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BESSEL PROPERTY AND BASICITY OF THE SYSTEM OF ROOT VECTOR–FUNCTIONS OF DIRAC OPERATOR WITH SUMMABLE COEFFICIENT

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Abstract. In the paper we study one-dimensional Dirac operator

$$Dy = By' + P(x)y, \ \ y = (y_1, \ y_2)^{\mathrm{T}},$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x) = diag\,(p(x), q(x))$, $p(x)$ and $q(x)$ are complex valued functions from the class $L_1(G)$, $G = (0, 2\pi)$.

Necessary and sufficient conditions of Bessel property and unconditional basicity (the Riesz basicity) of the system of root-functions of the operator D in $L_2^2(G)$ are set up. A theorem on equivalent basicity for these systems in $L_2^2(G)$ is proved.

1. Introduction and statement of results

In the paper we study Bessel property and basicity of the system of root vector-functions of one-dimensional Dirac operator with summable coefficient (potential). The root vector-functions are understood in generalized representation, i.e. without regard to boundary conditions (see [8]). Under this sense of root functions, necessary and sufficient conditions of unconditional basicity in L_2 of the system conditions of unconditinal basicity in L_2 of the system of root functions of the operator $Lu = -u'' + q(x)u$ were first set up by V. A. Il'in in [8]. Further, in the papers [2, 11, 13, 14, 15, 21, 28] these and other issues were studied for an ordinary differential operator of arbitrary order, and the criteria of Bessel property, Riesz property and unconditional basicity were set up. Criterion of Bessel property and unconditional basicity for the Dirac operator with the potential $P(x) \in L_2(G)$ were set up in the paper [16]. The papers [3, 17, 18, 19, 20] were devoted to componentwise uniform equiconvergence on a compact, uniform convergence, the Riesz property of the system of root vector functions of the Dirac operator.

Rich references [1, 4, 5, 7, 12, 23, 24, 25, 26, 27, 29] deal with properties of basicity and other spectral properties of root vector-functions of the Dirac operator (with boundary conditions). In the paper [29] the Riesz basicity is set up in the case when the potential belongs to the class L_2 and boundary conditions are separated. The

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Paper OaM-12-57

case with regular boundary conditions and with potential from the class L_2 was studied in the paper [5]. The Riesz basicity from subspaces, and in the case of strongly regular boundary conditions the Riesz basicity is proved. The Dirac operator with a potential from L_p, $p \geqslant 1$ was studied in the papers [26], [27] and the Riesz basicity was proved for the case of strongly regular boundary conditions while in the case of regular (but not strongly regular) boundary conditions, the Riesz basicity from subspaces is proved.

Let $L_p^2(G)$, $p \geqslant 1$, $G=(0,2\pi)$ be the space of two-component vector-functions $f(x)=(f_1(x),f_2(x))^{\mathrm{T}}$ with the norm

$$\|f\|_p = \|f\|_{p,2} = \left\{\int_G |f(x)|^p\,dx\right\}^{\frac{1}{p}} = \left\{\int_G \left(\sum_{i=1}^{n} |f_i(x)|^2\right)^{\frac{p}{2}} dx\right\}^{\frac{1}{p}}.$$

In the case $p=\infty$ the norm is defined by the equality $\|f\|_\infty = \|f\|_{\infty,2} = \sup\limits_{x\in\bar{G}} vrai\,|f(x)|$.

For $f(x)\in L_p^2(G)$, $g(x)\in L_q^2(G)$, $p^{-1}+q^{-1}=1$, $1\leqslant p\leqslant\infty$, there exist $(f,g)=\int_G \sum_{j=1}^2 f_j(x)\overline{g_j(x)}dx$.

Let us consider one-dimensional Dirac operator

$$Dy = By' + P(x)y,\ \ y(x) = (y_1(x),\ \ y_2(x))^{\mathrm{T}},\ \ x\in G=(0,2\pi),$$

where $B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x)=diag\,(p(x),q(x))$, $p(x)$ and $q(x)$ are complex-valued functions from the class $L_1(G)$.

Following [8] we will understand the root vector-functions of the operator D without regard to the form of boundary conditions, more exactly under eigen vector-function of the Dirac operator D responding to the eigen value λ, we will understand any identically nonzero two-component vector-function $\overset{0}{y}(x)$ that is absolutely continuous on $\bar{G}$ and almost everywhere in G satisfies the equation $D\overset{0}{u}=\lambda\overset{0}{u}$. Under the associated vector-function of order ℓ, $\ell\geqslant 1$ responding to the same λ and associated vector-function $\overset{0}{y}(x)$, we will understand any two-component vector-function $\overset{\ell}{y}(x)$ that is absolutely continuous $\bar{G}$ and almost everywhere in G satisfies the equation $D\overset{\ell}{y}=\lambda\overset{\ell}{y}+\overset{\ell-1}{y}$.

