4190-3.7759i$, which is related to the spectral radius, $\rho(C_{11,7})$, of $C_{11,7}$, that is, $\rho(C_{11,7})=|-15.4190-3.7759i|$, where $|\cdot|$ denotes the $\mathbb{C}$-norm, the list shows that if λ is an eigenvalue of $C_{11,7}$, then its conjugate $\overline{\lambda}$ also is another one. This suggests that the coefficients of the characteristic polynomial of $C_{11,7}$ are almost all real.

• **Test 3:** $n=54,\ g=3$, and the generating function f is defined over $(-\pi,\pi)$ by $f(x)=(1-\cos(x))^3$.

Table 3. *Eigenvalues of* $C_{54,3}$.

283.50	67.50	0

It is obvious that "0" is an eigenvalue of $C_{54,3}$, of algebraic and geometrical multiplicity equals, $54-2=52$. This case corresponds to Lemma 4.3. Indeed, $n=g^p n_0$, where $n=54,\ g=3,\ p=3$, and $n_0=2$, with $(n_0,g)=1$. In addition, we observe that the non null eigenvalues of $C_{54,3}$ are real. Hence, one can thinks that this follows from the fact that the generating function, f, is a trigonometry polynomial.

• **Test 4:** $n=54$ and $g=37$ (the integers n and g are coprime), and the generating function f is given in Test 3.

Table 4. *Eigenvalues of* $C_{54,37}$.

283.50	67.50	$264.54+66.51i$	$264.54-66.51i$	$216.22+110.81i$
$158.62-122.76i$	$158.62+122.76i$	$111.15+107.72i$	$111.15-107.72i$	$82.74+80.01i$
$216.22-110.81i$	$-71.74+118.54i$	$-71.74+118.54i$	$-70.94+119.85i$	$-68.33+121.36i$
$-66.79+121.40i$	$-71.74-118.54i$	$-71.88-115.45i$	$-71.15-114.01i$	$-70.94-119.85i$
$-68.33-121.36i$	$-66.79-121.40i$	$-64.04+119.98i$	$-71.88+115.45i$	$-64.04-119.98i$
$-68.40-112.26i$	$-71.15+114.01i$	$-66.72-112.22i$	$-63.16-118.62i$	$-63.16+118.62i$
$-63.02-115.37i$	$-68.40+112.26i$	$-66.72+112.22i$	$-63.02+115.37i$	$-63.83-113.89i$
$82.74-80.01i$	$-63.83+113.89i$	$70.87+52.61i$	$70.87-52.61i$	$67.85+31.04i$
$67.85-31.04i$	$67.51+14.44i$	$67.51-14.44i$	$138.53-2.86i$	$138.53+2.86i$
$139.27-1.51i$	$139.27+1.51i$	$135.92-4.53i$	$134.31-4.61i$	$135.92+4.53i$
$134.31+4.61i$	$130.54-1.67i$	$130.54+1.67i$	$131.42-3.11i$	$131.42+3.11i$

The above values are complex numbers, except the two numbers 283.50 and 67.50, and zero is not an eigenvalue of $C_{54,37}$. In addition, this list suggests that for any complex eigenvalue of $C_{54,37}$, its conjugate is also an eigenvalue.

• **Test 5:** $n=28,\ g=16$, and the generating function f is a polynomial defined over $(-\pi,\pi)$, by $f(x)=1+x^3$.

Table 5. *Eigenvalues of* $C_{28,16}$.

$119.85+49.59i$	$60.79+104.72i$	$-102.87+78.99i$	$-121.08+0.29i$
$-16.98-128.59i$	$28.00-45.11i$	$60.29-105.01i$	0

Here "0" is an eigenvalue of algebraic and geometrical multiplicity equals $28-7=21$. It is obvious to see that all the non null eigenvalues of $C_{28,16}$ are complex numbers. Furthermore, we observe that any non null eigenvalue of $C_{28,16}$ has its conjugate as eigenvalue. This test case meets the theoretical result stated in Lemma 4.4.

- **Test 6:** $n = 28,\ g = 9$, and the generating function f is given in **Test 5**.

Table 6. *Eigenvalues of* $C_{28,9}$.

$28+873.57i$	$-133.37+116.71i$	$-176.23+18.23i$	$103.90+143.51i$	$167.76+57.15i$
$-34.39-173.86i$	$72.33-161.73i$	$-128.7+78.99i$	$119.85+49.59i$	$-121.08+0.29i$
$60.79+104.72i$	$-96.77-27.70i$	$24.40+97.65i$	$-62.21+56.03i$	$-65.51+30.75i$
$29.09+74.44i$	$59.39+41.36i$	$79.63+25.86i$	$-79.01-12.03i$	$60.29-105.01i$
$77.34-81.95i$	$-16.98-128.59i$	$72.37-69.95i$	$49.92-62.41i$	$28-45.11i$
$6.13-72.11i$	$-21.34-81.95i$	$-17.42-81.89i$		

It is obvious to see that any eigenvalue of $C_{28,9}$ is not real and zero is not an eigenvalue. Furthermore, we observe that any eigenvalue of $C_{28,9}$ doesn't have its conjugate as eigenvalue.

