Operators
and
Matrices
Volume 12, Number 3 (2018), 837–853
doi:10.7153/oam-2018-12-50

ON THE WEAKLY CLOSED ALGEBRA GENERATED BY A UNITARY OPERATOR IN A PONTRYAGIN SPACE

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(*Communicated by I. M. Spitkovsky*)

Abstract. This work is devoted to a study of a weakly closed algebra generated by polynomials of a unitary operator in a Pontryagin space. In particular, the inverse operator can belong or not to this algebra. The corresponding criteria is obtained.

1. Introduction

Let us consider a normal operator N acting in a Hilbert space, calculate all operator values of polynomials, where the value of the independent variable is taken to be equal to N and, finally, close in weak topology this operator algebra denoting the new algebra by $\mathrm{Alg}\,N$. Depending on some properties of the normal operator, the algebra $\mathrm{Alg}\,N$ can (or not) be a star algebra, see [16] for detail analysis. It is obvious that in the special case of a unitary operator U the algebra $\mathrm{Alg}\,U$ is a star algebra if and only if the inverse operator belongs to $\mathrm{Alg}\,U$. In [22] it was shown that $U^{-1} \in \mathrm{Alg}\,U$ if and only if U does not contain a part equivalent to the bilateral shift. In the present paper we consider a similar problem for a unitary operator in a Pontryagin space. Section 2 contains mainly well known notions and results used in the course of the paper with the unique exception presented by Theorem 1. Section 3 deals with some function sets related with functional calculus for π-unitary operators. Section 4 is devoted to the case of a π-unitary operator U with unbounded spectral function such that $U^{-1} \in \mathrm{Alg}\,U$. The main result is given by Theorem 7 in Section 5.

2. Preliminaries

This paper continues the theme treated in the papers [17], [18] (see also the short communication [19]). The symbols $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ mean complex plane, the open unit disk in $\mathbb{C}$ and its border respectively. For the terminology and the history of the operator theory in indefinite metric spaces, see the monographs [1] or [2]. In what follows, the term "Krein space" means a separable Hilbert space $\mathfrak{H}$ with an ordinary scalar product $(\cdot,\cdot)$ and a Hermitian sesquilinear indefinite form (inner product) $[\cdot,\cdot]$: $(\forall x, y \in$

Mathematics subject classification (2010): 46C20, 47B50, 47A60, 47L75.
Keywords and phrases: Pontryagin spaces, unitary operators, inverse operator, weakly closed operator algebra.

Paper OaM-12-50

$\mathfrak{H}$) $[x,y] = (Jx,y)$, $J = J^{-1}$; if $\dim \mathrm{Ker}(I+J) = \kappa < \infty$, then this space is called a Pontryagin one. Operator J is called the fundamental symmetry (of $[\cdot,\cdot]$ with respect to $(\cdot,\cdot)$). As usual the symbol $[\perp]$ means the J- or π-orthogonality, i.e. the orthogonality with respect to $[\cdot,\cdot]$ and the symbol $\perp$ means the orthogonality with respect to the Hilbert scalar product. This paper is devoted mainly to Pontryagin spaces, but at the same time we use some previous results from [20] and [21] which require a familiarity with the following notions related to Krein spaces.

The set of all maximal non-negative subspaces of a Krein space $\mathfrak{H}$ is denoted by $\mathfrak{M}^+(\mathfrak{H})$. A subspace $\mathscr{M}$ of $\mathfrak{H}$ is said to be *regular* if there is a constant $c > 0$ such that $\operatorname*{Sup}_{y\in\mathscr{M},\|y\|=1} \{|[x,y]|\} \geqslant c\|x\|$ for every $x \in \mathscr{M}$. A subspace $\mathscr{L}$ of $\mathfrak{H}$ is called *pseudo-regular* ([8]) if it can be presented in the form

$$\mathscr{L} = \hat{\mathscr{L}} \dot{+} \mathscr{L}_1, \tag{1}$$

where $\hat{\mathscr{L}}$ is a regular subspace and $\mathscr{L}_1$ is a neutral subspace (i.e. $\mathscr{L}_1$ is the isotropic part of $\mathscr{L}$: $\mathscr{L}_1 = \mathscr{L} \cap \mathscr{L}^{[\perp]}$).

Let us define a special case of pseudo-regular subspaces: a non-negative (non-positive) subspace $\mathscr{L}$ is called *a subspace of the class* h^+ (h^-) if it is pseudo-regular and $\dim \mathscr{L}_1 < \infty$ for $\mathscr{L}_1$ as in (1). In Pontryagin spaces, every subspace is pseudo-regular and every semi-definite subspace belongs to the class h^+ or h^-.

Here the term “operator” means “bounded linear operator”. By the symbol $B^\#$ we denote the operator J-adjoint (J-a.) to B. If $B = B^\#$, then B is called J-s.a or (in the case of Pontryagin spaces) π-s.a. For an operator A the symbols $\sigma(A)$ and $\rho(A)$ mean respectively its *spectrum* and the set of *regular points* treated in the same way as in [6] or [1].

If an operator family $\mathfrak{Y}$ is such that the condition $A \in \mathfrak{Y}$ implies $A^\# \in \mathfrak{Y}$, then this family is said to be J-*symmetric*. An operator algebra $\mathfrak{A}$ is said to be WJ^*-algebra if it is closed in the weak operator topology, J-symmetric and it contains the identity I.

A subspace $\mathscr{L}$ is said to be A-*invariant* ($\mathfrak{Y}$-*invariant*) if it is invariant with respect to the operator A (operator family $\mathfrak{Y}$).

DEFINITION 1. A J-symmetric operator family $\mathfrak{Y}$ belongs to the class D^+_κ with $\kappa < \infty$ if there is a subspace $\mathscr{L}_+$ of $\mathfrak{H}$ such that

- $\mathscr{L}_+$ is $\mathfrak{Y}$-invariant,
- $\mathscr{L}_+ \in \mathfrak{M}^+(\mathfrak{H}) \cap h^+$,
- $\dim(\mathscr{L}_+ \cap \mathscr{L}_+^{[\perp]}) = \kappa$.

Everywhere below U means a π-unitary operator, i.e., $U\mathfrak{H} = \mathfrak{H}$, $[Ux,Uy] = [x,y]$ for all $x,y \in \mathfrak{H}$ and $\mathfrak{H}$ is a Pontryagin space. Due to well known Pontryagin’s theorem for every π-symmetric family $\mathfrak{Y} = \{U,U^\#\}$ there is a $\mathfrak{Y}$-invariant maximal non-negative subspace, hence, this family automatically belongs to some $D^+_{\kappa'}$-class with $\kappa' \leqslant \kappa$, where κ is the same as in the definition of Pontryagin space. Thus, the results from [20] and [21] are applicable to U.