Let $\{u_k(x)\}_{k=1}^\infty$ be an arbitrary system comprised from the root vector-functions of the operator D, $\{\lambda_k\}_{k=1}^\infty$ be the appropriate system of eigen values. Furthermore, the vector-function $u_k(x)$ is included into the system $\{u_k(x)\}_{k=1}^\infty$ together with appropriate associated vector-functions of less order.

DEFINITION 1.1. The system $\{\varphi_k(x)\}_{k=1}^\infty \subset L_2^2(G)$ is said to be Bessel if there exists a constant M such that for any vector-function $f(x)\in L_2^2(G)$ the equality

$$\sum_{k=1}^\infty |(f,\varphi_k)|^2 \leqslant M\,\|f\|_{2,2}^2$$

is fulfilled.

DEFINITION 1.2. The system $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_2^2(G)$ is quadratically close to the system $\{\psi_k(x)\}_{k=1}^{\infty} \subset L_2^2(G)$ if $\sum\limits_{k=1}^{\infty} \|\varphi_k - \psi_k\|_{2,2}^2 < \infty$.

DEFINITION 1.3. Two sequences of elements in Hilbert space H are equivalent if there exists an operator H that is linear bounded and boundedly inversible in H and takes one of these sequences to another one.

In the paper the following theorems are proved.

THEOREM 1.4. (Criterion of Bessel property) *Let $P(x) \in L_1(G)$, the lengths of the chains of root vector-functions be uniformly bounded and there exist a constant C_0 such that*

$$|Im\lambda_k| \leqslant C_0, \ \ k = 1,2,\dots. \tag{1.1}$$

Then for the Bessel property of the system $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ in $L_2^2(G)$ it is necessary and sufficient the existence of a constant M_1 such that

$$\sum_{|Re\lambda_k - \tau| \leqslant 1} 1 \leqslant M_1, \tag{1.2}$$

where τ is an arbitrary real number.

REMARK 1.5. In sufficient part of theorem 1.4 the condition of uniform boundedness of the length of chains of root vector-functions is fulfilled in all cases, because it follows from inequality (1.2).

Let D^* be an operator formally associated to the operator D:

$$D^* = B\frac{d}{dx} + P^*(x).$$

Let $\{\vartheta_k(x)\}_{k=1}^{\infty}$ be a system biorthogonally associated to $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2^2(G)$ and consists of the root vector-functions of the operator D^*.

THEOREM 1.6. (On unconditional basicity) *Let $P(x) \in L_1(G)$, the lengths of the chains of root vector-functions be uniformly bounded, one of the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\vartheta_k(x)\}_{k=1}^{\infty}$ be complete in $L_2^2(G)$ and condition (1.1) be fulfilled. Then necessary and sufficient condition for unconditional basicity in $L_2^2(G)$ of each of these systems is the existence of constants M_1 and M_2 that ensure validity of inequality (1.2) and*

$$\|u_k\|_2\,\|\vartheta_k\|_2 \leqslant M_2, \ \ k = 1,2,\dots \tag{1.3}$$

Note that under the conditions of theorem 1.5 satisfaction of inequalities (1.2) and (1.3) is the necessary and sufficient condition for Riesz basicity of each of the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\vartheta_k(x)\,\|\vartheta_k\|_2^{-1}\right\}_{k=1}^{\infty}$ in $L_2^2(G)$.

THEOREM 1.7. (On basicity) *Let $P(x) \in L_1(G)$, the lengths of the chains of root vector-functions be uniformly bounded, condition (1.1)–(1.3) be fulfilled and the system $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ be quadratically close to some basis $\{\psi_k(x)\}_{k=1}^{\infty}$ of the space $L_2^2(G)$. Then the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\{\vartheta_k(x)\,\|u_k\|_2\}_{k=1}^{\infty}$ are the bases in $L_2^2(G)$, and they are equivalent to the basis $\{\psi_k(x)\}_{k=1}^{\infty}$ and its biorthogonally associated system, respectively.*

REMARK 1.8. Theorems 1.4, 1.5, and 1.6 remain valid also in the case when $P(x) \in L_1(G)$ is an arbitrary but not necessarily a diagonal matrix-function.

2. Some auxiliary statements

Cite some necessary statements that will be used when proving theorems 1.4–1.6.

