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THE NORM OF BACKWARD DIFFERENCE OPERATOR $\Delta^{(n)}$ ON CERTAIN SEQUENCE SPACES

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Abstract. Let $p \geqslant 1$ and n be a non-negative integer and $A = (a_{m,k})_{m,k\geqslant 0}$ be a non-negative matrix. In this paper the norm of backward difference operators $\Delta^{(n)}$ and $\Delta^{(-n)}$ from the sequence space l_p into the certain sequence space A_p are computed, where A_p is the space of all real sequences $x = (x_k)_{k=0}^{\infty}$ such that

$$\sum_{m=0}^{\infty}\left|\sum_{k=0}^{\infty} a_{m,k}x_k\right|^p < \infty.$$

Moreover, the results are applied for well known matrices such as Cesàro matrix of order n and Hilbert and also new matrices which are introduced in this study.

1. Introduction

Let $p \geqslant 1$ and ω denote the set of all real-valued sequences. Any vector subspace of ω is called a sequence space. The classical space l_p is the set of all real sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that

$$\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} < \infty.$$

Let $A = (a_{m,k})_{m,k\geqslant 0}$ be a matrix. We define the matrix domain A_p by

$$\begin{aligned} A_p &= \left\{ x = (x_k) : Ax \in l_p \right\} \\ &= \left\{ x = (x_k) : \sum_{m=0}^{\infty}\left|\sum_{k=0}^{\infty} a_{m,k}x_k\right|^p < \infty \right\}, \end{aligned} \tag{1.1}$$

which is a sequence space. The new sequence space A_p generated by the limitation matrix A from a sequence space l_p can be the expansion or the contraction and or the overlap of the original space l_p [3].

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The matrix domain which plays an important role to construct a new sequence space of classical space l_p, has been studied by several authors. For instance, the matrix domains of the difference operator are investigated in [1, 3, 14, 15, 16] and the matrix domains of fractional difference operator are introduced in [2, 10, 11, 12, 13]. In these works topological properties, inclusion relations, duals and matrix transformations of these spaces are investigated, but the norm of matrix operators on these matrix domains are not studied.

Although the norm of matrix operators on the sequence space l_p have computed by many mathematicians such as Hardy, Bennett and Borwein [4, 6, 7, 8, 9], the problem of finding the norm of operators on matrix domains has not studied extensively. The authors recently computed norm of operators on some matrix domains [17]. In this present paper, we try to solve this problem for backward difference operator from l_p into A_p.

The semi-norm on the matrix domain A_p, $\|.\|_{A_p}$, is defined by

$$\|x\|_{A_p} = \left(\sum_{m=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{m,k} x_k \right|^p \right)^{\frac{1}{p}}.$$

Note that this function will be not a norm, since if $x = (1, -1, 0, 0, \cdots)$ and the matrix A is defined such that $a_{0,0} = a_{0,1} = 1$ and the remaining entries be zero, then $\|x\|_{A_p} = 0$ while $x \neq 0$. Consider that $A_p = l_p$ and $\|.\|_{A_p} = \|.\|_p$, for $A = I$.

Throughout this paper, we suppose that n is an arbitrary non-negative integer and $\binom{-1}{0} = 1$ and $\binom{n}{k} = 0$ for $k > n \geqslant 0$. The backward difference operators $\Delta^{(n)} = (\delta_{k,j}^{(n)})$ of order n and $\Delta^{(-n)} = (\delta_{k,j}^{(-n)})$ of order $-n$ are defined as below, respectively,

$$\delta_{k,j}^{(n)} = \begin{cases} (-1)^{k-j}\binom{n}{k-j} & j \leqslant k \leqslant n+j, \\ 0 & otherwise, \end{cases}$$

and

$$\delta_{k,j}^{(-n)} = \begin{cases} \binom{n+k-j-1}{k-j} & 0 \leqslant j \leqslant k, \\ 0 & otherwise. \end{cases}$$

Note that $\Delta^{(n)} = \Delta^{(-n)} = I$, when $n = 0$ and I is the identity matrix.