It is well known that the spectrum $\sigma(U)$ of the operator U can have a non-unitary part that contains finitely many points. Temporally we assume that

$$\sigma(U) \subset \mathbb{T}. \tag{2}$$

If $\sigma(U)$ lies on the unit circle only then U has the spectral function with finitely many so-called critical points [10]. To reach our main goal we need to show some steps from the existence proof of this spectral function. To do so we follow [12] where the corresponding theorem was proved in more general setting, a similar approach for a different class of operators was considered in [13], an alternative approach can be find in [9]. First, due to Pontryagin's theorem U has a maximal non positive invariant subspace $\mathfrak{L}_-$ (so its dimension is κ), therefore there is a polynomial $Q(\xi)$ such that for every vector $x \in \mathfrak{H}$ its image $Q(U)x$ is not a negative one. This polynomial is not uniquely defined, but if ξ_0 is zero of the minimal polynomial $N(\xi)$ of finite-dimensional operator $U|_{\mathfrak{L}_-}$, then it is zero of $Q(\xi)$ too. Of course, one can take $N(\xi)$ instead of $Q(\xi)$, but, because of some reasons, it is more convenient for us to keep a possibility to increase the grade of $Q(\xi)$. Due to (2), with no loss of generality, we can suppose that

$$\text{if } Q(\xi) = 0\text{ , then } \xi \in \mathbb{T}. \tag{3}$$

Next, for the new sesquilinear form $<x,y>:=[Q(U)x,Q(U)y]$, we have $<x,x>\geqslant 0$ $\forall x \in \mathfrak{H}$. Consider the sequence $\{c_k =<U^k x,x>\}_{k=-\infty}^{+\infty}$. Since the quadratic form $<\cdot,\cdot>$ is not negative, there is a unique solution of the trigonometric moment problem

$$c_k = \int_0^{2\pi} e^{ikt}\, d\sigma^{(x)}(t),$$

where $\sigma^{(x)}(t)$ is a non-decreasing function, $\sigma^{(x)}(0) = 0$, $\sigma^{(x)}(t-0) = \sigma^{(x)}(t)$ $\forall t \in [0,2\pi]$. For a fixed t the function $\sigma^{(x)}(t)$ defines a non negative quadratic form with respect to x. The latter provides a possibility to introduce the Hermitian sesquilinear form

$$\prec x,y \succ^{(t)} := \frac{1}{4}\sum_{m=0}^{3} i^m \cdot \sigma^{(x+i^m y)}(t).$$

for all x, the expression $\prec x,x \succ^{(t)}$ is a non-decreasing function of t, $\prec x,x \succ^{(0)} = 0$, $\prec x,x \succ^{(2\pi)} = [Q(U)x,Q(U)y]$. Thus, $\prec x,y \succ^{(t)}$ is continuous with respect to x and y, and there is an operator valued function G_t such that

- $G_o = 0$, $G_t = s - \lim_{\tau\to t-0} G_\tau$ for every $t \in (0,2\pi]$;
- for all $t \in (0,2\pi]$ and $\tau \in (0,t)$ the operator $(G_t - G_\tau)$ is π-non-negative;
- $Q(U)Q(U)^{\#}U^k x = \int_0^{2\pi} e^{ikt} dG_t x$ for every $x \in \mathfrak{H}$ and $k = 0, \pm 1, \pm 2, \dots$.

Using the last item and representations of the resolvent $R_\xi(U)$ in neighbourhoods of infinity and zero, it is easy to prove the following representation

$$Q(U)Q(U)^{\#}R_\xi(U) = \int_0^{2\pi} \frac{1}{t-\xi}\, dG_t x\ . \tag{4}$$

Let

$$\Lambda := \{\lambda : \lambda \in (0, 2\pi],\ N(e^{i\lambda}) = 0\}\ . \tag{5}$$

Using Cauchy principal value integrals of the resolvent one can obtain the spectral function E_t by the formula

$$\text{if } 0 < \alpha < \beta \leqslant 2\pi,\ [\alpha, \beta] \cap \Lambda = \emptyset, \text{ then } E(\Delta) = \int_\alpha^\beta |Q(e^{it})|^{-2} dG_t\ , \tag{6}$$

where $\Delta = [\alpha, \beta)$, $E(\Delta) = E_\beta - E_\alpha$. Of course, G_t depends on the polynomial Q, but in any case it has the above properties and E_t does not depend on Q. In particular, if $t_0 \in [0, 2\pi]$ is such that $Q(e^{it_0}) = 0$ and $t_0 \notin \Lambda$, then the integral in (6) converges as an improper integral. Generally speaking the operator-valued function G_t is not continuous in the strong operator topology and, in particular, it can be discontinuous at some points of the set Λ. Let us consider an example.

EXAMPLE 1. Let $\{g_j\}_{j=-2}^{+\infty}$ be an orthonormal basis in Pontryagin space $\mathfrak{H}$ and a fundamental symmetry J be given by the expressions

$$Jg_{-2} = g_{-2},\ Jg_{-1} = g_0,\ Jg_0 = g_{-1},\ Jg_m = g_m \text{ for } m = 1, 2, \dots\ .$$

Then $\mathfrak{H}$ is a Pontrygin space with $\kappa = 1$. The following operator U

$$Ug_{-1} = ig_{-2} + g_{-1} + \left(-\frac{1}{2} + \sum_{m=1}^{\infty}\left(e^{i/m} - 1 - \frac{i}{m}\right)\right) g_0 + \sum_{m=1}^{\infty}(e^{i/m} - 1)g_m\ ,$$

$$Ug_{-2} = g_{-2} + ig_0,\ Ug_0 = g_0,\ Ug_m = e^{i/m}g_m + (e^{i/m} - 1)g_0 \text{ for } m = 1, 2, \dots$$

is π-unitary, its maximal non negative invariant subspace is unique and corresponds to the eigenvector g_0, so the range of the operator $(U - I)$ is non negative, $\Lambda = \{0\}$. The direct calculation brings

$$G_0 = 0,\ \text{for } t > 1\ G_t \equiv G_{1+0} = (U - I)(U^{[*]} - I),$$

$$G_{1+0}g_{-2} = 0,\ G_{1+0}g_0 = 0,\ G_{1+0}g_m = (2 - e^{i/m} - e^{-i/m})(g_0 + g_m),\ m = 1, 2, \dots\ ,$$

$$G_{1+0}g_{-1} = \big(1 + \sum_{m=1}^{\infty}(2 - e^{i/m} - e^{-i/m})\big)g_0 + \sum_{m=1}^{\infty}(2 - e^{i/m} - e^{-i/m})g_m\ ,$$

$$\text{for } t \in (0, 1]\ G_t = G_{1+0} - \sum_{m:\ t \leqslant \frac{1}{m}}(2 - e^{i/m} - e^{-i/m})[\cdot, g_0 + g_m](g_0 + g_m)\ .$$

Thus, $G_{+0} = [\cdot, g_0]g_0$. Hence, the analysis of the example is finished.