STATEMENT 2.1. (see [16]) *If $p(x)$ and $q(x)$ belong to the class $L_1^{\ell oc}(G)$ and the points $x-t$, x, $x+t$ are in the domain G, then the following formulas are valid for the root vector-function $u_k(x)$:*

$$\begin{aligned} u_k(x\pm t) = (\cos\lambda_k t\cdot I \mp \sin\lambda_k t\cdot B)\,u_k(x) + \int_x^{x\pm t} (\sin\lambda_k\,(t-|\xi-x|)\,I \\ \pm\cos\lambda_k\,(t-|\xi-x|)\,B)\,[P(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]\,d\xi; \end{aligned} \tag{2.1}$$

$$\begin{aligned} u_k(x-t)+u_k(x+t) = 2u_k(x)\cos\lambda_k t + \int_{x-t}^{x+t} (\sin\lambda_k(t-|\xi-x|)I \\ +sgn(\xi-x)\cos\lambda_k(t-|x-\xi|)B)\,[P(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]\,d\xi, \end{aligned} \tag{2.2}$$

where I is a unit operator in E^2; $\theta_k=0$, if $u_k(x)$ is an eigen vector-function; $\theta_k=1$, if $u_k(x)$ is an associated vector-function, and in the last case it is assumed that $\lambda_k=\lambda_{k-1}$.

STATEMENT 2.2. (see [16]) *Let the functions $p(x)$ and $q(x)$ belong to the class $L_1(G)$. Then there exist the constants $G_i(n_k,G)$, $i=1,2,$independent of λ_k, such that the following estimates are valid*

$$\|\theta_k u_{k-1}\|_\infty \leqslant C_1\,(n_k,G)\,(1+|Im\lambda_k|)\,\|u_k\|_\infty, \tag{2.3}$$

$$\|u_k\|_\infty \leqslant C_2\,(n_k,G)\,(1+|Im\lambda_k|)^{\frac{1}{r}}\,\|u_k\|_r, \tag{2.4}$$

where n_k is the order of the associated vector-function u_k, $r \geqslant 1$.

3. Proof of the Bessel property criterion

In this section of the paper we will prove theorem 1.4 on Bessel property of root vector-function of Dirac's operator.

Necessity. Let us fix an arbitrary number $\tau \in (-\infty, +\infty)$ and introduce the indices set $J_\tau = \{k : |Re\lambda_k - \tau| \leqslant 1,\ |Im\lambda_k| \leqslant C_0\}$, where C_0 is a constant from condition (1.1). Choose a number $n_0 \geqslant 1$ so that $R = \left(n_0(1+C_0)^{\frac{3}{2}}\right)^{-1} \leqslant \frac{mesG}{4} = \frac{\pi}{2}$ and for any set $E \subset \bar{G}$, $mesE \leqslant 2R$, the following inequality be fulfilled:

$\omega(R) = \sup\limits_{E\subset\bar{G}} \left\{\|P\|_{1,E}\right\} \leqslant L^{-1}$, where $\|P\|_{1,E} = \int_E \{|p(x)| + |q(x)|\}\, dx$,

L is a positive number and choice of its value will be defined later.

Let $k \in J_\tau$, $x \in [0,\pi]$. Write the mean value formula (2.2) for the points x, $x+t$, $x+2t$, where $t \in [0,R]$:

$$\begin{aligned} u_k(x) = {} & 2u_k(x+t)\cos\lambda_k t - u_k(x+2t) + \int_x^{x+2t} (\sin\lambda_k(t-|x+t-\xi|)I \\ & + sgn(\xi - x - t)\cos\lambda_k(t-|x+t-\xi|)B)\,[P(\xi)u_k(\xi) - \theta_k u_{k-1}(\xi)]\, d\xi. \end{aligned}$$

Adding and subtracting $2u_k(x+t)\cos\tau t$ in the right hand side of this equality, and using the operation $R^{-1}\int_0^R dt$ we get

$$\begin{aligned} u_k(x) = {} & \frac{2}{R}\int_0^R u_k(x+t)\cos\tau t\,dt - \frac{1}{R}\int_0^R u_k(x+2t)\,dt \\ & + \frac{4}{R}\int_0^R u_k(x+t)\sin\frac{\lambda_k+\tau}{2}t\sin\frac{\tau-\lambda_k}{2}t\,dt \\ & + \frac{1}{R}\int_0^R\int_x^{x+2t} (\sin\lambda_k(t-|x+t-\xi|)I \\ & + sgn(\xi-x-t)\cos\lambda_k(t-|x+t-\xi|)B)\,[P(\xi)u_k(\xi) - \theta_k u_{k-1}(\xi)]\, d\xi\, dt. \end{aligned}$$

