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THE SPECTRUM AND FINE SPECTRUM OF GENERALIZED RHALY–CESÀRO MATRICES ON c_0 AND c

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Abstract. The generalized Rhaly Cesàro matrices A_α are the triangular matrix with nonzero entries $a_{nk} = \alpha^{n-k}/(n+1)$ with $\alpha \in [0,1]$. In [Proc. Amer. Math. Soc. 86 (1982), 405-409], Rhaly determined boundedness, compactness of generalized Rhaly Cesàro matrices on ℓ_2 Hilbert space and shown that its spectrum is $\sigma(A_\alpha, \ell_2) = \{1/n\} \cup \{0\}$. Also in [32], lower bounds for these classes were obtained under certain restrictions on ℓ_p by Rhoades. In this paper, boundedness, compactness, spectra, the fine spectra and subdivisions of the spectra of generaled Rhaly Cesàro operator on c_0 and c have been determined.

1. Introduction

Let $x = (x_n)$, $y = (y_n)$ be complex sequences. The generalized Rhaly-Cesàro transform $A_\alpha x = y$ of a sequence $x = (x_k)$ is defined by

$$y_n = \frac{1}{n+1} \sum_{k=0}^{n} \alpha^{n-k} x_k, \; n = 0, 1, 2, \ldots \tag{1.1}$$

where $\alpha \in (0,1]$. It is clear that if $\alpha = 0$, then A_0 is a diagonal matrix and if $\alpha = 1$, then $A = C_1$ is Cesàro matrix. Boundness and spectrum on various sequences spaces of C_1 matrix were considered by several authors [21, 28]. Throughout the article we will get $\alpha \in (0,1)$.

In 1982, Rhaly [29] determined the spectrum of generalized Rhaly-Cesàro matrix A_α on the Hilbert space ℓ_2. The main purpose of this paper is to present boundness, compactness, spectrum, fine spectrum and subdivision of the spectrum of continuous linear operators on the spaces c_0 and c of all null and convergent sequences of complex numbers, respectively.

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2. Boundness of generalized Rhaly-Cesàro operator

In 1982, Rhaly [29] showed that generalized Rhaly-Cesàro operator A_α is a bounded linear operator on the Hilbert space ℓ_2. We will show that A_α is bounded linear operator on c_0 and c.

When $A=(a_{nk})$ is an infinite matrix, necessary and sufficient conditions for boundness of A on various sequence spaces were considered by several authors.

From [27], it is known that

$$A=(a_{nk})\in B(c_0)\Longleftrightarrow\begin{cases}\text{i) } \|A\|=\sup\limits_n\sum\limits_k|a_{nk}|<\infty\\ \text{ii) } \lim\limits_n a_{nk}=0\end{cases} \tag{2.1}$$

$$A=(a_{nk})\in B(c)\Longleftrightarrow\begin{cases}\text{i) } \|A\|=\sup\limits_n\sum\limits_k|a_{nk}|<\infty\\ \text{ii) } \lim\limits_n \sum\limits_{k=p}^{\infty} a_{nk}=a_p \text{ (for all fixed } p)\end{cases}. \tag{2.2}$$

Now, first let us show that generalized Rhaly-Cesàro matrix is bounded linear operator on sequence spaces c_0 and c and then calculate norm of this operator.

THEOREM 1. *$A_\alpha\in B(c_0)$ and $\|A_\alpha\|_{B(c_0)}=1$ for $\alpha\in(0,1)$.*

Proof. From (2.1), we have

i) $\|A_\alpha\|=\sup_n\sum_k|a_{nk}|=\sup_n\sum_{k=0}^n\left|\frac{\alpha^{n-k}}{n+1}\right|\leqslant\sup_n\frac{1}{n+1}\sum_{k=0}^n 1=1$ and

ii) $\lim_n a_{nk}=\lim_n\frac{\alpha^{n-k}}{n+1}=0$;i.e, $\|A_\alpha\|\leqslant 1$ and hence $A_\alpha\in B(c_0)$.

Then, since

$$\begin{aligned}\|A_\alpha\|&=\sup_{x\neq\theta}\frac{\|Ax\|_{c_0}}{\|x\|_{c_0}}=\sup_{x\neq\theta}\frac{\left\|\left(x_0,\frac{\alpha x_0+x_1}{2},\frac{\alpha^2x_0+\alpha x_1+x_2}{3},\frac{\alpha^3x_0+\alpha^2x_1+\alpha x_2+x_3}{4},\ldots\right)\right\|}{\|x\|}\\ &\geqslant\frac{\left\|\left(1,\frac{\alpha}{2},\frac{\alpha^2}{3},\ldots\right)\right\|_{c_0}}{1}=\sup_n\left|\frac{\alpha^n}{n+1}\right|=1,\end{aligned}$$

we obtain $\|A_\alpha\|_{B(c_0)}=1$. □

THEOREM 2. *$A_\alpha\in B(c)$ and $\|A_\alpha\|_{B(c)}=1$ for $\alpha\in(0,1)$.*

Proof. It is similar to the proof of the previous Theorem. □

3. Compactness of generalized Rhaly-Cesàro operator

Compact linear operators have a great deal with application in practice. For instance, they play a central role in the theory of integral equations and in various problems of mathematical physics.

