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NONLINEAR MAPS PRESERVING CONDITION SPECTRUM OF JORDAN SKEW TRIPLE PRODUCT OF OPERATORS

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Abstract. Let $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with $\dim \mathscr{H} \geqslant 3$. Let $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathscr{B}(\mathscr{H})$ which contain all rank-one operators. Denote by $r_\varepsilon(A)$ the condition spectral radius of $A \in \mathscr{B}(\mathscr{H})$. We determine the form of surjective maps $\phi : \mathscr{W} \to \mathscr{V}$ satisfying $r_\varepsilon(AB^*A) = r_\varepsilon(\phi(A)\phi(B)^*\phi(A))$ for all A, B in $\mathscr{W}$, we characterize also the structure of surjective maps $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ with $\sigma_\varepsilon(AB^*A) = \sigma_\varepsilon(\phi(A)\phi(B)^*\phi(A))$ for all A, B in $\mathscr{B}(\mathscr{H})$ where $\sigma_\varepsilon(A)$ is the ε-condition spectrum of an operator A in $\mathscr{B}(\mathscr{H})$.

1. Introduction

Throughout this note, $\mathscr{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathscr{B}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$ with unit I. For $A \in \mathscr{B}(\mathscr{H})$, $\|A\|$ is the norm of A. Let $0 < \varepsilon < 1$, the ε-*condition spectrum* of A, $\sigma_\varepsilon(A)$, is defined by

$$\sigma_\varepsilon(A) := \{z \in \mathbb{C} : \|z - A\| \|(z-A)^{-1}\| \geqslant \varepsilon^{-1}\},$$

with the convention that $\|z - A\| \|(z-A)^{-1}\| = \infty$ when $z - A$ is not invertible, and is a nonempty compact set contains the usual spectrum $\sigma(A)$ of $A \in \mathscr{B}(\mathscr{H})$, [13]. The ε-*condition spectral radius* of A, $r_\varepsilon(A)$, is given by

$$r_\varepsilon(A) := \sup\{|z|, z \in \sigma_\varepsilon(A)\}.$$

The ε-*pseudo spectrum* of A, for $\varepsilon > 0$, $\Lambda_\varepsilon(A)$, is defined by

$$\Lambda_\varepsilon(A) := \{z \in \mathbb{C} : \|(z-A)^{-1}\| \geqslant \varepsilon^{-1}\},$$

with the convention that $\|(z-A)^{-1}\| = \infty$ when $z - A$ is not invertible. The definition of the ε-*pseudo spectral radius* of A, $r'_\varepsilon(A)$, is given by

$$r'_\varepsilon(A) := \sup\{|z|, z \in \Lambda_\varepsilon(A)\}.$$

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For more details and basic properties about the condition spectrum and pseudo spectra we refer the readers to [13, 14, 16]. The inner product on $\mathscr{H}$ will be denoted by $\langle .,. \rangle$. As usual for $x, f \in \mathscr{H}$, $x \otimes f$ denotes the rank at most one operator on $\mathscr{H}$ defined by $z \mapsto \langle z, f \rangle x$ for every $z \in \mathscr{H}$ and every operator of rank at most one on $\mathscr{B}(\mathscr{H})$ can be written in this form. For an operator $T \in \mathscr{B}(\mathscr{H})$, T^* denotes its adjoint.