Having used formula (2.1) in the third addend, we have

$$\begin{aligned} u_k(x) = {} & R^{-1}\int_0^{2\pi} u_k(t)w(t)dt + 4R^{-1}\int_0^R (\cos\lambda_k tI - \sin\lambda_k tB) \\ & \times \sin\frac{\lambda_k+\tau}{2}t\sin\frac{\tau-\lambda_k}{2}t\,dt\,u_k(x) + 4R^{-1}\cdot\int_0^R\int_x^{x+t}(\sin\lambda_k(t-|x+t-\xi|)I \\ & + \cos\lambda_k(t-|\xi-x|)\,B)\cdot[P(\xi)u_k(\xi) - \theta_k u_{k-1}(\xi)]\,d\xi\sin\frac{\tau+\lambda_k}{2}t\sin\frac{\tau-\lambda_k}{2}t\,dt \\ & + R^{-1}\int_0^R\int_x^{x+2t}(\sin\lambda_k(t-|x+t-\xi|)I \\ & + \cos\lambda_k(t-|x+t-\xi|)B)\,[P(\xi)u_k(\xi) - \theta_k u_{k-1}(\xi)]\,d\xi\,dt \\ = {} & R^{-1}\int_0^{2\pi} u_k(t)w(t)\,dt + I_1 + I_2 + I_3 \end{aligned} \tag{3.1}$$

where $w(t) = 2\cos\tau(x-t) - \frac{1}{2}$ for $x \leqslant t \leqslant x+R$, $w(t) = -\frac{1}{2}$ for $x+R < t \leqslant x+2R$ and $w(t) = 0$ for $t \notin [x, x+2R]$.

Taking $k \in J_\tau$ into account and using the inequalities $|\sin z| \leqslant 2$, $|\cos z| \leqslant 2$ for $|Imz| \leqslant 1$ and $|\sin z| \leqslant 2|z|$ for $|Imz| \leqslant 1$ we find

$$\begin{aligned}
|I_1| &\leqslant 16R\,|\tau-\lambda_k|\,|u_k(x)| \leqslant 16R\,(1+|Im\lambda_k|)\,|u_k(x)| \\
&\leqslant 16R(1+C_0)\,|u_k(x)| \leqslant 16n_0^{-1}\,|u_k(x)|\,; \\
|I_2| &\leqslant 64\left(\omega(R)\,\|u_k(x)\|_\infty + \frac{R}{2}\,\|\theta_k u_{k-1}\|_\infty\right); \\
|I_3| &\leqslant 8\,(\omega(R)\,\|u_k(x)\|_\infty + R\,\|\theta_k u_{k-1}\|_\infty)\,.
\end{aligned}$$

Taking these estimations into account in equality (3.1), we get the inequality

$$|u_k(x)| \leqslant R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| + 16n_0^{-1}\,|u_k(x)| + 72\omega(R)\,\|u_k\|_\infty + 40R\,\|\theta_k u_{k-1}\|_\infty\,.$$

Requiring $n_0 \geqslant 32$, hence we find that for $x \in [0,\pi]$ it is valid

$$|u_k(x)| \leqslant 2R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| + 144\omega(R)\,\|u_k\|_\infty + 80R\,\|\theta_k u_{k-1}\|_\infty\,. \tag{3.2}$$

In the same way, inequality (3.2) is proved in the case $x \in [\pi,2\pi]$. In this case the function $w(t)$ is defined in the following way: $w(t) = -\frac{1}{2}$ for $x-2R \leqslant t < x-R$, $w(t) = 2\cos v(x-t) - \frac{1}{2}$ for $x-R \leqslant t \leqslant x$, $w(t) = 0$ for $t \in [x-2R,x]$. Therefore, inequality (3.2) is fulfilled for any $x \in [0,2\pi]$. Having used estimations (2.3) and (2.4), allowing for $1+|Im\lambda_k| \leqslant 1+C_0$, from inequality (3.2) we have

$$\begin{aligned}
|u_k(x)| \leqslant 2R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| &+ \Big[144C_2\,(n_k,G)\,(1+C_0)^{\frac{1}{2}}\,\omega(R) \\
&+80R\theta_k C_1\,(n_k,G)\,c_2\,(n_k,G)\,(1+C_0)^{\frac{3}{2}}\Big]\,\|u_k\|_2\,.
\end{aligned}$$