In this paper, we consider the inequalities of the forms

$$\|\Delta^{(n)} x\|_{A_p} \leqslant U\|x\|_p, \qquad \|\Delta^{(-n)} x\|_{A_p} \leqslant V\|x\|_p,$$

for all sequence $x \in l_p$. The constants U and V are not depending on x, and the norms of $\Delta^{(n)}$ and $\Delta^{(-n)}$ are the smallest possible values of U and V, respectively. Note that in the above inequalities, we choose the matrix domains A_p which satisfy boundedness of the operators $\Delta^{(n)}$ and $\Delta^{(-n)}$.

We use the notation $\|.\|_{A_p}$ for the norm of operators from l_p into A_p, and $\|.\|_p$ for the norm of operators from l_p into itself.

In this study, we focus on computing the norm of operator $\Delta^{(n)}$ from l_p into A_p, for $p = 1$ in Section 2 and for $p > 1$ in Section 3. Moreover the norm of operator $\Delta^{(-n)}$ from l_p into A_p is considered in Section 4.

2. The norm of operator $\Delta^{(n)}$ from l_1 into A_1

In this section, we try to solve the problem of finding norm of operator $\Delta^{(n)}$ from l_1 into A_1, where A are Cesàro, Hilbert, identity and backward difference matrices. We may begin with the following theorem which is essential in the study.

THEOREM 2.1. *Let $A=(a_{k,i})_{k,i\geqslant 0}$ be a matrix and $B=(b_{k,i})_{k,i\geqslant 0}$ be a lower triangular matrix. If $M=\sup_j u_j<\infty$ where*

$$u_j=\sum_{k=0}^{\infty}\left|(AB)_{k,j}\right|,$$

for $j=0,1,\cdots$, then B is a bounded operator from l_1 into A_1 and

$$\|B\|_{A_1}=M.$$

Proof. Let x be a sequence in l_1. We have

$$\begin{aligned}\|Bx\|_{A_1}&=\sum_{k=0}^{\infty}\left|\sum_{j=0}^{\infty}\sum_{i=0}^{j}a_{k,j}b_{j,i}x_i\right|=\sum_{k=0}^{\infty}\left|\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}a_{k,i}b_{i,j}x_j\right|\\&=\sum_{k=0}^{\infty}\left|\sum_{j=0}^{\infty}(AB)_{k,j}x_j\right|\leqslant\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|(AB)_{k,j}\right||x_j|\\&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|(AB)_{k,j}\right||x_j|=\sum_{j=0}^{\infty}u_j|x_j|\leqslant M\|x\|_1,\end{aligned}$$

which says that $\|B\|_{A_1}\leqslant M$. Let m be a non-negative integer. We take $x=e_m$ which e_m denotes the sequence having 1 in place m and 0 elsewhere, then $\|x\|_1=1$ and $\|Bx\|_{A_1}=u_m$. Hence

$$u_m=\frac{\|Bx\|_{A_1}}{\|x\|_1}\leqslant\|B\|_{A_1},$$

and $M=\sup_m u_m\leqslant\|B\|_{A_1}$. Therefore we have the desired result. □

In the following, we investigate the norms of Cesàro matrix of order 1, $C^1=(c_{k,j})$, which is defined by

$$c_{k,j}=\begin{cases}\frac{1}{k+1} & 0\leqslant j\leqslant k\\ 0 & j>k.\end{cases}$$

To do this, the following two lemmas are needed.