Theory of compact linear operators served as a model for the early work in functional analysis. Their properties closely resemble those of operators on finite dimensional spaces. For a compact linear operator, spectral theory can be treated fairly completely in the sense that Fredholm's famous theory of linear integral equations may be extended to linear functional equations $Tx-\lambda x=y$ with a complex parameter λ. This generalized theory is called the *Riesz-Schauder theory*.

Definition of compact linear operator is as follows;

Let X and Y be normed spaces. An operator $T:X\to Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact, that is, the closure $\overline{T(M)}$ is compact.

From the definition of compactness of a set we readily obtain a useful criterion for operators:

THEOREM 3. [25] *Let X and Y be normed spaces and $T:X\to Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.*

The compact linear operators from X into Y form a vector space.

Furthermore, the following Theorem also implies that certain simplifications take place in the finite dimensional case:

THEOREM 4. [25] *Let X and Y be normed spaces and $T:X\to Y$ a linear operator. Then:*

(a) If T is bounded and $\dim T(X)<\infty$, the operator T is compact.

(b) If $\dim X<\infty$, the operator T is compact.

THEOREM 5. [25] *Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, then the limit operator T is compact.*

The compactness of the Rhaly operator was discussed in [38], [39], [40]. In 1982, Rhaly [29] showed that generalized Rhaly-Cesàro operator A_α on the Hilbert space ℓ_2 were compact linear operator. Our aim is to show that A_α is compact linear operator on c_0 and c.

THEOREM 6. *A_α is compact on c_0 for $\alpha\in(0,1)$.*

Proof. Let

$$A_\alpha^{(r)}(x):=\left(x_0,\frac{1}{2}(\alpha x_0+x_1),\frac{1}{3}(\alpha^2x_0+\alpha x_1+x_2),\ldots,\frac{1}{r+1}\sum_{k=0}^{r}\alpha^{r-k}x_k,0,0,\ldots\right).$$

Since $\dim(A_\alpha^r(c_0))=r+1<\infty$ for all $r\in\mathbb{N}$, from Theorem 4, A_α^r is compact linear operator on c_0 for all $r\in\mathbb{N}$. For each $x\in c_0$, we have

$$\begin{aligned}
\left\|\left(A_\alpha - A_\alpha^{(r)}\right)(x)\right\|_{c_0} &= \left\|\left(0,0,\dots,\frac{1}{r+2}\sum_{k=0}^{r+1}\alpha^{n-r-1}x_k,\frac{1}{r+3}\sum_{k=0}^{r+2}\alpha^{n-r-2}x_k,\dots\right)\right\|_{c_0} \\
&= \sup_{n\geqslant r}\left|\frac{1}{n+1}\sum_{k=0}^{n}\alpha^{n-k}x_k\right| \leqslant \left(\sup_{n\geqslant r}\frac{1}{n+1}\sum_{k=0}^{n}\alpha^{n-k}\right)\|x\|_{c_0} \\
&= \sup_{n\geqslant r}\frac{1}{n+1}\left(\alpha^n+\alpha^{n-1}+\dots+\alpha+1\right)\|x\|_{c_0} \\
&= \|x\|_{c_0}\sup_{n\geqslant r}\frac{1}{n+1}\frac{1-\alpha^{n+1}}{1-\alpha} \\
&= \frac{\|x\|_{c_0}}{1-\alpha}\sup_{n\geqslant r}\left(\frac{1-\alpha^{n+1}}{n+1}\right) \longrightarrow 0, \text{ as } r\to\infty.
\end{aligned}$$

Hence

$$\left\|A_\alpha - A_\alpha^{(r)}\right\| \leqslant \sup_{x\neq\theta}\frac{\left\|\left(A_\alpha - A_\alpha^{(r)}\right)(x)\right\|_{c_0}}{\|x\|_{c_0}} \leqslant \frac{1}{1-\alpha}\sup_{n\geqslant r}\frac{1-\alpha^{n+1}}{n+1} \longrightarrow 0, \text{ as } r\to\infty.$$

Therefore

$$A_\alpha^{(r)} \longrightarrow A_\alpha, \text{ as } r\to\infty \text{ (U.O.C)}$$

and from Theorem 5 A_α is compact linear operator on c_0. □

THEOREM 7. *A_α is compact on c for $\alpha\in(0,1)$.*

Proof. It is similar to the proof of the previous Theorem. □

4. Spectrum of generalized Rhaly-Cesàro operator

Let $X\neq\{0\}$ be a complex normed space and $T:D(T)\to X$ a linear operator with domain $D(T)\subset X$. A complex number λ that satisfies the conditions