By uniform boundedness of the length of the chain of the root vectors $C_2\,(n_k,G) \leqslant \gamma_1 = const$, $C_1\,(n_k,G)\,C_2\,(n_k,G) \leqslant \gamma_2 = const$. Consequently,

$$|u_k(x)| \leqslant 2R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| + \left[144\gamma_1(1+C_0)^{\frac{1}{2}}L^{-1} + 80\theta_k\gamma_2 n_0^{-1}\right]\|u_k\|_2$$

Hence, by inequalities $|\sum_{i=1}^m a_i|^2 \leqslant m\sum_{i=1}^m |a_i|^2$ we have

$$\begin{aligned}
|u_k(x)|^2\,\|u_k\|_2^{-2} \leqslant \frac{16}{R^2}&\left\{\left|\int_0^{2\pi} u_k^1(t)w(t)dt\right|^2\|u_k\|_2^{-2} + \left|\int_0^{2\pi} u_k^2(t)w(t)dt\right|^2\|u_k\|_2^{-2}\right\} \\
&+4\left\{\left(144\gamma_1(1+C_0)^{\frac{1}{2}}L^{-1}\right)^2 + \left(80\theta_k\gamma_2 n_0^{-1}\right)^2\right\},
\end{aligned}$$

where $u_k(t) = \left(u_k^1(t), u_k^2(t)\right)^T$.

Having applied the Bessel inequality, we get that for any finite set $J \subset J_\tau$ the following inequality is fulfilled

$$\sum_{k\in J} |u_k(x)|^2 \|u_k\|_2^{-2} \leqslant \frac{32M}{R^2} \|w\|_2^2 + 4\left\{\left(144\gamma_1(1+C_0)^{\frac{1}{2}}L^{-1}\right)^2 + \left(80\theta_k\gamma_2 n_0^{-1}\right)^2\right\}\sum_{k\in J} 1. \tag{3.3}$$

Taking into account $\|w\|_2^2 = O(R)$ and chosing the number n_0 so large that

$$4\left\{\left(144\gamma_1(1+C_0)^{\frac{1}{2}}L^{-1}\right)^2 + \left(80\gamma_2 n_0^{-1}\right)^2\right\} \leqslant \frac{1}{4\pi}$$

be fulfilled, from (3.3) we find

$$\sum_{k\in J} 1 \leqslant const R^{-1} = const n_0 (1+C_0)^{\frac{3}{2}} = const.$$

By arbitrariness of the set $J \subset J_\tau$, hence we get inequality (1.2). The part of necessity of theorem 1.1 is proved.

Sufficiency. Writing the shift formula (2.1) for $u_k(x+t)$ for $x=0$ and scalarly multiplying by the vector-function $f(t) = (f_1(t), f_2(t))^T \in L_2^2(0,2\pi)$, we conclude that for the Bessel inequality to be fulfilled for the system $\varphi_k(t) = u_k(t)\|u_k\|_2^{-1}$, $k = 1,2,\ldots$, it suffices to set up validity of the following inequalities:

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\cos\lambda_k t\,dt\right|^2 \left|\varphi_k^i(0)\right|^2 \leqslant C\|f\|_2^2, \quad i=1,2; \tag{3.4}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\sin\lambda_k t\,dt\right|^2 \left|\varphi_k^{3-i}(0)\right|^2 \leqslant C\|f\|_2^2, \quad i=1,2; \tag{3.5}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t p(\xi)\varphi_k^1(\xi)\sin\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2; \tag{3.6}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t q(\xi)\varphi_k^2(\xi)\cos\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2; \tag{3.7}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\int_0^t p(\xi)\varphi_k^1(\xi)\cos\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2; \tag{3.8}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\int_0^t q(\xi)\varphi_k^2(\xi)\sin\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2; \tag{3.9}$$

$$\sum_{k=1}^{\infty}\left|\theta_k\int_0^{2\pi}\overline{f_i(t)}\int_0^t \frac{u_{k-1}^i(\xi)}{\|u_k\|_2}\sin\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2, \quad i=1,2; \tag{3.10}$$

$$\sum_{k=1}^{\infty}\left|\theta_k\int_0^{2\pi}\overline{f_i(t)}\int_0^t \frac{u_{k-1}^{3-i}(\xi)}{\|u_k\|_2}\cos\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \leqslant C\|f\|_2^2, \quad i=1,2; \tag{3.11}$$

where $\varphi_k^i(\xi) = u_k^i(\xi)\,\|u_k\|_2^{-1}$.