If effectively G_t has a jump at a point $t = \lambda$, $\lambda \in \Lambda$, then we redefine Q taking the new polynomial $\tilde{Q}(\xi) = Q(\xi)(\xi - e^{i\lambda})$. As a result we are eliminating the corresponding jump, so the new operator-valued function has the form $\tilde{G}_t = \int_0^t |\tau - e^{i\lambda}|^2 dG_\tau$. Therefore, without loss of generality. we can assume that

$$G_{\lambda+0} = G_{\lambda-0} \text{ for every } \lambda \in \Lambda.$$

The latter means that

$$\text{for every } x \in \mathfrak{H} \text{ and } k = 0, \pm 1, \pm 2, \dots ,$$
$$\int_0^{2\pi} e^{ikt} dG_t x = \lim_{\delta \to +0} \int_{[0,2\pi] \setminus O_\delta} e^{ikt} dG_t x \,, \tag{7}$$

where $O_\delta = \cup_{\lambda \in \Lambda} (\lambda - \delta, \lambda + \delta)$, $\delta > 0$. Formulae (6) and (7) bring ($\mathrm{s}-\lim$ means the limit in the strong operator topology)

$$Q(U)Q(U)^{[*]}U^k = \mathrm{s}-\lim_{\delta \to +0} \int_{[0,2\pi] \setminus O_\delta} |Q(e^{it})|^2 e^{ikt} dE_t \,. \tag{8}$$

Due to (2) $\overline{Q(e^{it})} = c \cdot e^{-ipt} \cdot Q(e^{it})$, where c is a number, $|c| = 1$ and p is the degree of Q, so (8) can be re-written as

$$c \cdot Q(U)^2 U^{k-p} = c \cdot \mathrm{s}-\lim_{\delta \to +0} \int_{[0,2\pi] \setminus O_\delta} (Q(e^{it}))^2 e^{i(k-p)t} dE_t$$

or

$$Q(U)^2 U^k = \mathrm{s}-\lim_{\delta \to +0} \int_{[0,2\pi] \setminus O_\delta} (Q(e^{it}))^2 e^{ikt} dE_t \,,\ k = 0, \pm 1, \pm 2, \dots . \tag{9}$$

Now we suppose that (2) is not true. Let us present $\mathfrak{H}$ in the form

$$\mathfrak{H} = \mathfrak{H}_{\mathbb{T}} [+] \mathfrak{H}_{\mathbb{T}'}, \quad U\mathfrak{H}_{\mathbb{T}} \subset \mathfrak{H}_{\mathbb{T}}, \quad U\mathfrak{H}_{\mathbb{T}'} \subset \mathfrak{H}_{\mathbb{T}'}, \tag{10}$$

where $\sigma(U|_{\mathfrak{H}_{\mathbb{T}}}) \subset \mathbb{T}$, $\sigma(U|_{\mathfrak{H}_{\mathbb{T}'}}) \subset \mathbb{C} \setminus \mathbb{T}$. The subspace $\mathfrak{H}_{\mathbb{T}'}$ has a finite dimension. Let E_λ be the π-orthogonal spectral function generated by $U|_{\mathfrak{H}_{\mathbb{T}}}$ and let Λ be the set of its critical points. Let us define the domain extension of the operators E_λ as

$$E_\lambda|_{\mathfrak{H}_{\mathbb{T}'}} = 0 \,, \lambda \in [0, 2\pi] \setminus \Lambda \,. \tag{11}$$

Denote (CLin = closed linear span)

$$\widetilde{\mathfrak{H}} := \mathrm{CLin}\{E(\Delta)\mathfrak{H}\}, \tag{12}$$

where Δ runs through the set of all intervals $\Delta \subset [0, 2\pi]$ like in (6), so

$$\text{the subspace } E(\Delta)\mathfrak{H} \text{ is positive.} \tag{13}$$

The subspace $\widetilde{\mathfrak{H}}$ is non-negative and generally speaking has a non-trivial isotropic part. Everywhere below with only one exception (Section 5) we shall suppose

$$\widetilde{\mathfrak{H}} \cap \widetilde{\mathfrak{H}}^{[\perp]} \neq \{0\}. \tag{14}$$

Let

$$\mathfrak{G}_1 = \widetilde{\mathfrak{H}} \cap \widetilde{\mathfrak{H}}^{[\perp]}, \ \mathfrak{G}_0 = J\mathfrak{G}_1, \ \mathfrak{G}_2 = \widetilde{\mathfrak{H}} \cap \mathfrak{G}_1^{\perp}, \ \mathfrak{G}_3 = (\widetilde{\mathfrak{H}} \oplus \mathfrak{G}_0)^{[\perp]}. \tag{15}$$

All considerations would be only simpler if $\mathfrak{G}_3$ is trivial, so we will suppose $\mathfrak{G}_3 \neq \{0\}$. Without loss of generality one can suppose that $\mathfrak{G}_3 = (\widetilde{\mathfrak{H}} \oplus \mathfrak{G}_0)^{\perp}$ and that the Hilbert scalar product on $\mathfrak{G}_2$ coincides with $[\cdot,\cdot]$. Thus,

$$\mathfrak{H}_{\mathbb{T}} \ = \ \mathfrak{G}_0 \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \mathfrak{G}_3 \tag{16}$$

and

$$J|_{\mathfrak{H}_{\mathbb{T}}} = \begin{pmatrix} 0 & V^{-1} & 0 & 0 \\ V & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & J_3 \end{pmatrix}, \quad U|_{\mathfrak{H}_{\mathbb{T}}} = \begin{pmatrix} U_{00} & 0 & 0 & 0 \\ U_{10} & U_{11} & U_{12} & U_{13} \\ U_{20} & 0 & U_{22} & 0 \\ U_{30} & 0 & 0 & U_{33} \end{pmatrix}. \tag{17}$$

In Representation (17) V is an isometry, I_2 is the identity in $\mathfrak{G}_2$, J_3 is a fundamental symmetry in $\mathfrak{G}_3$ and the elements of the matrix representation for $U|_{\mathfrak{H}_{\mathbb{T}}}$ have the following relations

$$\begin{aligned} (U_{00})^{-1} &= \ V^{-1}(U_{11})^{*}V, \\ U_{10} &= -U_{11}V\left((U_{20})^{*}U_{20} + (U_{30})^{*}J_3U_{30} + iA\right), \\ U_{20} &= -U_{22}(U_{12})^{*}VU_{00}, \\ U_{30} &= -U_{33}J_3(U_{13})^{*}VU_{00}, \end{aligned} \tag{18}$$

where A is a selfadjoin operator. Moreover, the operators U_{22} and U_{33} are, respectively, unitary and J_3-unitary in the corresponding subspaces. Let us denote

$$\widetilde{U}: \ = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}, \quad \widetilde{U}^{\uparrow}: \ = \begin{pmatrix} U_{00} & 0 \\ U_{20} & U_{22} \end{pmatrix}. \tag{19}$$

Operators $\widetilde{U}$ and $\widetilde{U}^{\uparrow}$ act in the spaces $\widetilde{\mathfrak{H}} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ and $\widetilde{\mathfrak{H}}^{\uparrow}: \ = \mathfrak{G}_0 \oplus \mathfrak{G}_2$ respectively. Since U_{22} is a unitary operator, its model e^{iT}, i.e. the multiplication operator by e^{it} in a suitable function space, can be defined in a classical way, as it is done in the theory of multiplicity of self-adjoint and unitary operators, so let us pass to some notations relating to direct integrals of Hilbert spaces and model descriptions of corresponding operators (see [14], §41; [4], Chapter 7; [5], Chapter 4.4; [15], Chapter VII). Let $\mathfrak{E}$ be some separable Hilbert space ($\mathfrak{E}$ can be finite-dimensional), let $\sigma(t)$ be a non-decreasing scalar function defined on the segment $[0,2\pi]$, continuous from the left in all points of the segment and having an infinite number of growth points, $\sigma(0) = 0$ and let μ_σ be the corresponding Lebesgue-Stieltjes measure. Consider a mapping $t \mapsto \mathfrak{E}_t$, $t \in [0,2\pi]$, where $\mathfrak{E}_t \subset \mathfrak{E}$, $\dim(\mathfrak{E}_t)$ is a μ_σ-measurable (but not necessarily finite a.e.) function, if $\dim(\mathfrak{E}_{t_1}) = \dim(\mathfrak{E}_{t_2})$, then $\mathfrak{E}_{t_1} = \mathfrak{E}_{t_2}$ and if $\dim(\mathfrak{E}_{t_1}) < \dim(\mathfrak{E}_{t_2})$, then $\mathfrak{E}_{t_1} \subset \mathfrak{E}_{t_2}$, $\mu_\sigma(\{t: \ \dim \mathfrak{E}_t = 0\}) = 0$. Note that under these conditions there is a non-zero vector $d \in \mathfrak{E}$ such that