The problem of characterizing maps on matrices or operators that preserve certain functions, subsets and relations has attracted the attention of many mathematicians in the last decade; for example we can see [1, 2, 3, 4, 5, 6, 9, 10, 11, 14, 15] and their references. In recent years, a great activity has occurred in characterising maps preserving condition spectrum, pseudo spectra, condition spectral radius and pseudo spectral radius. The study of linear case was done by Kumar and Kulkarni in [14]. The non-linear case has also been studied in recent papers [2, 3, 6, 7, 8], the authors changed the linearity of such map ϕ by the weaker condition $F(\phi(A \diamond B)) = F(A \diamond B)$ where $A \diamond B$ is one of different kinds of operations such as sum $A+B$, the usual product AB, triple product ABA, the Jordan product $AB+BA$, Lie product $AB-BA$ and skew product A^*B. $F(.)$ is one of the following objects $\sigma_\varepsilon(.)$, $r_\varepsilon(.)$, $\Lambda_\varepsilon(.)$ or $r'_\varepsilon(.)$. In [3], Bendaoud et al. characterised surjective maps preserving the condition spectrum and condition spectral radius of product and Jordan triple product of operators. In particular they showed that a surjective map ϕ from $\mathscr{B}(\mathscr{H})$ into itself, where $\mathscr{H}$ is a Hilbert space with $dim\mathscr{H} \geqslant 3$, satisfies

$$r_\varepsilon(\phi(A)\phi(B)\phi(A)) = r_\varepsilon(ABA), \quad (A, B \in \mathscr{B}(\mathscr{H})),$$

if and only if there exist a functional $h : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$, with $|h(A)| = 1$ for all $A \in \mathscr{B}(\mathscr{H})$, and a unitary or anti-unitary operator U on $\mathscr{H}$ such that either $\phi(A) = h(A)UAU^*$ or $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathscr{B}(\mathscr{H})$. They showed also that a surjective map ϕ from $\mathscr{B}(\mathscr{H})$ into itself, where $\mathscr{H}$ is Hilbert space with $\dim \mathscr{H} \geqslant 3$, satisfies

$$\sigma_\varepsilon(\phi(A)\phi(B)\phi(A)) = \sigma_\varepsilon(ABA), \quad (A, B \in \mathscr{B}(\mathscr{H}))$$

if and only if there is a third root of unity c and a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that either $\phi(A) = cUAU^*$ or $\phi(A) = cUA^{tr}U^*$ for all $A \in \mathscr{B}(\mathscr{H})$, where A^{tr} is the transpose of A with respect to an arbitrary but fixed orthogonal basis of $\mathscr{H}$.

Let $\mathscr{W}$, $\mathscr{V}$ be two subsets of $\mathscr{B}(\mathscr{H})$ which contain all rank-one operators. The aim of this note concern the study of the surjective maps $\phi : \mathscr{W} \to \mathscr{V}$ satisfying

- $r_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = r_\varepsilon(AB^*A)$ for all $A, B \in \mathscr{W}$.

As a consequence we obtain the description of surjective maps $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ with

- $\sigma_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \sigma_\varepsilon(AB^*A)$ for all $A, B \in \mathscr{B}(\mathscr{H})$.

The proof of such a promised results uses some arguments that are influenced by ideas from [2, 8, 10].

2. Preliminaries

In this section, we collect some lemmas that will be used in the proof of our main results. In the sequel we denote by ε a positive number such that $0<\varepsilon<1$. We begin by gathering some properties of the ε-condition spectral radius.

LEMMA 2.1. *Let $\mathscr{H}$ be a complex Hilbert space and $T\in\mathscr{B}(\mathscr{H})$. Then we have*

1. $r_\varepsilon(T)=0$ *if and only if* $T=0$.

2. $r_\varepsilon(UTU^*)=r_\varepsilon(T)$, *for all unitary operators* $U\in\mathscr{B}(\mathscr{H})$.

3. $r_\varepsilon(\lambda T)=|\lambda|r_\varepsilon(T)$ *for* $\lambda\in\mathbb{C}$.

4. $r_\varepsilon(T^*)=r_\varepsilon(T)$.

Proof. For the first three propositions see [2, Proposition 2.6], the fourth is an immediate consequence of the definition. □

The second result gives link between rank-one operators which have the same ε-condition spectral radius in particular case.

LEMMA 2.2. *Let $u,v\in\mathscr{H}$. If $r_\varepsilon(u\otimes f)=r_\varepsilon(v\otimes f)$ for every $f\in\mathscr{H}$, then the vectors u and v are linearly dependent.*

Proof. See [2, Proposition 2.6]. □

The following lemma characterizes the ε-condition spectrum and the ε-condition spectral radius of nontrivial projection.