Prove estimation (3.4). As by estimation (2.4) and conditions (1.1), (1.2)

$$\left|\varphi_k^i(0)\right| = \left|u_k^i(0)\right|\,\|u_k\|_2^{-1} \leqslant \|u_k\|_\infty\,\|u_k\|_2^{-1}$$
$$\leqslant C_2(n_k,G)(1+C_0)^{\frac{1}{2}}\,\|u_k\|_2\,\|u_k\|_2^{-1} = C_2(n_k,G)(1+C_0)^{\frac{1}{2}} \leqslant const,$$

as by (1.2) the quantity $C_2(n_k,G)$ is bounded, then for validity of (3.4) it is sufficient the following inequality to be fulfilled

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\cos\lambda_k t\,dt\right|^q \leqslant C\,\|f\|_2^2. \tag{3.12}$$

This inequality was set up in the paper [8] provided $Re\lambda_k \geqslant 0$, (1.1) and (1.2) for $\tau \geqslant 1$. Hence inequality (3.12) follows for $Re\lambda_k \in (-\infty,+\infty)$, $|Im\lambda_k| \leqslant C_0$, as by the theorem condition (1.2) is fulfilled for any $\tau \in (-\infty,+\infty)$. Inequality (3.5) is set up just in the same way.

Now let us be convinced in validity of inequalities (3.6)–(3.9). They are proved in the same way. Therefore we prove only the inequality (3.6). Denote

$$g_i(t,\xi) = \begin{cases} f_i(t+\xi) & \text{for } 0 \leqslant t \leqslant 2\pi-\xi \\ 0, & \text{for } 2\pi-\xi < t \leqslant 2\pi, \end{cases} \quad i=1,2,$$

where $\xi \in [0,2\pi]$. Then by estimation (2.4) and conditions (1.1), (1.2), we have

$$\begin{aligned}
I_k &= \left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t p(\xi)\varphi_k^1(\xi)\sin\lambda_k(t-\xi)\,d\xi\,dt\right|^2 \\
&= \int_0^{2\pi}\overline{f_1(t)}\int_0^t p(\xi)\varphi_k^1(\xi)\sin\lambda_k(t-\xi)\,d\xi\,dt \\
&\quad\times\int_0^{2\pi} f_1(t)\int_0^t \overline{p(\xi)}\,\overline{\varphi_k^1(\xi)}\,\overline{\sin\lambda_k(t-\xi)}\,d\xi\,dt \\
&= \int_0^{2\pi} p(\xi)\varphi_k^1(\xi)\int_0^{2\pi}\overline{g_1(t,\xi)}\sin\lambda_k t\,dt\,d\xi \\
&\quad\times\int_0^{2\pi}\overline{p(\tau)}\,\overline{\varphi_k^1(\tau)}\int_0^{2\pi} g_1(r,\tau)\,\overline{\sin\lambda_k r}\,dr\,d\tau \\
&= \int_0^{2\pi}\int_0^{2\pi} p(\xi)\overline{p(\tau)}\varphi_k^1(\xi)\overline{\varphi_k^1(\tau)}\int_0^{2\pi}\overline{g_1(t,\xi)} \\
&\quad\times\sin\lambda_k t\,dt\cdot\int_0^{2\pi} g_1(r,\tau)\,\overline{\sin\lambda_k r}\,dr\,d\xi\,d\tau \\
&\leqslant C_2^2(n_k,G)(1+C_0)\int_0^{2\pi}\int_0^{2\pi}|p(\xi)|\,|p(\tau)| \\
&\quad\times\left|\int_0^{2\pi}\overline{g_1(t,\xi)}\sin\lambda_k t\,dt\right|\left|\int_0^{2\pi} g_1(r,\tau)\,\overline{\sin\lambda_k r}\,dr\right|d\xi\,d\tau \\
&\leqslant const\int_0^{2\pi}\int_0^{2\pi}|p(\xi)|\,|p(\tau)|\left|\int_0^{2\pi}\overline{g_1(t,\xi)}\sin\lambda_k t\,dt\right|\left|\int_0^{2\pi}\overline{g_1(r,\tau)}\sin\lambda_k r\,dr\right|d\xi\,d\tau.
\end{aligned}$$

Hence for an arbitrary natural number N we have

$$\sum_{k=1}^{N} I_k \leqslant const \int_0^{2\pi}\int_0^{2\pi} |p(\xi)|\,|p(\tau)|$$
$$\times \left(\sum_{k=1}^{N} \left|\int_0^{2\pi} \overline{g_1(t,\xi)} \sin\lambda_k t dt\right| \left|\int_0^{2\pi} \overline{g_1(r,\tau)} \sin\lambda_k r dr\right|\right) d\xi d\tau$$
$$\leqslant const \int_0^{2\pi}\int_0^{2\pi} |p(\xi)|\,|p(\tau)|\,\|g_1(\cdot,\xi)\|_2\,\|g_1(\cdot,\tau)\|_2\, d\xi d\tau.$$

Taking into account that $\|g_1(\cdot,\xi)\|_2 \leqslant \|f_1\|_2$ is fulfilled for any fixed $\xi \in [0,2\pi]$, we get $\sum\limits_{k=1}^{N} I_k \leqslant const\,\|p\|_1^2\,\|f_1\|_2^2 \leqslant const\,\|f\|_2^2$.