$$d \in \mathfrak{E}_t \text{ for almost all } t \in [0,2\pi]. \tag{20}$$

Denote by $\mathscr{M}_{\vec{\sigma}}(\mathfrak{E})$ the space of the vector-valued functions $f(t)\colon \ t \mapsto \mathfrak{E}_t$ μ_σ-measurable in the weak sense, defined a.e. and finite a.e. on the segment $[0,2\pi]$. Next, the symbol $L^2_{\vec{\sigma}}(\mathfrak{E})$ means here the Hilbert space of functions $f(t) \in \mathscr{M}_{\vec{\sigma}}(\mathfrak{E})$ such that $\int_0^{2\pi} \|f(t)\|^2_{\mathfrak{E}} d\sigma(t) < \infty$.

Let $\{\widetilde{g}_j(t)\}_{j=1}^k$ be a finite set of functions from $\mathscr{M}_{\vec{\sigma}}(\mathfrak{E})$ with Properties

$$\left.\begin{array}{l}\bullet\ \text{the system } \{\widetilde{g}_j(t)\}_{j=1}^k \text{ is linearly independent modulo } L^2_{\vec{\sigma}}(\mathfrak{E});\\ \bullet\ \text{for every } j=1,2,\dots,k \text{ the function } e^{it}\widetilde{g}_j(t) \text{ has the representation}\\ \quad e^{it}\widetilde{g}_j(t)=\eta_j(t)+\sum_{l=1}^k c_{lj}\widetilde{g}_l(t), \text{ where } \eta_j(t)\in L^2_{\vec{\sigma}}(\mathfrak{E});\\ \bullet\ \text{eigenvalues of the matrix } C=\left(c_{lj}\right)_{k\times k} \text{ are all located on the unit circle.}\end{array}\right\} \tag{21}$$

Denote $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ the Hilbert functional space generated by the linear span of $L^2_{\vec{\sigma}}(\mathfrak{E})$ and $\{\widetilde{g}_j(t)\}_{j=1}^k$, where functions in the set $\{\widetilde{g}_j(t)\}_{j=1}^k$ are supposed by definition to be normalized, pairwise orthogonal, and orthogonal to $L^2_{\vec{\sigma}}(\mathfrak{E})$.

THEOREM 1. *For a π-unitary operator U satisfying the condition (14) there are a Hilbert space $L^2_{\vec{\sigma}}(\mathfrak{E})$, a function set $\{\widetilde{g}_j(t)\}_{j=1}^k$, $k=\dim\mathfrak{H}_1$, satisfying Conditions (21), and an isometric operator $W:\ \widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})\mapsto\widetilde{\mathfrak{H}}$, $WL^2_{\vec{\sigma}}(\mathfrak{E})=\mathfrak{G}_2$ such that*

$$\widetilde{U}=W(e^{-iT})^*W^{-1},\qquad \widetilde{U}^{\uparrow}=W^{\uparrow}e^{iT}(W^{\uparrow})^{-1}, \tag{22}$$

where $W^{\uparrow}=(I_2\oplus V^{-1})W$, I_2 is the identical operator on $\mathfrak{G}_2$, V is an isometric operator mapping $\mathfrak{G}_0$ onto $\mathfrak{G}_1$: $Vx=Jx$ for every $x\in\mathfrak{G}_0$, $\sigma(t)$ is continuous for every t such that $e^{it}\in\sigma(U_{11}^{\mathbb{T}})\cup\sigma(U_{33}^{\mathbb{T}})$. The space $L^2_{\vec{\sigma}}(\mathfrak{E})$ and the system $\{\widetilde{g}_j(t)\}_{j=1}^k$ can be chosen such that for every $j=1,2,\dots,k$ and at almost all $t\in[-1,1]$ the condition $\widetilde{g}_j(t)\in\mathfrak{E}_1$ holds, where $\mathfrak{E}_1$ is some subspace of $\mathfrak{E}$ with the dimension no greater than k.

Proof. This theorem is in fact a direct corollary of Theorem 6.5 from [21], therefore here only a sketch of proof will be presented. The latter would be more transparent if we assume that the finite-dimensional operator U_{11} has one-point spectrum: $\sigma(U_{11})=\{-1\}$, so (see (5)) $\Lambda=\{\pi\}$. Next (see (16) and comments related with (19)), the operator U_{22} is unitary in the Hilbert space $\mathfrak{G}_2$ and due to this fact there is a Hilbert space $L^2_{\vec{\sigma}}(\mathfrak{E})$ and an isometric operator $W^{(2)}:\ W^{(2)}L^2_{\vec{\sigma}}(\mathfrak{E})=\mathfrak{G}_2$ such that $U_{22}=W^{(2)}e^{iT}(W^{(2)})^{-1}$. Now let us choose an orthonormal base $\{g_j\}_{j=1}^k$ in $\mathfrak{G}_1$ and put (see (17)) $\{h_j=V^{-1}g_j\}_{j=1}^k$. For every $\varepsilon\in(0,\pi)$ and $j=1,2,\dots,k$ the expression $[E([0,\pi-\varepsilon)\cup(\pi+\varepsilon],2\pi)x,h_j]$ represents a continuous linear functional on $\mathfrak{G}_2$, so there is the function $\widetilde{g}_j(t)$ (the same for every ε) such that

$$[E([0,\pi-\varepsilon)\cup(\pi+\varepsilon,2\pi))x,h_j]=\int_{[0,\pi-\varepsilon)\cup(\pi+\varepsilon,2\pi]}(f(t),\widetilde{g}_j(t))_{\mathfrak{E}}\,d\sigma(t)\,,$$

$$\int_{[0,\pi-\varepsilon)\cup(\pi+\varepsilon,2\pi]}||\widetilde{g}_j(t)||^2_{\mathfrak{E}}\,d\sigma(t)<\infty\,,$$

where $f(t) \in L^2_{\vec{\sigma}}(\mathfrak{E})$, $W^{(2)}f(t) = x$. Due to (14) and (15) the relation

$$\sum_{j=1}^{k} \alpha_j \widetilde{g}_j(t) \in L^2_{\vec{\sigma}}(\mathfrak{E})$$

brings $\alpha_j = 0,\ j = 1,2,\dots,k$. It is easy to check that $x =$

$$\lim_{\varepsilon\to+0}\{E([0,\pi-\varepsilon)\cup(\pi+\varepsilon,2\pi))x - \sum_{j=1}^{k} g_j \int_{[0,\pi-\varepsilon)\cup(\pi+\varepsilon,2\pi]} (f(t),\widetilde{g}_j(t))_{\mathfrak{E}}\, d\sigma(t)\} \quad (23)$$

for every $x \in \mathfrak{G}_2$, $W^{(2)}f(t) = x$. Let $U_{11}g_j = \sum_{m=1}^{k} \gamma_{mj} g_m$. Then (23) brings

$$[Ux, h_j] = \int_0^{2\pi} \left(f(t), (\widetilde{g}_j(t)e^{-it} - \sum_{l=1}^{k} \bar{\gamma}_{jl}(\widetilde{g}_l(t))\right)_{\mathfrak{E}} d\sigma(t).$$

$f(t) \in L^2_{\vec{\sigma}}(\mathfrak{E})$ is an arbitrary function, therefore $(\widetilde{g}_j(t)e^{-it} - \sum_{l=1}^{k} \bar{\gamma}_{jl}(\widetilde{g}_l(t)) \in L^2_{\vec{\sigma}}(\mathfrak{E})$. The rest is a simple calculation based on (18). $\square$

DEFINITION 2. The space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ from Theorem 1 is called a *basic model space* for the operator U and W is called an *operator of similarity* (generated by $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$).