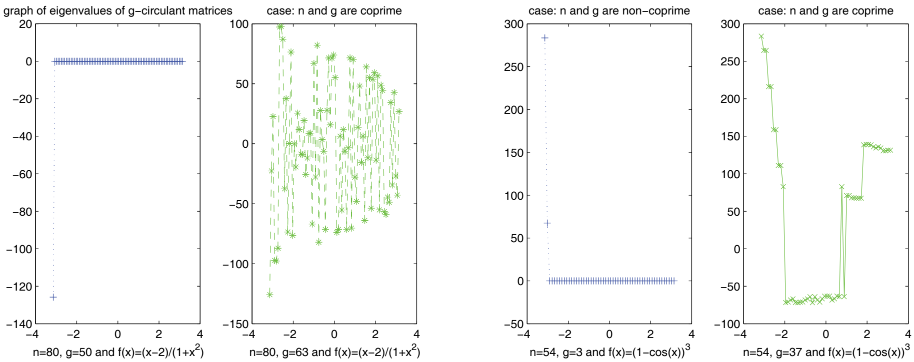
A combination of six tests shows the crucial role played by the generating function in characterizing the eigenvalues of g-circulant matrices. Before dealing with the conditioning of g-circulant structures, it is quite interesting to notice that the eigenvalues of $C_{n,g}$ agree with the corresponding theoretical results (as already proven by numerical tests).

In fact, for $g \geqslant 2$, if $\delta = (n,g) = 1$, then the eigenvalues are bounded away from zero (see tests and Figure 1), and if $\delta > 1$, some of the first $n_g = \frac{n}{\delta}$ eigenvalues are always bounded away from zero, while the remaining are null, as stated in Lemmas 4.3, 4.4 and 4.6. In particular, for each couple: $n = 80$ and $g = 63$, $n = 11$ and $g = 7$, $n = 54$ and $g = 37$ or $n = 28$ and $g = 9$, the greater common divisor , δ, equals 1. So, the eigenvalues of $C_{n,g}$ are bounded away from zero in the well conditioned tests (see both smallest and greatest eigenvalues (in modulus) in tests $2,\ 4$ and 6 above, and the graphs in green). On the other hand, for each case: $n = 80$ and $g = 50$, or $n = 54$ and $g = 3$, $n = 28$ and $g = 16$, more than one half of the eigenvalues of $C_{n,g}$ are null, which imply that the g-circulant matrices are singular, so are very ill-conditioned.

It is now interesting to analyze the clustering of the spectrum of g-circulant matrices. Any non-coprime case (graphs (in blue) in Figure 1 along with tests $1,\ 3$, and 5) give rise to a good clustered at zero, in the last $n - n_g$ eigenvalues, while the remaining ones, perhaps, are non null. This is a result which was expected in the light of Lemmas 4.3, 4.4 and 4.6. The g-circulant matrices $C_{n,g}$ guarantee a good clustering in a subspace which is the most large possible (remember that the rank of $C_{n,g}$ tends to n_g, since $n - n_g$ eigenvalues vanish as n increases, so that the rank of $C_{n,g}$ cannot be greater or equal than n_g). This good clustering at zero of $C_{n,g}$ occurs in the ill-conditioned cases (some of eigenvalues are null). We can also observe, from six tests above that the g-circulant structures are not necessary related to a distribution function (in the sense of eigenvalues). Indeed, the g-circulant matrices have only non null complex eigenvalues.

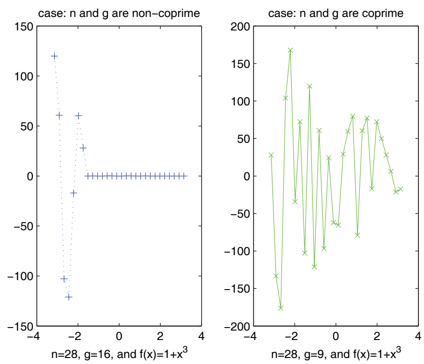
The situation is different in the coprime case, as tests $2,\ 4$, and 6, together with graphs (in green) in Figure 1, suggest. The lists of eigenvalues corresponding to these tests show that when the generating function has a complex root, any eigenvalue does not admit its conjugate as eigenvalue (see test 6, case of the polynomial generating function). This observation is quite different when the symbol related to $C_{n,g}$ only has

Eigenvalues of g-circulant matrices when n and g are both non-coprime and coprime.



$f(x) = \frac{x-2}{1+x^2}$ $\qquad$ $f(x) = (1-\cos(x))^3$

Spectrum of g-circulant matrices when n and g are both non-coprime and coprime.



$$f(x) = 1 + x^3$$

Figure 1: *The graphs representing the eigenvalues of $C_{n,g}$ in both non-coprime case (in blue) and coprime case (in green).*

real roots (one can refer to tests 2 and 4, to see that, except the eigenvalue associated with the spectral radius of $C_{n,g}$ (test 2), each eigenvalue has its conjugate as another one). We can say that the g-circulant structure has a lot of cyclically repeated (i.e., a number and its conjugate), hence linearly independent, columns, which means that a good clustering at zero is no longer possible. However, the numerical experiments corresponding to coprime case confirm the result stated in Lemma 4.2.

7. Conclusion

This paper has studied in detail the eigenvalues of the g-circulant matrices and has provided an iterative analysis that computes the eigenvalues in recursive way. The obtained results generalize those of Willian F. Trench (case where the positive integers n and g are coprime) and they also represent an improvement and an extension of the work due to S. Serra Capizzano and D. Sesana which concerns closed form expressions of such values. We have presented and discussed various numerical evidences which have confirmed our theoretical results. Future works would study the more involved eigenvector behavior for the g-circulant structures.

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