LEMMA 2.2. *If $n\in\mathbb{N}$, then*

$$\sum_{j=0}^{m}(-1)^j\binom{n}{j}=\begin{cases}(-1)^m\binom{n-1}{m} & m<n\\ 0 & m=n.\end{cases}$$

Proof. Use the identity $\binom{n}{j}=\binom{n-1}{j-1}+\binom{n-1}{j}$ for $j \geqslant 1$, for $m<n$ and note that the left hand side of the equality is the summation of the coefficients of binomial $(1-z)^n$ for $m=n$. $\square$

LEMMA 2.3. *If $n \in \mathbb{N}$, then*

$$\binom{n-1}{0}+\frac{1}{2}\binom{n-1}{1}+\cdots+\frac{1}{n}\binom{n-1}{n-1}=\frac{2^n-1}{n}.$$

Proof. By integrating from 0 to 1 of both sides of the identity

$$(1+z)^{n-1}=\sum_{j=0}^{n-1}\binom{n-1}{j}z^j,$$

the proof is obvious. $\square$

THEOREM 2.4. *Let C^1 be the Cesàro matrix of order 1. Then $\Delta^{(n)}$ is a bounded operator from l_1 into C_1^1 and*

$$\|\Delta^{(n)}\|_{C_1^1}=\frac{2^n-1}{n}.$$

Proof. According to above theorem and Lemma 2.2, we deduce that

$$\begin{aligned} u_j &= \sum_{k=j}^{\infty}\left|\left(C^1\Delta^{(n)}\right)_{k,j}\right| = \sum_{k=j}^{n+j}\left|\sum_{i=j}^{k}c_{k,i}\delta_{i,j}^{(n)}\right| \\ &= \sum_{k=j}^{n+j}\left|\sum_{i=j}^{k}\frac{(-1)^{i-j}}{k+1}\binom{n}{i-j}\right| = \sum_{k=j}^{n+j}\frac{1}{k+1}\left|\sum_{i=0}^{k-j}(-1)^i\binom{n}{i}\right| \\ &= \frac{1}{j+1}\binom{n-1}{0}+\frac{1}{j+2}\binom{n-1}{1}+\cdots+\frac{1}{j+n}\binom{n-1}{n-1}. \end{aligned}$$

So by Lemma 2.3

$$\|\Delta^{(n)}\|_{C_1^1}=\sup_j u_j = u_0 = \frac{2^n-1}{n}. \quad \square$$

Consider the Hilbet matrix $H=(h_{j,k})$ whose entries are $h_{j,k}=\frac{1}{j+k+1}$ for all $j,k \geqslant 0$. For the next theorem, we need the definition of β function and the following lemma

$$\beta(m,n)=\int_0^1 z^{m-1}(1-z)^{n-1}dz,$$

where $m,n \in \mathbb{N}$.

LEMMA 2.5. *For* $n \in \mathbb{N}$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{j+m}=\int_0^1 z^{m-1}(1-z)^n dz=\beta(m,n+1).$$

Proof. By using identity

$$(1-z)^n=\sum_{j=0}^{n}(-1)^j\binom{n}{j}z^j,$$

and multiplying both sides in term z^{m-1} also integrating from 0 to 1, we get the result. □

THEOREM 2.6. *If H is the Hilbert matrix, then $\Delta^{(n)}$ is a bounded operator from l_1 into H_1 and*

$$\|\Delta^{(n)}\|_{H_1}=\frac{1}{n}.$$

Proof. By the notation of Theorem 2.1 and Lemma 2.5

$$\begin{aligned}
u_j &= \sum_{k=0}^{\infty}\left|\left(H\Delta^{(n)}\right)_{k,j}\right| = \sum_{k=0}^{\infty}\left|\sum_{i=j}^{n+j} h_{k,i}\delta_{i,j}^{(n)}\right| \\
&= \sum_{k=0}^{\infty}\sum_{i=j}^{n+j}(-1)^{i-j}\binom{n}{i-j}\frac{1}{k+i+1} = \sum_{k=0}^{\infty}\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\frac{1}{k+i+j+1} \\
&= \sum_{k=0}^{\infty}\int_0^1 z^{k+j}(1-z)^n dz = \int_0^1\sum_{k=0}^{\infty} z^{k+j}(1-z)^n dz \\
&= \int_0^1 \frac{z^j}{1-z}(1-z)^n dz = \beta(j+1,n),
\end{aligned}$$

hence $u_j=\beta(j+1,n)$. Since the function $\beta(j,n)$ is decreasing with respect to j for all n, so

$$\|\Delta^{(n)}\|_{H_1}=\sup_j u_j = u_0=\beta(1,n)=\frac{1}{n}. \quad \square$$