It is clear that a basic model space and, generated by the latter, an operator of similarity are not uniquely determined (see [21], Subsection 6.2 for details).

3. Some function sets

First, we need to introduce some notations. Let us choose and fix a basic model space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ for U, where $\sigma(t)$ is the corresponding non decreasing continuous from the left function given on the segment $[0,2\pi]$. Then $\sigma(t)$ has the unique representation

$$\sigma(t) = \sigma_c(t) + \sigma_s(t), \quad (24)$$

where $\sigma_c(t)$ and $\sigma_s(t)$ are non-decreasing functions such that $\sigma_c(t)$ generates absolutely continuous and $\sigma_s(t)$ generates singular (including the atomic component) measures with respect to the standard Lebesgue measure.

Next, let $\{\widetilde{g}_j(t)\}_{j=1}^{k}$ be a set of unbounded elements generating the space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$. In concordance with the notations given in [20] (formula (4.4)) (compare also with [21], formula (6.79)) we define

$$\iota_{jq}(t) = (\widetilde{g}_j(t), \widetilde{g}_q(t))_{\mathfrak{E}}, \quad j,q = 1,2,\dots,k, \quad G(t) = 1 + \sum_{j=1}^{k} \iota_{jj}(t).$$

For simplicity we assume that

$$1 \notin \sigma(U_{11}) \cup \sigma(U_{33}). \quad (25)$$

Condition (25) means that the operator $E_{2\pi}$ is correctly defined and projects all the space on $\mathfrak{H}_{\mathbb{T}}$.

Next, let numbers $0=\alpha_0<\alpha_1<\alpha_2<\ldots<\alpha_m<\alpha_{m+1}=2\pi$ be such that $\{e^{i\alpha}j\}_{j=1}^m=\sigma(U_{11})\cup\sigma(U_{33})$, and let $\beta_j\in(\alpha_j,\alpha_{j+1})$ be some fixed numbers, $j=0,1,2,\ldots,m$. Then for $t\in(\alpha_j,\alpha_{j+1})$ we define $\nu(t)=\int_{\beta_j}^t G(t)d\sigma(t)$.

Similarly we introduce a function $\eta(t)=\int_0^t(1/G(t))d\sigma(t)$, where $t\in[0,2\pi]$.

The space L^2_ν is by definition the natural closure of the set of all continuous functions given on $[0,2\pi]$ and vanishing on some neighbourhood of the set $\{\alpha_j\}_{j=1}^m$, the space L^2_η is defined as usual.

Note that the spaces L^∞_σ and L^2_ν, as well as the spaces L^1_σ and L^2_η, form compatible pairs or, so called Banach pairs (for details see [3] or [11]). Therefore, the spaces $L^1_\sigma+L^2_\eta$ and $L^\infty_\sigma\cap L^2_\nu$ are well defined. In particular, the standard norm on $L^1_\sigma+L^2_\eta$ is given by the formula

$$\|f\|=\inf_{f_1+f_2=f}\{\|f_1\|_{L^1_\sigma}+\|f_2\|_{L^2_\eta}\}.$$

The space $L^\infty_\sigma\cap L^2_\nu$ can be considered as the adjoin one to the space $L^1_\sigma+L^2_\eta$ if the duality between these two spaces is given by the formula $\langle f(t),g(t)\rangle=\int_{-1}^1 f(t)\overline{g(t)}d\sigma(t)$, where $f(t)\in L^1_\sigma+L^2_\eta$ and $g(t)\in L^\infty_\sigma\cap L^2_\nu$.

Next, as a consequence of the definition of $\sigma_c(t)$, there is a Lebesgue integrable function $\omega(t)\geqslant 0$ defined a.e. in the segment $[-\pi;\pi]$ such that for every $t\in[-\pi;\pi]$

$$\sigma_c(t)=\int_0^t\omega(t)dt. \tag{26}$$

Below the symbol μX means the standard Lebesgue measure of a set X.

We will study the structure of $\mathrm{Alg}U$ in two steps. For the beginning we will consider the closure of operators $M(U)$, where $M(\xi)$ is a polynomial such that $M(U)$ is completely defined by the basic model space of the operator U. First, let us note that the spectrum of the operator U_{33} is a subset of the set Λ and therefore forms a finite set. Therefore $\mathfrak{G}_3$ is a Pontryagin space, it is a finite-dimensional space or can be presented as a π-orthogonal sum of two subspaces invariant with respect to U_{33} such that the first one is finite-dimensional and the other one is positive. Thus, U_{33} has the minimal polynomial $M_3(\xi)$: $M_3(U_{33})=0$. Next, let $M_0(\xi)$ $(\equiv M_1(\xi))$ be the minimal polynomial for the finite-dimensional operator U_{00} (or U_{11}), and $M_{\mathbb{T}'}(\xi)$ be the minimal polynomial for the operator $U|_{\mathfrak{H}_{\mathbb{T}'}}$. In this case for any polynomial $M(\xi)$ of the form

$$M(\xi)=M_0^2(\xi)M_3^2(\xi)M_{\mathbb{T}'}(\xi)P(\xi), \tag{27}$$

where $P(\xi)$ is an arbitrary polynomial, all matrix elements of $M(U)$ with respect to the decomposition

$$\mathfrak{H}=\mathfrak{G}_0\oplus\mathfrak{G}_1\oplus\mathfrak{G}_2\oplus\mathfrak{G}_3\oplus\mathfrak{H}_{\mathbb{T}'} \tag{28}$$

are equal to zero with exception, maybe, for ones corresponding to the mappings $\mathfrak{G}_0\to\mathfrak{G}_1$, $\mathfrak{G}_0\to\mathfrak{G}_2$, $\mathfrak{G}_2\to\mathfrak{G}_1$, $\mathfrak{G}_2\to\mathfrak{G}_2$. The exceptional elements $\mathfrak{G}_0\to\mathfrak{G}_2$, $\mathfrak{G}_2\to\mathfrak{G}_1$

and $\mathfrak{G}_2 \to \mathfrak{G}_2$ can be calculated via the basic model space of U, but this way does not work for $\mathfrak{G}_0 \to \mathfrak{G}_1$ (see Example 1). The latter problem can be resolved using some additional steps. Let W be an operator of similarity generated by $L^2_{\vec{\sigma}}(\mathfrak{E})$. Let us define (see (22)) $h_j = W^{\uparrow}\widetilde{g}_j(\xi)$, $e_j = W\widetilde{g}_j(\xi)$, $j = 1,2,\dots,k$. Due to the equality $M_0(U_{00}) = 0$, we obtain $M(e^{it})\widetilde{g}(t) \in L^2_{\vec{\sigma}}(\mathfrak{E})$, hence the integrals $\int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t)$, $\quad j,q = 1,2,\dots,k$ converge but, generally speaking,

$$(M(U)h_j, e_q) \neq \int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t), \quad j,q = 1,2,\dots,k.$$