LEMMA 2.3. *Let $P\in\mathscr{B}(\mathscr{H})$ be a nontrivial projection. Then*

$$\sigma_\varepsilon(P)=\overline{D}\Big(\frac{1}{1-\varepsilon^2},\frac{\varepsilon}{1-\varepsilon^2}\Big)\cup\overline{D}\Big(\frac{-\varepsilon^2}{1-\varepsilon^2},\frac{\varepsilon}{1-\varepsilon^2}\Big),$$

where $\overline{D}(a,r)$ is the closed disc centered at $a\in\mathbb{C}$ with radius $r\geqslant 0$.
In particular

$$r_\varepsilon(P)=\frac{1}{1-\varepsilon}.$$

Proof. See [2, Proposition 2.4] □

We close this section by the following lemma which characterises the maps preserving zero Jordan skew triple products of operators in both directions.

LEMMA 2.4. *Let $\mathscr{H}$ be a complex Hilbert space with dim $\mathscr{H} \geqslant 3$. Let $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathscr{B}(\mathscr{H})$ which contain all rank-one operators. Suppose that $\phi : \mathscr{W} \to \mathscr{V}$ is a surjective map. Then ϕ satisfies $AB^*A = 0 \Leftrightarrow \phi(A)\phi(B)^*\phi(A) = 0$ for all $A, B \in \mathscr{W}$ if and only if there exist unitary or anti-unitary operators U, V on $\mathscr{H}$ and a functional $h : \mathscr{W} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms*

$$\phi(A) = h(A)UAV \ for \ all \ A \in \mathscr{W},$$

or

$$\phi(A) = h(A)UA^*V \ for \ all \ A \in \mathscr{W}.$$

Proof. See [10, Corollary 3.5]. □

3. Main results

We begin by stating and proving the promised main results, since all the necessary ingredients are collected in the preliminary section. The first theorem characterizes surjective maps preserving the ε-condition spectral radius of Jordan skew triple product of operators.

THEOREM 3.1. *Let $\mathscr{H}$ be a complex Hilbert space with dim $\mathscr{H} \geqslant 3$. Let $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathscr{B}(\mathscr{H})$ which contain all rank-one operators. Suppose that $\phi : \mathscr{W} \to \mathscr{V}$ is a surjective map. Then ϕ satisfies*

$$r_\varepsilon(AB^*A) = r_\varepsilon(\phi(A)\phi(B)^*\phi(A)) \ for \ all \ A, \ B \in \mathscr{W} \tag{3.1}$$

if and only if one of the following statements holds.

1. *There exist a unitary or an anti-unitary operator U on $\mathscr{H}$ and a functional $\varphi : \mathscr{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathscr{W}$.*
2. *There exist a unitary or an anti-unitary operator U on $\mathscr{H}$ and a functional $\varphi : \mathscr{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that $\phi(A) = \varphi(A)UA^*U^*$ for all $A \in \mathscr{W}$.*

Proof. According to Lemma 2.1 it is easy to check that if ϕ takes one of the forms (1) or (2) then $r_\varepsilon(AB^*A) = r_\varepsilon(\phi(A)\phi(B)^*\phi(A))$ for all A, $B \in \mathscr{W}$ and so the sufficiency condition is obvious.

Conversely, assume that $\phi : \mathscr{W} \to \mathscr{V}$ is a surjective map satisfying the equation (3.1). We divide the proof into three steps.

Step 1: $\phi(A) = h(A)UAV$ or $\phi(A) = h(A)UA^*V$ for every $A \in \mathscr{W}$ with U, V unitary or anti-unitary operators on $\mathscr{H}$ and $|h(A)|$ is constant.