(Rl) $R_\lambda(T) := T_\lambda^{-1} := (T-\lambda I)^{-1}$ resolvent operator exists,

(R2) $R_\lambda(T)$ is bounded, and

(R3) $R_\lambda(T)$ is defined on a set which is dense in X.

is called a regular value of T.

$$\rho(T) := \{\lambda\in\mathbb{C} : \lambda \text{ is a regular values of } T\}$$

is called resolvent set of T. $\sigma(T)=\mathbb{C}-\rho(T)$ is called spectrum set of T.

Furthermore, the spectrum $\sigma(T)$ is divided into three disjoint sets, some of them may be empty, as follows.

- The point spectrum or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda\in\sigma_p(T)$ is called an eigenvalue of T.

- The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and satisfies (R3) but not (R2), that is, $R_\lambda(T)$ is unbounded.
- The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of $R_\lambda(T)$ is not dense in X.

Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking, it is concerned with certain inverse operators, their general properties and their relations to the original operators. Such inverse operators arise quite naturally in connection with the problem of solving equations (systems of linear algebraic equations, differential equations, integral equations). For instance, the investigations of boundary value problems by Sturm and Liouville and Fredholm's famous theory of integral equations were important to the development of the field. For more information on spectrum, see [25].

The following theorem tells us that the point spectrum of a compact linear operator is not complicated. In fact, it is known from the following theorem that every nonzero spectral value of a compact linear operator is an eigenvalue. The spectrum of a compact linear operator largely resembles the spectrum of an operator on a finite dimensional space.

THEOREM 8. [25] *The set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable (perhaps finite or even empty), and the only possible point of accumulation is $\lambda = 0$. Every spectral value $\lambda \neq 0$ of T is an eigenvalue of T. However, if X is infinite dimensional, then $0 \in \sigma(T)$.*

4.1. Spectrum of generalized Rhaly-Cesàro operator on c_0

Spectrum of compact Rhaly operator was specified in [38], [39] and [40]. The spectrum of generalized Rhaly-Cesàro operator A_α on the Hilbert space ℓ_2 was examined by Rhaly [29] in 1982. Now we determine spectrum of A_α on c_0.

THEOREM 9. $\sigma_p(A_\alpha, c_0) = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} =: S$ *for* $0 < \alpha < 1$.

Proof. Let

$$A_\alpha x = \lambda x \Longleftrightarrow \begin{cases} x_0 & = \lambda x_0 \\ \frac{1}{2}(\alpha x_0 + x_1) & = \lambda x_1 \\ \frac{1}{3}\left(\alpha^2 x_0 + \alpha x_1 + x_2\right) & = \lambda x_2 \\ \frac{1}{4}\left(\alpha^3 x_0 + \alpha^2 x_1 + \alpha x_2 + x_3\right) & = \lambda x_3 \\ & \vdots \\ \frac{1}{n+1}\left(\sum_{k=0}^{n} \alpha^{n-k} x_k\right) & = \lambda x_n \\ & \vdots \end{cases}. \tag{4.1}$$

i) From (4.1), we have $(1-\lambda)x_0 = 0$. If $x_0 \neq 0$, then $\lambda = 1$. From (4.1), we get

$$\begin{aligned}
&\tfrac{1}{2}(\alpha x_0 + x_1) = x_1 &&\Rightarrow \tfrac{1}{2}\alpha x_0 = \tfrac{1}{2}x_1 \Rightarrow x_1 = \alpha x_0\\
&\tfrac{1}{3}\left(\alpha^2 x_0 + \alpha x_1 + x_2\right) = x_2 &&\Rightarrow \tfrac{2}{3}\alpha^2 x_0 = \tfrac{2}{3}x_2 \Rightarrow x_2 = \alpha^2 x_0\\
&&&\vdots\\
&\tfrac{1}{n+1}\left(\sum_{k=0}^{n} \alpha^{n-k} x_k\right) = x_n &&\Rightarrow x_n = \alpha^n x_0,\ x_0 \neq 0,\ 0 < \alpha < 1.
\end{aligned}$$

Hence we take $x_0 = 1$. Since

$$\left|\frac{x_{n+1}}{x_n}\right| \to |\alpha| < 1,$$

The series $\sum_n |x_n|$ converges, therefore $x_n \longrightarrow 0$; i.e, $x = (x_n) \in c_0$. Therefore we have $\lambda = 1 \in \sigma_p(A_\alpha, c_0)$.

ii) In (4.1), let $x_0 = 0$. Then

$$\frac{1}{2}x_1 = \lambda x_1 \Rightarrow \left(\lambda - \frac{1}{2}\right)x_1 = 0 \Rightarrow \text{ if } x_1 \neq 0, \text{ then } \lambda = \frac{1}{2}.$$