Let us redefine (27)

$$M(\xi) = M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)P(\xi). \tag{29}$$

Then

$$(M(U)h_j, e_q) = \int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t), \quad j,q = 1,2,\dots,k. \tag{30}$$

Comparing (30) with (9), we have (note that here $E_{2\pi}\mathfrak{H} = \mathfrak{H}_{\mathbb{T}}$)

$$M(U)U^k = \mathrm{s}-\lim_{\delta\to+0} \int_{[0,2\pi]\setminus O_\delta} M(e^{it})e^{ikt}dE_t \ , \ k = 0, \pm 1, \pm 2, \dots \ , \tag{31}$$

with O_δ as in (7). Using the equalities like (30) let us introduce values $\varphi(U)$ for some class of functions $\varphi(\xi)$. Note that we do not make the assumption that the condition $\varphi(U) \in \mathrm{Alg}\, U$ is satisfied. Thus, let the function $\varphi(\xi)$ be such that the function $\varphi(e^{it})$ is measurable, defined a.e., uniformly bounded on $[0,2\pi]$ and

$$\int_0^{2\pi} |\varphi(e^{it})|G(t)d\sigma(t) < \infty. \tag{32}$$

Let us introduce the operator $\overset{\diamond}{\varphi}(U)$ as it is described (see (22) and (30)):

$$\left.\begin{array}{l}
\bullet\ P^{\uparrow}\,\overset{\diamond}{\varphi}(U)|_{\widetilde{\mathfrak{H}}^{\uparrow}} = W^{\uparrow}\Phi(W^{\uparrow})^{-1}\text{, where } P^{\uparrow} \text{ is the ortoprojection on } \widetilde{\mathfrak{H}}^{\uparrow}\text{,}\\
\quad\text{and } \Phi \text{ is the multiplication operator by function } \varphi(e^{it}) \text{ acting on } \widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E});\\
\bullet\ \text{the subspace } \widetilde{\mathfrak{H}} \text{ is invariant with respect to } \overset{\diamond}{\varphi}(U) \text{ and } \overset{\diamond}{\varphi}(U)|\widetilde{\mathfrak{H}} =\\
\quad W\bar{\Phi}^*(W)^{-1}\text{, where } \bar{\Phi} \text{ is the multiplication operator by } \bar{\varphi}(e^{it});\\
\bullet\ (\overset{\diamond}{\varphi}(U)h_j, e_q) = \int_0^{2\pi} \varphi(e^{it})\iota_{jq}(t)d\sigma(t),\ j,q = 1,2,\dots,k;\\
\bullet\ \text{all other elements of the matrix representation of the operator } \overset{\diamond}{\varphi}(U)\\
\quad\text{with respect to the decomposition (28) are equal to zero.}
\end{array}\right\} \tag{33}$$

Due to (32) the operator $\overset{\diamond}{\varphi}(U)$ is well defined by Conditions (33). The answer to the natural question concerning the relation between $\overset{\diamond}{\varphi}(U)$ and $\mathrm{Alg}\, U$ is directly connected (see below) with some properties of σ_c.

Condition (32) is satisfied if $\varphi(\xi)$ is such that

$$\varphi(\xi) = M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)\psi(\xi), \tag{34}$$

where $\psi(\xi)$ is a meromorphic on $\mathbb{D}$ and continuous on $\mathbb{D}\cup\mathbb{T}$ function. The next proposition follows directly from Theorem 1 and comparison with Formulae (30) and (33).

PROPOSITION 1. *If a function $\varphi(\xi)$ has Form (34), then the corresponding operator $\overset{\diamond}{\varphi}(U)$ belongs to the closure in norm topology of the set of elements $M(U)$, where polynomials $M(\xi)$ have Form (29).*

4. The case of $WJ^\star$-algebra

Let $\mathrm{Alg}_M U$ be the weak closure of the operator set $\{M(U)\}$, where $M(\xi)$ runs through the set of all polynomials of the type (29). Due to Proposition 1 the relation $\overset{\diamond}{\varphi}(U) \in \mathrm{Alg}_M U$ is fulfilled for every function $\varphi(\xi)$ that have Representation (34).

LEMMA 1. *If $\varphi(\xi) = M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)\xi^{-1}$ and*

$$\mu\{t : \omega(t) = 0\} > 0, \tag{35}$$

then $\overset{\diamond}{\varphi}(U) \in \mathrm{Alg}_M U$.

Proof. Let F_0 be the set of all polynomials of the type $Q(\xi) = \sum_{l=1}^p \alpha_l \xi^l$, where p is an arbitrary natural number. Under Condition (35) due to the theorem of Szegö (see, for instance, Garneet [1], Chap. IV, Theorem 3.1) the equality $\inf_{Q\in F_0}\int_{-\pi}^{\pi}|1-Q(e^{it})|^2|\psi(t)|d\sigma(t) = 0$ is true for every function $\psi(t)\in L^1_\sigma$, thus,

$$\inf_{Q\in F_0}\int_{-\pi}^{\pi}|1-Q(e^{it})||\psi(t)|d\sigma(t) = 0. \tag{36}$$

If $f(t) = W^{-1}x$, $x\in\mathfrak{G}_2$, then for every $j,q = 1,2,\dots,k$ and the function

$$\psi(t) = M_0^4(e^{it})M_3^4(e^{it})M_{\mathbb{T}'}(e^{it})\left\{(f(t),f(t))_{\mathfrak{E}} + |(\widetilde{g}_j(t),f(t))_{\mathfrak{E}}| + |\iota_{qj}(t)|\right\}$$

we have $\psi(t)\in L^1_\sigma$, hence for this function Condition (36) is true and, moreover, the same condition is fulfilled for every linear combination of functions of such type. The rest follows from Theorem 1, the structure of F_0 and the trivial equality $|e^{-it}| = 1$. □

THEOREM 2. *Let $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ be a basic model space for U. Then $U^{-1}\in \mathrm{Alg}\,U$ if and only if Condition (35) is fulfilled.*

Proof. Let Condition (35) be fulfilled. Due to the equalities

$$M_{\mathbb{T}'}(U)|_{\mathfrak{H}_{\mathbb{T}'}} = 0$$

and (11) Representation 9 is valid. Then $M_0^4(U)M_3^4(U)M_{\mathbb{T}'}(U)U^{-1} \in \mathrm{Alg}\,U$ (see (9) and Lemma 1). With no loss of generality, we can assume that $M_0^4(0)M_3^4(0)M_{\mathbb{T}'}(0) = 1$, so in this case the rational function $\xi^{-1} - M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)\xi^{-1}$ is in fact a polynomial. Thus, the conclusion $U^{-1} \in \mathrm{Alg}\,U$ is evident. Now let us assume that $U^{-1} \in \mathrm{Alg}\,U$ and simultaneously $\mu\{t : \ \omega(t) = 0\} = 0$. The latter condition means that the operator U_{22} contains bilateral right shift and the operator U_{22}^{-1} contains bilateral left shift. At the same time the left shift does not belong to the algebra generated by the right shift, because there are some subspaces invariant with respect to right sift and not invariant with respect to left shift. This is a contradiction. □

COROLLARY 1. *If for the operator* U *Condition* (35) *is fulfilled, then for every* $\xi \in \rho(U)$, $R_\xi(U) \in \mathrm{Alg}\,U$, *and* $E(\Delta) \in \mathrm{Alg}\,U$ *for every* $\Delta = [\alpha, \beta] \subset [0, 2\pi]$ *such that* $\alpha, \beta \notin \Lambda$.