For establishing this step we borrow the idea from [10, Theorem 3.1]. Indeed, since ϕ satisfies the equation (3.1), it follows from Lemma 2.1 that the map ϕ preserves zero Jordan skew triple product of operators in both directions, which means that

$$AB^*A = 0 \iff \phi(A)\phi(B)^*\phi(A) = 0 \ (A, B \in \mathscr{W}).$$

Consequently by Lemma 2.4 there exist a unitary or an anti-unitary operators U, V on $\mathscr{H}$ and a functional $h:\mathscr{W}\to\mathbb{C}\setminus\{0\}$ such that either $\phi(A)=h(A)UAV$ or $\phi(A)=h(A)UA^{*}V$ for all $A\in\mathscr{W}$. It remains to prove that $|h(A)|$ is constant for every $A\in\mathscr{W}$ in the both cases.

• Assume that $\phi(A)=h(A)UAV$ for all $A\in\mathscr{W}$. It follows by equation (3.1) that for every A, B in $\mathscr{W}$, $r_{\varepsilon}(AB^{*}A)=|\,h(A)^{2}h(B)\,|\,r_{\varepsilon}(UAB^{*}AV)$. In particular if $A=x\otimes f$ for x and f in $\mathscr{H}$ with $\langle B^{*}x,f\rangle\neq 0$, we get

$$r_{\varepsilon}(x\otimes fB^{*}x\otimes f)=|\,h(x\otimes f)^{2}h(B)\,|\,r_{\varepsilon}(Ux\otimes fB^{*}x\otimes fV),$$

which is equivalent to

$$|\,\langle B^{*}x,f\rangle\,|\,r_{\varepsilon}(x\otimes f)=|\,h(x\otimes f)^{2}h(B)\langle B^{*}x,f\rangle\,|\,r_{\varepsilon}(Ux\otimes V^{*}f),$$

thus

$$|h(B)|=\frac{r_{\varepsilon}(x\otimes f)}{|h(x\otimes f)^{2}|r_{\varepsilon}(Ux\otimes V^{*}f)}.$$

Since for any non-zero B_1, $B_2\in\mathscr{W}$, there exist x, $f\in\mathscr{H}$ such that $\langle B_i^{*}x,f\rangle\neq 0$, $i=1,2$, one gets $|h(B_1)|=|h(B_2)|$. Finally we conclude that there exists a constant $\beta>0$ such that $|h(A)|=\beta$ for all $A\in\mathscr{W}$.

• Assume that $\phi(A)=h(A)UA^{*}V$ for all $A\in\mathscr{W}$. It follows by equation (3.1) that $r_{\varepsilon}(AB^{*}A)=|\,h(A)^{2}h(B)\,|\,r_{\varepsilon}(UA^{*}BA^{*}V)$ for all A, B in $\mathscr{W}$. In particular if $A=x\otimes f$ for x and f in $\mathscr{H}$ with $\langle B^{*}x,f\rangle\neq 0$, we have

$$r_{\varepsilon}(x\otimes fB^{*}x\otimes f)=|\,h(x\otimes f)^{2}h(B)\,|\,r_{\varepsilon}(V^{*}(x\otimes f)B^{*}(x\otimes f)U^{*}),$$

which is equivalent to

$$|\,\langle B^{*}x,f\rangle\,|\,r_{\varepsilon}(x\otimes f)=|\,h(x\otimes f)^{2}h(B)\langle B^{*}x,f\rangle\,|\,r_{\varepsilon}(V^{*}x\otimes Uf),$$

thus

$$|h(B)|=\frac{r_{\varepsilon}(x\otimes f)}{|h(x\otimes f)^{2}|r_{\varepsilon}(V^{*}x\otimes Uf)}.$$

With the similar reasoning as above one gets $|h(A)|=\beta$, for all $A\in\mathscr{W}$ with β being a positive constant.

Step 2: $U=\alpha V^{*}$ for some nonzero scalar α.

For establishing this point we will discuss the forms given in the previous step. Note that $|h(A)|=\beta$ is constant in both cases for all $A\in\mathscr{W}$.