THEOREM 2.7. *The backward difference operator $\Delta^{(n)}$ is a bounded operator from l_1 into l_1 and*

$$\|\Delta^{(n)}\|_1 = 2^n.$$

Proof. According to notations of Theorem 2.1 for identity matrix, we obtain

$$u_j=\sum_{k=j}^{n+j}\left|(-1)^{k-j}\binom{n}{k-j}\right|=\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

So

$$\|\Delta^{(n)}\|_1 = \sup_j u_j = 2^n. \quad \square$$

THEOREM 2.8. *The difference operator* $\Delta^{(n)}$ *is a bounded operator from* l_1 *into* $\Delta_1^{(m)}$ *and*

$$\|\Delta^{(n)}\|_{\Delta_1^{(m)}} = 2^{n+m}.$$

Proof. It is easy, so we omit the proof. $\square$

3. Upper bounds of the operator $\Delta^{(n)}$ from l_p into A_p

The purpose of this section is to find the norm of operator $\Delta^{(n)}$ from l_p space into C_p^n and H_p spaces. To do this, we need the Schur's Theorem and a lemma which are essential in the study.

THEOREM 3.1. ([9], Theorem 275) *Let* $p > 1$ *and* $T = (t_{m,k})$ *be a matrix operator with* $t_{m,k} \geqslant 0$ *for all* m, k. *Suppose that* K, R *are two strictly positive numbers such that*

$$\sum_{m=0}^{\infty} t_{m,k} \leqslant K \quad for\ all\ k, \qquad \sum_{k=0}^{\infty} t_{m,k} \leqslant R \quad for\ all\ m,$$

(bounds for column and row sums respectively). Then

$$\|T\|_p \leqslant R^{1/p^*} K^{1/p},$$

where p^* *is the conjugate of* p *i.e.* $\frac{1}{p} + \frac{1}{p^*} = 1$.

The above theorem is known as Schur's theorem.

LEMMA 3.2. *Let* $p \geqslant 1$ *and* $A = (a_{m,k})$ *be a matrix and* $T = (t_{m,k})$ *be a lower triangular matrix. If* AT *is a bounded operator on* l_p, *then* T *will be a bounded operator from* l_p *into* A_p *and*

$$\|T\|_{A_p} = \|AT\|_p.$$

Proof. For every $x \in l_p$,

$$\begin{aligned} \|Tx\|_{A_p}^p &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=0}^{j} a_{k,j} t_{j,i} x_i \right|^p = \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} a_{k,i} t_{i,j} x_j \right|^p \\ &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} (AT)_{k,j} x_j \right|^p = \|ATx\|_p^p, \end{aligned}$$

so the proof is finished. $\square$

Let the sequences $a^n = (a^n_j)_{j=0}^{\infty}$ be defined by

$$a^n_j = \binom{n+j-1}{j}, \tag{3.1}$$

these sequences play an essential role in this study, hence we bring the first four of these sequences in below:

$$\begin{aligned} a^0 &: 1\ 0\ 0\ \ 0\ \cdots, \\ a^1 &: 1\ 1\ 1\ \ 1\ \cdots, \\ a^2 &: 1\ 2\ 3\ \ 4\ \cdots, \\ a^3 &: 1\ 3\ 6\ 10\ \cdots. \end{aligned}$$

One can note that the relation $a^{n+1}_j = \sum_{k=0}^{j} a^n_k$ is hold for these sequences which is stated in the following lemma.