COROLLARY 2. *If for the operator* U *Condition* (35) *is fulfilled, then* $\mathrm{Alg}\,U$ *is a* WJ^**-algebra.*

THEOREM 3. *If for the operator* U *Condition* (35) *is fulfilled, then there is a* π*-self-adjoin operator* A *such that* $\mathrm{Alg}\,U = \mathrm{Alg}\,A$.

Proof. If there is a point $\theta \in \mathbb{T}$, $\theta \in \rho(U)$, then the statement of the theorem is trivial because in this case one can apply the Möbius transform

$$A = c \cdot (U + \bar{\theta} I)(U - \theta I)^{-1} \tag{37}$$

with $c : \ c^2 = -\theta/\bar{\theta}$. If such a point does not exist, the corresponding case can be reduced in fact to the above one. Indeed, if $\alpha, \beta, \tau \in (0, 2\pi)$, $\alpha < \tau < \beta$ are some constant such that $\Delta = [\alpha, \beta] \cap \Lambda = \emptyset$, then the operator $A_1 = c \cdot (U + \bar{\theta} I)(U - \theta I)^{-1}(I - E(\Delta))$ with $\theta = e^{i\tau}$ and c as in (37) is well defined and π-s.a. The latter is true also for the operator $A_2 = \gamma \cdot (U + \bar{\zeta} I)(U - \zeta I)^{-1} E(\Delta)$, where the corresponding constants are defined by the conditions $\zeta = e^{it}$, $t \in (0, 2\pi) \backslash \Delta$, $\gamma^2 = -\zeta/\bar{\zeta}$. Now one can put

$$A = \frac{1}{\|A_1\|} \cdot A_1 + 2E(\Delta) + \frac{1}{\|A_2\|} \cdot A_2\,. \tag{38}$$

□

REMARK 1. The projection $E(\Delta)$ in (38) is such that the subspace $E(\Delta)\mathfrak{H}$ is a Hilbert space with the scalar product $[\cdot, \cdot]$, so the operators $U|_{E(\Delta)\mathfrak{H}}$ and $A_1|_{E(\Delta)\mathfrak{H}}$ can be treated as, respectively, unitary and s.a. ones (in the ordinary Hilbert sense).

Theorem 3 opens a possibility to use for a description of $\operatorname{Alg} U$ some results concerning so-called monogenic WJ^*-algebras [20], i.e. algebras generated by a single J-s.a. operator and the identity. Let us introduce some notions and notations from [20].

Every operator $B \in \operatorname{Alg} U$ can be associated with a scalar function $f_B(\lambda)$ (the *portrait* of B) such that

$$BE(\Delta) = \int_\Delta f_B(\lambda) dE_\lambda,$$

where Δ runs over the set of all closed subintervals of $[0, 2\pi]$ disjoint from Λ. If f_B is the portrait of B then B is called an *original* for f_B (the same function has different originals).

Now we proceed with descriptions of some operator subalgebras of $\operatorname{Alg} U$. Let $\varphi(t)$ be a continuous scalar function vanishing near Λ. Set

$$B_\varphi = \int_0^{2\pi} \varphi(t) dE_\lambda, \tag{39}$$

where the improper integral has the obvious meaning.

Let $\mathfrak{S}(U)$ be the collection of operators $\{S\}$ that can be presented (see Decomposition (28)) as

$$S = \begin{pmatrix} 0 & 0\,0\,0\,0 \\ S_{10} & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \end{pmatrix}, \tag{40}$$

where $S_{01} \colon \mathfrak{G}_0 \to \mathfrak{G}_1$ runs through all operators such that (compare with (30))

$$\text{if } \sum_{j,q=1}^{k} \alpha_{jq} \iota_{jq}(t) \in L^1_\sigma + L^2_\eta, \text{ then } \sum_{j,q=1}^{k} \alpha_{jq}(S_{10} h_j, e_q) = 0. \tag{41}$$

As in [20] we denote by $\mathfrak{A}_\Lambda$ the weak closure of the operator set $\{B_\varphi\}$ generated by (39). Let $\mathscr{G}_\psi(U)$ be the subset of the operators from $\mathfrak{A}_\Lambda$, which are originals for $\psi(t)$ with respect to E_λ.

PROPOSITION 2. *If Condition* (35) *is fulfilled, then* $\mathfrak{A}_\Lambda = \operatorname{Alg}_M U$.

Proof. Under Condition (35) we have $E(\Delta) \in \operatorname{Alg}_M U$ for every closed interval such that Λ and Δ are disjoint, therefore $\mathfrak{A}_\Lambda \subset \operatorname{Alg}_M(U)$. On the other hand due to Representation (31) $M_0^4(U) M_3^4(U) M_{\mathbb{T}'}(U) P(U) \in \mathfrak{A}_\Lambda$. The rest is straightforward. $\square$

Proposition 2 and Theorem 5.1 from [20] bring the following result:

THEOREM 4. *If Condition (35) is fulfilled, an operator B belongs to* $\mathrm{Alg}_M U$ *if and only if the following conditions hold:*

$$\left.\begin{array}{l}\bullet\ BE_\lambda = E_\lambda B \text{ for every } \lambda \in [-1;1]\backslash\Lambda;\\[2mm] \bullet\ B|_{\widetilde{\mathfrak{H}}^{[\perp]}} = 0,\ B\mathfrak{H} \subset \widetilde{\mathfrak{H}};\\[2mm] \bullet\ \text{there is a function } \varphi(t) \in L^\infty_\sigma \cap L^2_\nu \text{ such that } \varphi(t) \text{ is the portrait of}\\ \quad B \text{ with respect to } E_\lambda;\\[2mm] \bullet\ \text{if } \mathfrak{H}_1 \neq \{0\} \text{ and } \psi(t) = \sum_{l,j=1}^k \alpha_{lj}\iota_{lj}(t) \in L^1_\sigma + L^2_\eta, \text{ then}\\[2mm] \qquad\qquad \displaystyle\sum_{l,j=1}^k \alpha_{lj}(Bh_l, e_j) = \int_{-1}^1 \varphi(t)\psi(t)d\sigma(t).\end{array}\right\} \tag{42}$$

The next theorem (it follows directly from Proposition 2 and [20], Corollary 5.2) describes the set of originals that correspond to the same function.

THEOREM 5. *If U is such that (35) is fulfilled, then $\mathscr{G}_\varphi(U) \cap \mathrm{Alg}_M U \neq \emptyset$ if and only if $\varphi(t) \in L^\infty_\sigma \cap L^2_\nu$; if B_0 is a fixed operator from $\mathscr{G}_\varphi(U) \cap \mathrm{Alg}_M U$, then $\mathscr{G}_\varphi(U) \cap$* $\mathrm{Alg}_M U = \{B_0 + S\}_{S \in \mathfrak{S}(U)}$.