• Assume that $\phi(A)=h(A)UAV$ for every $A\in\mathscr{W}$.

Let $x\in\mathscr{H}$ be a nonzero vector, then for every nonzero vector $f\in\mathscr{H}$, we have

$$\begin{aligned}r_{\varepsilon}(x\otimes f(x\otimes f)^{*}x\otimes f)&=r_{\varepsilon}(x\otimes f(f\otimes x)x\otimes f)\\&=\|x\|^{2}\|f\|^{2}r_{\varepsilon}(x\otimes f).\end{aligned}$$

On the other hand we have

$$\begin{aligned}r_{\varepsilon}(\phi(x\otimes f)(\phi(x\otimes f))^{*}\phi(x\otimes f))&=\beta^{3}r_{\varepsilon}(Ux\otimes fVV^{*}(f\otimes x)U^{*}Ux\otimes fV)\\&=\beta^{3}\|x\|^{2}\|f\|^{2}r_{\varepsilon}(Ux\otimes fV)\\&=\beta^{3}\|x\|^{2}\|f\|^{2}r_{\varepsilon}(VUx\otimes f).\end{aligned}$$

Combining this with equation (3.1), we get $\beta^3\|x\|^2\|f\|^2 r_\varepsilon(VUx\otimes f) = \|x\|^2\|f\|^2 r_\varepsilon(x\otimes f)$, so $r_\varepsilon(x\otimes f) = \beta^3 r_\varepsilon(VUx\otimes f)$, thus by Lemma 2.2, x and VUx are linearly dependent for every $x\in\mathscr{H}$. We conclude that $VU=\alpha I$ for some scalar α.

- Assume that $\phi(A) = h(A)UA^*V$ for every $A\in\mathscr{W}$.

Note that $r_\varepsilon(T) = r_\varepsilon(T^*)$ for every $T\in\mathscr{B}(\mathscr{H})$.

Let $f\in\mathscr{H}$ be a nonzero vector, then for every nonzero vector $x\in\mathscr{H}$, we have

$$\begin{aligned} r_\varepsilon(x\otimes f(x\otimes f)^* x\otimes f) &= r_\varepsilon(x\otimes f(f\otimes x)x\otimes f) \\ &= \|x\|^2\|f\|^2 r_\varepsilon(x\otimes f) \\ &= \|x\|^2\|f\|^2 r_\varepsilon(f\otimes x). \end{aligned}$$

On the other hand

$$\begin{aligned} r_\varepsilon(\phi(x\otimes f)(\phi(x\otimes f))^*\phi(x\otimes f)) &= \beta^3 r_\varepsilon(Uf\otimes xVV^*(x\otimes f)U^*Uf\otimes xV) \\ &= \beta^3\|x\|^2\|f\|^2 r_\varepsilon(Uf\otimes xV) \\ &= \beta^3\|x\|^2\|f\|^2 r_\varepsilon(VUf\otimes x). \end{aligned}$$

As before we have $r_\varepsilon(f\otimes x) = \beta^3 r_\varepsilon(VUf\otimes x)$. It follows by Lemma 2.2 that f and VUf are linearly dependent for every $f\in\mathscr{H}$. This implies that $VU=\alpha I$ for some scalar α.

Step 3: ϕ takes the desired form.

Recall that we already obtained that $\phi(A) = \varphi(A)UAU^*$ or $\phi(A) = \varphi(A)UA^*U^*$ for every $A\in\mathscr{W}$, with $\varphi(A) = \alpha h(A)$ and $|\varphi(A)|$ is constant for every $A\in\mathscr{W}$. It remains to show that $|\varphi(A)| = 1$ for every $A\in\mathscr{W}$. Indeed, let $x\in\mathscr{H}$ be a unit vector. If $\phi(A) = \varphi(A)UAU^*$ for every $A\in\mathscr{W}$, then by equation (3.1), we have

$$\begin{aligned} r_\varepsilon(x\otimes x) &= r_\varepsilon(x\otimes x(x\otimes x)^* x\otimes x) \\ &= r_\varepsilon(\phi(x\otimes x)\phi(x\otimes x)^*\phi(x\otimes x)) \\ &= |\varphi(x\otimes x)|^3 r_\varepsilon(Ux\otimes xU^*) \\ &= |\varphi(x\otimes x)|^3 r_\varepsilon(x\otimes x). \end{aligned}$$