Hence, the validity of inequality (3.5) follows from arbitrariness of the number N.

Prove inequality (3.10). By estimations (2.3), (2.4) and conditions (1.1), (1.2)

$$\theta_k \left|u_{k-1}^i(\xi)\right| \|u_k\|_2^{-1} \leqslant \theta_k C_1(n_k,G)\, C_2(n_k,G)\,(1+C_0)^{\frac{3}{2}}$$

$$\|u_k\|_2\,\|u_k\|_2^{-1} = \theta_k C_1(n_k,G)\, C_2(n_k,G) \leqslant C = const.$$

After changing the order of integration, the left hand side of (3.10) with regard to the last inequality is majorized from above with the series

$$C\sum_{k=1}^{\infty}\int_0^{2\pi}\left|\int_0^{2\pi}\overline{g_i(t,\xi)}\sin\lambda_k t dt\right|^2 d\xi.$$

This series converges and its some does not exceed the value $C\|f\|_2^2$. The validity of inequality (3.10) is proved. Inequality (3.11) is proved just in the same way. Theorem 1.4 is completely proved. □

4. Proof of theorems 1.2 and 1.3

Proof of theorem 1.5. Necessity. Let the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\vartheta_k(x)\}_{k=1}^{\infty}$ form an unconditional basis in $L_2^2(G)$. According to Lorch's theorem [22] the systems, $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\vartheta_k(x)\,\|\vartheta_k\|_2^{-1}\right\}_{k=1}^{\infty}$ are the Riesz basis in $L_2^2(G)$. Then by N. K. Barri's theorem [6, pp. 347–375] the series

$$\sum_{k=1}^{\infty}|(f,u_k)|^2\,\|u_k\|_2^{-2}, \quad \sum_{k=1}^{\infty}|(g,\vartheta_k)|^2\,\|\vartheta_k\|_2^{-2}$$

converge for any $f(x), g(x)$ from the class $L_2^2(G)$. Since the Bessel property of the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\vartheta_k(x)\,\|\vartheta_k\|_2^{-1}\right\}_{k=1}^{\infty}$ follows from convergence of these series (see [9], pp. 433–435), then necessity of condition (1.2) follows from theorem 1.4. The necessity of condition (1.3) is known well not only for unconditional but for ordinary basicity of the arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$ as well (see [6], p. 370).

Sufficiency. Let conditions (1.1)–(1.3) be fulfilled, and the system $\{u_k(x)\}$ be complete in $L_2^2(G)$. By theorem 1.4, the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\vartheta_k(x)\,\|\vartheta_k\|_2^{-1}\right\}_{k=1}^{\infty}$ are Bessel in $L_2^2(G)$.

As conditions (1.3) are fulfilled, then for arbitrary $g(x)\in L_2^2(G)$

$$\sum_{k=1}^{\infty}|(g,\vartheta_k\,\|u_k\|_2)|^2\leqslant M_2\sum_{k=1}^{\infty}\left|(g,\vartheta_k\,\|\vartheta_k\|_2^{-1})\right|^2\leqslant const\,\|g\|_2^2.$$

Therefore, the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\{\vartheta_k(x)\,\|u_k\|_2\}_{k=1}^{\infty}$ are Bessel in $L_2^2(G)$. Prove the unconditional basicity of the system $\{u_k(x)\}_{k=1}^{\infty}$. Let $P_nf=\sum\limits_{k=1}^{n}(f,\vartheta_k)\,u_k(x)$, $n=1,2,\ldots$. Then

$$\begin{aligned}\|P_nf\|_2&=\sup_{\|g\|_2=1}|(P_nf,g)|=\sup_{\|g\|_2=1}\left|\sum_{k=1}^{n}|(f,\vartheta_k)\,(u_k,g)|\right|\\&\leqslant\sup_{\|g\|_2=1}\sum_{k=1}^{n}|(f,\vartheta_k\,\|u_k\|_2)|\left|\left(g,u_k\,\|u_k\|^{-1}\right)\right|\leqslant const\,\|f\|_2.\end{aligned}$$