LEMMA 3.3. *We have*

$$\sum_{k=0}^{j} \binom{n+j-k-1}{j-k} = \sum_{k=0}^{j} \binom{n+k-1}{k} = \binom{n+j}{j}.$$

Proof. The proof is obvious. $\square$

Also the next useful lemma shows that the above sequences a^n are the coefficients of the binomial $(1-z)^{-n}$.

LEMMA 3.4. *For* $|z| < 1$, *we have*

$$(1-z)^{-n} = \sum_{j=0}^{\infty} a^n_j z^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j.$$

Proof. By differentiating $n-1$ times of the identity $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get the result. $\square$

If (a_k) is a non-negative sequence with $a_0 > 0$ and $A_j = a_0 + a_1 + \cdots + a_j$, the Nörlund matrix $N_a = (a_{j,k})$ is defined as follows:

$$a_{j,k} = \begin{cases} \frac{a_{j-k}}{A_j} & 0 \leqslant k \leqslant j, \\ 0 & otherwise. \end{cases} \tag{3.2}$$

The Cesàro matrix of order n, $C^n = (c^n_{j,k})$, is the Nörlund matrix N_{a^n} with the sequence a^n as in (3.1). So

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leqslant k \leqslant j \\ 0 & otherwise. \end{cases} \tag{3.3}$$

Note that C^0 and C^1 are the well known identity and Cesàro matrices, respectively.

Hardy in [8] has proved that C^n is a bounded operator on l_p and

$$\|C^n\|_p = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}, \tag{3.4}$$

where $p>1$. In particular for $n=1$, $\|C^1\|_p = p^*$.

The sequence space associated with C^n, according to relation (1.1), will be called C_p^n. So

$$C_p^n = \left\{ x=(x_k) : \sum_{k=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{j=0}^{k} \binom{n+j-k-1}{j-k} x_j \right|^p < \infty \right\}.$$

For proving our main theorem in this section, we need the following combinatoric lemma.

LEMMA 3.5. *For $j=1,2,\cdots$, we have*

$$\sum_{k=0}^{j} (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} = 0.$$

Proof. By using Lemma 3.4 and the following identity

$$\begin{aligned}
1 &= (1-z)^n(1-z)^{-n} \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^j \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} z^j \\
&= 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{j} (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} z^j,
\end{aligned}$$

the result is obvious. □

THEOREM 3.6. *Suppose that $p>1$ and C^n is the Cesàro matrix of order n. Then $\Delta^{(n)}$ is a bounded operator from l_p into C_p^n and*

$$\|\Delta^{(n)}\|_{C_p^n} = 1.$$

Proof. If $B = C^n\Delta^{(n)}$, we have

$$\begin{aligned}
b_{t,s} &= (C^n\Delta^{(n)})_{t,s} = \sum_{j=s}^{t} C^n_{t,j}\Delta^{(n)}_{j,s} \\
&= \frac{1}{\binom{t+n}{t}} \sum_{j=s}^{t} (-1)^{j-s} \binom{n}{j-s} \binom{n+t-j-1}{t-j} \\
&= \frac{1}{\binom{t+n}{t}} \sum_{j=0}^{t-s} (-1)^{j} \binom{n}{j} \binom{n+t-j-s-1}{t-j-s}.
\end{aligned}$$

Now if $t=s$, then $t-s=0$ hence $j=0$ and $b_{s,t}=\frac{1}{\binom{t+n}{t}}$. Also if $t>s$, then $u=t-s\geqslant 1$ hence by using Lemma 3.5

$$b_{s,t}=\frac{1}{\binom{t+n}{t}}\sum_{j=0}^{u}(-1)^j\binom{n}{j}\binom{n+u-j-1}{u-j}=0.$$