By Lemma 2.3, $r_\varepsilon(x\otimes x) = \dfrac{1}{1-\varepsilon}$, so it follows that $|\varphi(x\otimes x)|^3 = 1$. Thus $|\varphi(A)| = 1$ for every $A\in\mathscr{W}$.

If $\phi(A) = \varphi(A)UA^*U^*$ for every $A\in\mathscr{W}$ then

$$\begin{aligned} r_\varepsilon(x\otimes x) &= r_\varepsilon(x\otimes x(x\otimes x)^* x\otimes x) \\ &= r_\varepsilon(\phi(x\otimes x)\phi(x\otimes x)^*\phi(x\otimes x)) \\ &= |\varphi(x\otimes x)|^3 r_\varepsilon(Ux\otimes xU^*) \\ &= |\varphi(x\otimes x)|^3 r_\varepsilon(x\otimes x). \end{aligned}$$

Again by Lemma 2.3, $r_\varepsilon(x\otimes x) = \dfrac{1}{1-\varepsilon}$ which implies that $|\varphi(x\otimes x)|^3 = 1$ and thus $|\varphi(A)| = 1$ for every $A\in\mathscr{W}$, which completes the proof. $\square$

Now we are able to characterize surjective maps on $\mathscr{B}(\mathscr{H})$ preserving ε-condition spectrum of the Jordan skew triple product of operators.

THEOREM 3.2. *Let $\mathscr{H}$ be a complex Hilbert space with $\dim \mathscr{H} \geqslant 3$. Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a surjective map. Then ϕ satisfies*

$$\sigma_\varepsilon(AB^*A) = \sigma_\varepsilon(\phi(A)\phi(B)^*\phi(A)) \text{ for all } A,\ B \in \mathscr{B}(\mathscr{H}) \tag{3.2}$$

if and only if one of the following statements holds.

1. *There exists a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that $\phi(A) = UAU^*$ for all $A \in \mathscr{B}(\mathscr{H})$.*
2. *There exists an anti-unitary operator U on $\mathscr{H}$ such that $\phi(A) = UA^*U^*$ for all $A \in \mathscr{B}(\mathscr{H})$.*

To prove this theorem, we need the following lemma summarizing some properties of the ε-condition spectrum proved in [13].

LEMMA 3.3. *Let $\mathscr{H}$ be a complex Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then we have*

1. $\sigma_\varepsilon(\alpha + \beta A) = \alpha + \beta \sigma_\varepsilon(T)$ *for all* $\alpha,\ \beta \in \mathbb{C}$.
2. $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T)$, *for all unitary* $U \in \mathscr{B}(\mathscr{H})$.
3. $\sigma_\varepsilon(UTU^*) = \overline{\sigma_\varepsilon(T)}$, *for all anti-unitary* U *on* $\mathscr{H}$.
4. $\sigma_\varepsilon(T^*) = \overline{\sigma_\varepsilon(T)}$.
5. $\sigma_\varepsilon(I) = \{1\}$.

Proof of Theorem 3.2. The "if" part is easily verified according to the previous lemma. To prove the "only if" part, suppose that ϕ satisfies the equation (3.2). It follows that ϕ preserves the ε-condition spectral radius of Jordan skew triple product of operators. Then by Theorem (3.1) there exist a unitary or an anti-unitary operator U on $\mathscr{H}$ and $\varphi : \mathscr{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that either

$$\phi(A) = \varphi(A)UAU^* \text{ for all } A \in \mathscr{B}(\mathscr{H}), \tag{3.3}$$

or

$$\phi(A) = \varphi(A)UA^*U^* \text{ for all } A \in \mathscr{B}(\mathscr{H}). \tag{3.4}$$

We will complete the proof by discussing two cases.