Consequently, the sequence $\{P_nf\}$ is uniformly bounded. Then by the theorem on bases (see [10, p. 11]), the system $\{u_k(x)\}_{k=1}^{\infty}$ is a basis in $L_2^2(G)$. It will be an unconditional basis as well by the Bessel property of the systems $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\{\vartheta_k(x)\,\|u_k\|_2\}_{k=1}^{\infty}$ in $L_2^2(G)$. The system $\{\vartheta_k(x)\}_{k=1}^{\infty}$ is also an unconditional basis in $L_2^2(G)$, because it is a system biorthogonally associated to $\{u_k(x)\}_{k=1}^{\infty}$ Theorem 1.5 is proved. □

Proof of theorem 1.6. As the system $\{\vartheta_k(x)\}_{k=1}^{\infty}$ consists of root vector-functions of the operator $D^*=B\frac{d}{dx}+P^*(x)$, then by theorem 1.4 conditions (1.1) and (1.2) provide Bessel property of the system $\left\{\vartheta_k(x)\,\|\vartheta_k\|_2^{-1}\right\}_{k=1}^{\infty}$ in $L_2^2(G)$, i.e.

$$\sum_{k=1}^{\infty}\left|\left(f,\vartheta_k\,\|\vartheta_k\|_2^{-1}\right)\right|^2\leqslant M\,\|f\|_2^2 \tag{4.1}$$

for any vector-function $f(x)\in L_2^2(G)$.

From inequality (4.1), condition (1.3) and guadratical closeness of the system $\left\{u_k\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ it follows that the seties

$$\sum_{k=1}^{\infty}\tilde f_k\,\|u_k\|_2\,\|\vartheta_k\|_2\left(u_k(x)\,\|u_k\|_2^{-1}-\psi_k(x)\right)$$

converges in $L_2^2(G)$ for any $f(x)\in L_2^2(G)$, where $\tilde f_k=\left(f,\vartheta_k\,\|\vartheta_k\|_2^{-1}\right)$. Let us denote the sum of this series by Kf, where K is some linear operator acting in $L_2^2(G)$. This follows from fundamental property of the sequence

$$K_n f = \sum_{k=1}^{n} \tilde{f}_k \|u_k\|_2 \|\vartheta_k\|_2 \left(u_k(x) \|u_k\|_2^{-1} - \psi_k(x) \right)$$

in the space $L_2^2(G)$.

Obviously, the relation $\|Kf - K_n f\|_2 = o(1)\|f\|_2$ is fulfilled for any $f(x) \in L_2^2(G)$. Consequently, the sequence of finite-dimensional operators K_n converges to the operator K, i.e. $\|K - K_n\|_{L_2^2 \to L_2^2} \to 0 \;\; n \to \infty$. Hence, it follows the compactness of the operator K in $L_2^2(G)$.

It is clear that $K u_k \|u_k\|_2^{-1} = u_k \|u_k\|_2^{-1} - \psi_k$, i.e. $(I-K) u_k \|u_k\|_2^{-1} = \psi_k$, $k = 1,2,\ldots$.

Show that the operator $I-K$ is continuously inversible. As K is a compact operator, then from the Fredholm alternative it follows that if the operator $I-K$ is irreversible, then there exists a nonzero element $g \in L_2^2(G)$ such that $(I-K)^* g = 0$. This element g satisfies the relation

$$(g,\psi_k) = \left(g,(I-K)u_k \|u_k\|_2^{-1}\right) = \left((I-K)^* g, u_k \|u_k\|_2^{-1}\right) = 0, \;\; k = 1,2,\ldots.$$

Hence, by the basicity of the system $\{\psi_k(x)\}_{k=1}^{\infty}$ in $L_2^2(G)$ it follows that $g \equiv 0$. The obtained contradiction shows that the operator $I-K$ is reversible. Consequently, the system $\left\{u_k(x)\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ is a basis in $L_2^2(G)$ equivalent to the basis $\{\psi_k(x)\}_{k=1}^{\infty}$.

If by $\{z_k(x)\}_{k=1}^{\infty}$ we denote a system biorthogonal to $\{\psi_k(x)\}_{k=1}^{\infty}$, then $\vartheta_k(x)\|u_k\|_2 = (I-K)^* z_k(x)$. Hence it follows that the system $\{\vartheta_k(x)\|u_k\|_2\}_{k=1}^{\infty}$ is a basis equivalent to the basis $\{z_k(x)\}_{k=1}^{\infty}$ in $L_2^2(G)$.

Theorem 1.6 is proved. □

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