So

$$b_{s,t}=\begin{cases}\frac{1}{\binom{t+n}{t}} & s=t\\ 0 & s\neq t,\end{cases}$$

and by applying Lemma 3.2, we have $\|\Delta^{(n)}\|_{C_p^n}=\|B\|_p$. Since by Theorem 3.1 for B, $R\leqslant 1$ and $C\leqslant 1$, we obtain $\|\Delta^{(n)}\|_{C_p^n}\leqslant 1$. Now let $x=e_1$, we have $\|e_1\|_p=1$ and $\|\Delta^{(n)}e_1\|_{C_p^n}=1$, so $\|\Delta^{(n)}\|_{C_p^n}=1$. □

4. Upper bounds of the operator $\Delta^{(-n)}$ from l_p into A_p

In this section, we introduce four type of sequence spaces H_p^n, E_p^n, $D_p^{[m,n]}$ and B_p^n and will find the norm of operator $\Delta^{(-n)}$ from l_p into these spaces.

THEOREM 4.1. *([9], Theorem 323) Let $p>1$ and H be the Hilbert matrix. Then H is a bounded operator on l_p and*

$$\|H\|_p=\pi\csc(\pi/p).$$

THEOREM 4.2. *Suppose that $p>1$ and the matix H^n is defined by*

$$h_{j,k}^n=\frac{n!}{\prod_{i=0}^{n}(j+k+1+i)},\qquad (for\ j,k=0,1,\cdots). \tag{4.1}$$

Then $\Delta^{(-n)}$ is a bounded operator from l_p into H_p^n and

$$\|\Delta^{(-n)}\|_{H_p^n}=\pi\csc(\pi/p).$$

Proof. According to Lemma 3.2 and Theorem 4.1, it is sufficient to prove $H^n\Delta^{(-n)}=H$. By Lemma 3.4, we have

$$\begin{aligned}(H^n\Delta^{(-n)})_{k,m}&=\sum_{j=m}^{\infty}h_{k,j}^n\delta_{j,m}^{(-n)}\\&=\sum_{j=m}^{\infty}\binom{n+j-m-1}{j-m}\frac{n!}{(j+k+1)(j+k+2)\cdots(j+k+n+1)}\\&=\sum_{j=0}^{\infty}\binom{n+j-1}{j}\frac{(j+k+m)!n!}{(j+m+k+n+1)!}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \binom{n+j-1}{j} \beta(j+m+k+1, n+1) \\
&= \sum_{j=0}^{\infty} \binom{n+j-1}{j} \int_0^1 z^{j+m+k} (1-z)^n dz \\
&= \int_0^1 \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j z^{m+k} (1-z)^n dz \\
&= \int_0^1 (1-z)^{-n} z^{m+k} (1-z)^n dz \\
&= \frac{1}{k+m+1} = h_{k,m}. \quad \square
\end{aligned}
$$

Note that for $n = 0$, we have $H^n = H$.

In the following, we define matrix $E^n = (e^n_{j,k})$ by

$$
e^n_{j,k} = \begin{cases} \frac{(-1)^{j-k}}{j+1} \binom{n-1}{j-k} & k \leqslant j \leqslant k+n-1, \\ 0 & otherwise. \end{cases} \tag{4.2}
$$

For computing the norm of operator $\Delta^{(-n)}$ from l_p into sequence space E^n_p, we need the following lemma.

LEMMA 4.3. *For $j = 0, 1, 2, \cdots$ and $n \in \mathbb{N}$, we have*

$$
\sum_{k=0}^{j} (-1)^{j-k} \binom{n-1}{j-k} \binom{n+k-1}{k} = 1.
$$

Proof. Let $|z| < 1$. By using identity

$$
\begin{aligned}
\sum_{j=0}^{\infty} z^j &= (1-z)^{-1} = (1-z)^{-n}(1-z)^{n-1} \\
&= \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} z^j \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^{j-k} \binom{n-1}{j-k} \binom{n+k-1}{k} z^j,
\end{aligned}
$$

the proof is trivial. $\square$

THEOREM 4.4. *Let the matrix E^n be defined as in (4.2). Then $\Delta^{(-n)}$ is a bounded operator from l_p into E^n_p and $\|\Delta^{(-n)}\|_{E^n_p} = p^*$.*