Case 1. ϕ takes the form (3.3).

• Suppose that U is unitary. Let us prove that $\varphi(A) = 1$ for every $A \in \mathscr{B}(\mathscr{H})$. We will proceed as in the proof of [10, Corollary 5.4]. Indeed, if $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathscr{B}(\mathscr{H})$ with U unitary. It follows from equation (3.2) that for all $A \in \mathscr{B}(\mathscr{H})$ we have

$$\begin{aligned}
\sigma_\varepsilon(A) &= \sigma_\varepsilon(\phi(I)\phi(A^*)^*\phi(I)) \\
&= \sigma_\varepsilon(\varphi(I)^2\overline{\varphi(A^*)}UAU^*) \\
&= \varphi(I)^2\overline{\varphi(A^*)}\sigma_\varepsilon(A).
\end{aligned}$$

In particular for $A = I$, since $\sigma_\varepsilon(I) = \{1\}$, we have $\varphi(I)^2\overline{\varphi(I)} = 1$, thus $\varphi(I) = 1$. So

$$\sigma_\varepsilon(A) = \overline{\varphi(A^*)}\sigma_\varepsilon(A) \text{ } for \text{ } all \text{ } A \in \mathscr{B}(\mathscr{H}).$$

Which implies that, for every $A \in \mathscr{B}(\mathscr{H})$,

$$z \in \sigma_\varepsilon(A) \Leftrightarrow \overline{\varphi(A^*)}z \in \sigma_\varepsilon(A).$$

If $\sigma_\varepsilon(A)$ is not a disc centred at zero, we deduce that $\overline{\varphi(A^*)} = 1$. Therefore $\varphi(A) = 1$ for every $A \in \mathscr{B}(\mathscr{H})$. Otherwise if $\sigma_\varepsilon(A)$ is a disc centred at zero. Choose x, f in $\mathscr{H}$ such that $\langle x, f\rangle \neq 0$ and $\langle A^*x, f\rangle \neq 0$. By Lemma 2.3, $\sigma_\varepsilon(x \otimes f)$ is not a disc centred at zero, then by the above consideration, $\varphi(x \otimes f) = 1$. Furthermore by equation (3.2)

$$\begin{aligned}
\sigma_\varepsilon(x \otimes fA^*x \otimes f) &= \sigma_\varepsilon(\phi(x \otimes f)\phi(A)^*\phi(x \otimes f)) \\
&= \varphi(x \otimes f)^2\overline{\varphi(A)}\sigma_\varepsilon(x \otimes fA^*x \otimes f) \\
&= \overline{\varphi(A)}\langle A^*x, f\rangle\sigma_\varepsilon(x \otimes f).
\end{aligned}$$

On the other hand $\sigma_\varepsilon(x \otimes fA^*x \otimes f) = \langle A^*x, f\rangle\sigma_\varepsilon(x \otimes f)$. Thus for all $A \in \mathscr{B}(\mathscr{H})$ $\langle A^*x, f\rangle\sigma_\varepsilon(x \otimes f) = \overline{\varphi(A)}\langle A^*x, f\rangle\sigma_\varepsilon(x \otimes f)$. Since $\langle A^*x, f\rangle \neq 0$ and $\sigma_\varepsilon(x \otimes f)$ is not a circular disc centred at zero, we deduce that $\overline{\varphi(A)} = 1$. Finally $\varphi(A) = 1$ for all $A \in \mathscr{B}(\mathscr{H})$, and so $\phi(A) = UAU^*$ for all $A \in \mathscr{B}(\mathscr{H})$.

- Assume that U is anti-unitary.