THEOREM 6. *Let $B \in \mathrm{Alg}\,U$. Then $B = Q(U) + F$, where $Q(\xi)$ is a polynomial and $F \in \mathrm{Alg}_M U$.*

Proof. If $B \in \mathrm{Alg}\,A = \mathfrak{A}$, $A \in D^+_\kappa$, then, as it was shown in [20] (Theorem 4.23), $B = Q(A) + F$, where $Q(\xi)$ is a polynomial and $F \in \mathfrak{A}_\Lambda$. Due to Proposition 2, the same representation is valid for $B \in \mathrm{Alg}\,U$ and A given in (38). Thus, we need to prove that $Q(A)$ can be substituted by $\check{Q}(U)$, where $\check{Q}(\xi)$ also is a polynomial, and, simultaneously, F must be substituted by an operator of the same class. Due to the structure of A in (38) the algebra $\mathrm{Alg}\,U$ is the direct sum of $\mathrm{Alg}(A_1|_{I-E(\Delta)\mathfrak{H}})$ and $\mathrm{Alg}(A_2|_{E(\Delta)\mathfrak{H}})$. The second summand can be treated as $W^\star$-algebra, hence we consider only the first one. Thus,

$$B|_{(I-E(\Delta))\mathfrak{H}} = Q(A_1|_{(I-E(\Delta))\mathfrak{H}}) + F|_{(I-E(\Delta))\mathfrak{H}} =$$
$$Q(c\cdot(U+\bar{\theta}I)(U-\theta I)^{-1})|_{(I-E(\Delta))\mathfrak{H}} + F|_{(I-E(\Delta))\mathfrak{H}}\,.$$

$Q(c\cdot(\xi+\bar{\theta})(\xi-\theta)^{-1})$ is a rational function and its unique pole is θ, but $\theta \in \rho(U|_{(I-E(\Delta))\mathfrak{H}})$. Let r_2 be a positive number such that $\sigma(U|_{(I-E(\Delta))\mathfrak{H}}) \cap \{\xi : |\xi-\theta| \leqslant r_2\} = \emptyset$ and let $r_1 \in (0, r_2)$. Let us define a function $\psi(\xi)$ that is smooth in whole complex plane, $\psi(\xi) = 0$ on $\{\xi : |\xi-\theta| \leqslant r_1\}$ and $\psi(\xi) = Q(c\cdot(\xi+\bar{\theta})(\xi-\theta)^{-1})$ on $\{\xi : |\xi-\theta| \geqslant r_2\}$. Next, let $L(\xi)$ be an interpolation polynomial for $\psi(\xi)$ with the interpolation set $\Lambda \cup \sigma(U|_{\mathbb{T}'})$ and a set of multiplicities such that $(Q(c\cdot(U+\bar{\theta}I)(U-\theta I)^{-1}) - L(U))|_{\mathfrak{H}_{\mathbb{T}'}} = 0$ and for the function $\varphi(\xi) = \psi(\xi) - L(\xi)$ Condition (32) is fulfilled. Now (see (33)) $B =$

$$B(I-E(\Delta)) + BE(\Delta) = \big(Q(A_1)(I-E(\Delta)) + F(I-E(\Delta))\big) + BE(\Delta) =$$

$$\big(\overset{\diamond}{\varphi}(U)+L(U)\big)(I-E(\Delta))+F(I-E(\Delta))+BE(\Delta)=$$

$$L(U)+\big(\overset{\diamond}{\varphi}(U)(I-E(\Delta))-L(U)E(\Delta)+F(I-E(\Delta))+BE(\Delta)\big).$$

The rest is straightforward. □

5. General case

Up to this point it was assumed that the subspace $\widetilde{\mathfrak{H}}$ had a non-trivial isotropic subspace $\mathfrak{G}_1$ (see (12), (14) and (15)). This assumption is equivalent to the assumption of unboundedness of the spectral function E_λ. A critical point λ_0 is called *singular* if E_λ is unbounded in every neighbourhood of λ_0, so it is possible to characterize $\mathfrak{G}_1$ in terms of singular critical points. Now we describe how to introduce function and operator sets used above if there are no singular critical points in Λ including the case $\Lambda=\emptyset$. In particular, in the case $\widetilde{\mathfrak{H}}=\{0\}$ a basic model space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ for U is reduced to an ordinary Hilbert space $L^2_{\vec{\sigma}}(\mathfrak{E})$, thus Condition (35) that played a key role starting from Lemma 1, has a sense for the case $\widetilde{\mathfrak{H}}=\{0\}$ too. The results from Section 4 can be easily reformulated for the latter case taking into account some simplifications such that: $\nu(t)\equiv\eta(t)\equiv\sigma(t)$, so $L^\infty_\sigma\cap L^2_\nu=L^\infty_\sigma$ and $L^1_\sigma+L^2_\eta=L^1_\sigma$, moreover $\mathfrak{S}(E_\lambda)=\{0\}$. Note that Formula (39) can be applied independently to the characterization in question of Λ.

Condition (35) can be substituted by another condition not related directly with the notion of basic model space.

THEOREM 7. *Let U be an arbitrary π-unitary operator. Then $U^{-1}\in \operatorname{Alg}U$ if and only if there is no a positive vector x such that the system $\{U^m x\}_{m=0}^\infty$ is π-orthogonal.*

Proof. Let a positive vector x be such that the system $\{U^m x\}_{m=0}^\infty$ is π-orthogonal. Put $\mathfrak{L}=\operatorname{CLin}\{U^m x\}_{m=0}^\infty$. The subspace $\mathfrak{L}$ is invariant for U, so it is invariant for every $B\in\operatorname{Alg}U$. Under the same hypothesis the system $\{U^m x\}_{m=-1}^\infty$ is also π-orthogonal, therefore the positive vector $U^{-1}x$ is π-orthogonal to the non-negative subspace $\mathfrak{L}$. Thus, $U^{-1}x\notin\mathfrak{L}$, so $U^{-1}\notin\operatorname{Alg}U$. Now let us consider the opposite case, i.e. let a vector x with the above property does not exist. Consider $\mathfrak{G}_1$. If $\mathfrak{G}_1\neq\{0\}$, then for U there are a basic model space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ and a corresponding operator of similarity W. Our aim is to prove that in this case Condition (35) is fulfilled. Indeed, if $\mu\{t:\,\omega(t)=0\}=0$, then one can take the function

$$f_0(t)=\begin{cases}\frac{1}{\sqrt{\omega(t)}},\ \text{if } \omega(t)\neq 0,\\ 0,\ \text{if } t\in[0,2\pi]\backslash\{t:\,\omega(t)\neq 0\}\,.\end{cases} \tag{43}$$

and put $x=Wf_0(t)d$, where $d\in\mathfrak{E}$ is as in (20). A simple verification shows that the system $\{U^m x\}_{m=0}^\infty$ is π-orthogonal. The latter contradicts to our hypothesis and the rest follows from Theorem 2. If $\mathfrak{G}_1=\{0\}$, then the same reasoning can be applied changing the basic model space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ to $L^2_{\vec{\sigma}}(\mathfrak{E})$. □

Closing remarks

Theorems 2, 3 and 7 are presented here for the first time. A theorem like Theorem 6 can also be proved if Condition (35) is not fulfilled but in this case the operator classes $\mathrm{Alg}_M U$ and $\mathfrak{S}(U)$ will have different structures. These problems are to be studied in a next publication.

Acknowledgement. The author is very grateful to Professors M. Chugunova, P. Martin and I. Pesenson for their assistance in making the paper more readable. This paper was written while the author was visiting the University of Antofagasta, Chile. He would like to thank Professors A.Restuccia and A.Sotomayor for their kind invitation and the University for its hospitality and financial support.

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(Received January 19, 2017)

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