Proof. It is sufficient to prove $E^n\Delta^{(-n)} = C^1$ where C^1 is the Cesàro matrix. By Lemma 4.3

$$\begin{aligned}(E^n\Delta^{(-n)})_{j,m} &= \sum_{k=m}^{j} e^n_{j,k}\delta^{(-n)}_{k,m} \\ &= \frac{1}{j+1}\sum_{k=m}^{j}(-1)^{j-k}\binom{n-1}{j-k}\binom{n+k-m-1}{k-m} \\ &= \frac{1}{j+1}\sum_{k=0}^{j-m}(-1)^{j-m-k}\binom{n-1}{j-m-k}\binom{n+k-1}{k} \\ &= \frac{1}{j+1} = C^1_{j,m}.\end{aligned}$$

Hence from Lemma 3.2 and relation 3.4, we conclude the results. □

By the sequences a^n in relation (3.1), we define a lower triangular matrix $D^{[m,n]} = (d_{j,k})$ with

$$\begin{aligned} d_{j,k} &= \begin{cases} \frac{a^m_{j-k}}{a^{n+m+1}_j} & 0 \leqslant k \leqslant j, \\ 0 & k > j, \end{cases} \\ &= \begin{cases} \frac{\binom{m+j-k-1}{j-k}}{\binom{n+m+j}{j}} & 0 \leqslant k \leqslant j, \\ 0 & k > j, \end{cases} \end{aligned} \tag{4.3}$$

where m and n are non-negative integers.

Note that for $m = n = 0$, $D^{[0,0]}$ is the identity matrix, for $m = 1$, $n = 0$, $D^{[1,0]}$ is the Cesàro matrix, and also for $m = 0,1$ and $n = 2$, we have

$$D^{[0,2]} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad D^{[1,2]} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.4}$$

In sequel, we need the following lemma.

LEMMA 4.5. *For non-negative integers n, m and j, we have*

$$\sum_{k=0}^{j}\binom{n+k-1}{k}\binom{m+j-k-1}{j-k} = \binom{n+m+j-1}{j}.$$

Proof. Let $|z| < 1$. From the identities

$$\begin{aligned}(1-z)^{-n}(1-z)^{-m} &= \sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j\sum_{j=0}^{\infty}\binom{m+j-1}{j}z^j \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{n+k-1}{k}\binom{m+j-k-1}{j-k}z^j,\end{aligned}$$

and

$$(1-z)^{-n}(1-z)^{-m}=(1-z)^{-(n+m)}=\sum_{j=0}^{\infty}\binom{n+m+j-1}{j}z^{j},$$

we obtain the claim. □

Now, we are ready to obtain the norm of operator $\Delta^{(-n)}$ from l_p into $D_p^{[m,n]}$.

THEOREM 4.6. *Suppose that $D^{[m,n]}$ is defined as in (4.3). Then $\Delta^{(-n)}$ is a bounded operator from l_p into $D_p^{[m,n]}$ and*

$$\|\Delta^{(-n)}\|_{D_p^{[m,n]}}=\frac{\Gamma(n+m+1)\Gamma(1/p^*)}{\Gamma(n+m+1/p^*)}.$$

In particular, $\|I\|_{D_p^{[1,0]}}=p^*$.