Let us show that this case cannot occur. We will use an idea given in [2, Theorem 3.5]. Assume that $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathscr{B}(\mathscr{H})$ and U is an anti-unitary operator. Note that by Lemma 2.3, one can see that for every unit vector $x \in \mathscr{H}$

$$\sigma_\varepsilon(x \otimes x) = \overline{\sigma_\varepsilon(x \otimes x)}.$$

Following a similar discussion as in the previous case we get that for every $A \in \mathscr{B}(\mathscr{H})$,

$$\sigma_\varepsilon(A) = \varphi(A)\overline{\varphi(A^*)\sigma_\varepsilon(A)}.$$

In particular for $A = I$ we get $\varphi(I) = 1$. Moreover if $x \in \mathscr{H}$ is a unit vector, by equation (3.2), we get

$$\begin{aligned}
\sigma_\varepsilon(x \otimes x) &= \sigma_\varepsilon(x \otimes xIx \otimes x)) \\
&= \sigma_\varepsilon(\varphi(x \otimes x)^2\overline{\varphi(I)}Ux \otimes xU^*) \\
&= \varphi(x \otimes x)^2\overline{\sigma_\varepsilon(x \otimes x)} \\
&= \varphi(x \otimes x)^2\sigma_\varepsilon(x \otimes x).
\end{aligned}$$

Thus $\varphi(x \otimes x)^2 = 1$. Now take an arbitrary non self adjoint operator $A \in \mathscr{B}(\mathscr{H})$ and a unit vector $x \in \mathscr{H}$ such that $\langle A^*x, x\rangle \neq 0$. Then it follows that

$$\begin{aligned}
\sigma_\varepsilon(x \otimes xA^*x \otimes x) &= \sigma_\varepsilon(\phi(x \otimes x)\phi(A)^*\phi(x \otimes x)) \\
&= \varphi(x \otimes x)^2\overline{\varphi(A)}\sigma_\varepsilon(Ux \otimes xA^*x \otimes xU^*) \\
&= \overline{\varphi(A)\langle A^*x, x\rangle}\sigma_\varepsilon(x \otimes x).
\end{aligned}$$

This implies that

$$\overline{\varphi(A)\langle A^*x,x\rangle} = \langle A^*x,x\rangle,$$

and so

$$\frac{\langle A^*x,x\rangle}{\overline{\langle A^*x,x\rangle}} = \overline{\varphi(A)}.$$

In particular for $A\in\mathscr{B}(\mathscr{H})$ for which there exist nonzero vectors $x,y\in\mathscr{H}$ satisfying $A^*x=x$ and $A^*y=iy$ we get

$$1=\frac{\langle A^*x,x\rangle}{\overline{\langle A^*x,x\rangle}}=\overline{\varphi(A)}=\frac{\langle A^*y,y\rangle}{\overline{\langle A^*y,y\rangle}}=-1,$$

which is a contradiction.

Case 2. ϕ takes the form (3.4).

We have $\phi(A)=\varphi(A)UA^*U^*$ for all $A\in\mathscr{B}(\mathscr{H})$. With similar discussion as in the first case with U unitary or anti-unitary we deduce that $\varphi(I)=1$.

• Let U be unitary. By equation (3.2) and the fact that $\varphi(I)=1$, we have

$$\sigma_\varepsilon(A)=\overline{\varphi(A^*)}\sigma_\varepsilon(A^*)\quad\text{for all } A\in\mathscr{B}(\mathscr{H}).$$

Then following the same steps as the second discussion in the first case we deduce that this case cannot occur.

• Assume that U is anti-unitary. To complete the proof let us show that $\varphi(A)=1$ for every $A\in\mathscr{B}(\mathscr{H})$. We have $\phi(A)=\varphi(A)UA^*U^*$ for every $A\in\mathscr{B}(\mathscr{H})$. It follows that $\sigma_\varepsilon(A)=\varphi(A^*)\sigma_\varepsilon(A)$. As in the above consideration we conclude that $\varphi(A)=1$ for all $A\in\mathscr{B}(\mathscr{H})$. Then ϕ takes the desired form.

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