Proof. By using Lemma 4.5

$$\begin{aligned}(D^{[m,n]}\Delta^{(-n)})_{j,i}&=\sum_{k=i}^{j}d_{j,k}\delta_{k,i}^{(-n)}\\&=\frac{1}{\binom{n+m+j}{j}}\sum_{k=i}^{j}\binom{n+k-i-1}{k-i}\binom{m+j-k-1}{j-k}\\&=\frac{1}{\binom{n+m+j}{j}}\sum_{k=0}^{j-i}\binom{n+k-1}{k}\binom{m+j-i-k-1}{j-i-k}\\&=\frac{\binom{n+m+j-i-1}{j-i}}{\binom{n+m+j}{j}}=C_{j,i}^{n+m},\end{aligned}$$

so $D^{[m,n]}\Delta^{(-n)}=C^{n+m}$. Now by applying Lemma 3.2 and relation 3.4, we deduce the result. □

EXAMPLE 4.7. For both matrices in relation (4.4), we have

$$\|\Delta^{-2}\|_{D_p^{[0,2]}}=\frac{2p^{*2}}{p^*+1},\quad \|\Delta^{-2}\|_{D_p^{[1,2]}}=\frac{6p^{*3}}{(2p^*+1)(p^*+1)}.$$

Bennett in Theorem 11.5 from [4] investigated the following inequality

$$\|x\|_{C_p^1}\leqslant\|x\|_{H_p}\leqslant\frac{\pi}{p^*}\csc(\pi/p)\|x\|_{C_p^1},\tag{4.5}$$

for all $x\in l_p$. Similarly, we have the following inequality.

COROLLARY 4.8. *If* $p > 1$, *then*

$$\|x\|_{C_p^1} \leqslant \|\Delta^{(-n)}x\|_{H_p^n} \leqslant \frac{\pi}{p^*}\csc(\pi/p)\|x\|_{C_p^1},$$

for all $x \in l_p$.

Proof. By Lemma 3.2 and Theorem 4.2, we have

$$\|\Delta^{(-n)}x\|_{H_p^n} = \|H^n\Delta^{(-n)}x\|_p = \|Hx\|_p = \|x\|_{H_p}.$$

Hence relation (4.5) completes the proof. □

Bennett used the factorization $H = C^tB$, to prove the right hand side of relation (4.5), where B is given by

$$B_{j,k} = \frac{(j+1)}{(j+k+1)(j+k+2)}, \tag{4.6}$$

and C^t is the Copson matrix.

THEOREM 4.9. ([5], Proposition 2) *If* $p > 1$ *and the matix* B *is defined by* (4.6), *then* B *is a bounded operator on* l_p *and*

$$\|B\|_p = \frac{\pi}{p}\csc(\pi/p^*).$$

For H^n which is defined as in (4.1), we have a similar factorization of the form $H^n = C^tB^n$, where B^n is given by

$$b^n_{j,k} = \frac{(n+1)!(j+k)!(j+1)}{(j+k+n+2)!}. \tag{4.7}$$

If $n = 0$ in (4.7), then $B^n = B$.

THEOREM 4.10. *Suppose that* $p > 1$ *and the matix* B^n *is defined by* (4.7). *Then* $\Delta^{(-n)}$ *is a bounded operator from* l_p *into* B^n_p *and*

$$\|\Delta^{(-n)}\|_{B_p^n} = \frac{\pi}{p}\csc(\pi/p^*).$$

Proof. According to Lemma 3.2 and Theorem 4.9, it is sufficient to prove $B^n\Delta^{(-n)} = B$.

$$\begin{aligned}(B^n\Delta^{(-n)})_{k,m} &= \sum_{j=m}^{\infty} b^n_{k,j}\delta^{(-n)}_{j,m} = \sum_{j=m}^{\infty}\binom{n+j-m-1}{j-m}\frac{(n+1)!(j+k)!(k+1)}{(j+k+n+2)!}\\ &= (k+1)\sum_{j=0}^{\infty}\binom{n+j-1}{j}\beta(j+m+k+1,n+2)\end{aligned}$$

$$= (k+1)\sum_{j=0}^{\infty}\binom{n+j-1}{j}\int_0^1 z^{j+m+k}(1-z)^{n+1}dz$$
$$= (k+1)\int_0^1 (1-z)^{-n}z^{m+k}(1-z)^{n+1}dz$$
$$= \frac{(k+1)}{(m+k+1)(m+k+2)} = b_{k,m}. \quad \square$$

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