Hence from (4.1), we have

$$x_n = n\alpha^{n-1}x_1,\ \ x_0 = 0,\ \ x_1 = 1,\ \ \alpha \in (0,1).$$

Since

$$\left|\frac{x_{n+1}}{x_n}\right| \to |\alpha| < 1,$$

the series $\sum_n |x_n|$ converges, therefore $x_n \longrightarrow 0$; i.e, $x = (x_n) \in c_0$. Hence, we get $\lambda = \frac{1}{2} \in \sigma_p(A_\alpha, c_0)$.

iii) If m is the smallest integer for which $x_m \neq 0$, since

$$\frac{1}{m+1}\left(\sum_{k=0}^{m} \alpha^{m-k} x_k\right) = \lambda x_m,\ \ x_m \neq 0$$

and $x_0 = x_1 = \cdots = x_{m-1} = 0$, we have

$$\frac{1}{m+1}x_m = \lambda x_m,\ \ x_m \neq 0$$

i.e,

$$\lambda = \frac{1}{m+1}.$$

Thus, the equation (4.1) becomes

$$\frac{1}{n+1}\left(\sum_{k=m}^{n} \alpha^{n-k} x_k\right) = \frac{1}{m+1}x_n \text{ for all } n > m$$

equation. Therefore, we have

$$x_{m+n} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}\alpha^n x_m, \text{ for all } n \geqslant 1, \ \ x_m \neq 0.$$

Hence, since

$$\left|\frac{x_{m+n+1}}{x_{m+n}}\right| = \frac{(m+1)(m+2)\cdots(m+n)(m+n+1)}{(m+1)(m+2)\cdots(m+n)(n+1)}|\alpha| \to |\alpha| < 1,$$

we have $\sum_n |x_n| < \infty$ and therefore $(x_{m+n}) \in c_0$. Then we get

$$\lambda = \frac{1}{m+1} \in \sigma_p(A_\alpha, c_0).$$

As a result, eigenvalues for each m are simple and $\frac{1}{m} \in \sigma_p(A_\alpha, c_0)$, i.e;

$$\sigma_p(A_\alpha, c_0) = \left\{\frac{1}{m} : m = 1, 2, \ldots\right\} = S. \quad \square$$

LEMMA 1. [34, p. 221–223] *Each bounded linear operator $T : X \longrightarrow X$ is determined by an infinite matrix of complex numbers, where $X = c_0, \ c, \ \ell_1$.*

We will use the following Lemma to find the adjoint of a linear transform on the c_0 sequence space.

LEMMA 2. [42, p. 266] *Let $T : c_0 \longmapsto c_0$ be a linear map and define $T^* : \ell_1 \longmapsto \ell_1$, by $T^*g = g \circ T$, $g \in c_0^* \cong \ell_1$, then T must be given with a matrix by Lemma 1, moreover, $T^* : \ell_1 \longmapsto \ell_1$ is transposed matrix of T.*

THEOREM 10. $\sigma_p(A_\alpha^*, c_0^* \cong \ell_1) = S$ *for* $0 < \alpha < 1$.

Proof. From Lemma 2, it is clear that the matrix of $(A_\alpha)^*$ is transpose of matrix A_α, i.e;

$$a_{nk}^* = \begin{cases} \frac{\alpha^{k-n}}{k+1} & , \ 0 \leqslant n \leqslant k \\ 0 & , \ n > k \end{cases}. \tag{4.2}$$

Let $A_\alpha^* x = \lambda x$. Since A_α^* is transpoze of A, for $n \geqslant 1$, we have

$$\begin{aligned} x_0 + \frac{\alpha}{2}x_1 + \frac{\alpha^2}{3}x_2 + \frac{\alpha^3}{4}x_3 + \cdots &= \lambda x_0 \\ \frac{1}{2}x_1 + \frac{\alpha}{3}x_2 + \frac{\alpha^2}{4}x_3 + \cdots &= \lambda x_1 \\ \frac{1}{3}x_2 + \frac{\alpha}{4}x_3 + \cdots &= \lambda x_2 \\ \frac{1}{4}x_3 + \cdots &= \lambda x_3 \\ &\vdots \end{aligned}$$

where $x \neq 0$. Thus, for all $n \geqslant 1$,

$$x_n = \frac{1}{\alpha^n} \frac{\left(\lambda - \frac{1}{n}\right)\left(\lambda - \frac{1}{n-1}\right)\cdots(\lambda - 1)}{\lambda^n} x_0 = \frac{1}{\alpha^n} \prod_{k=1}^{n} \left(1 - \frac{1}{k\lambda}\right) x_0 \tag{4.3}$$

is valid. Hence, for all $n \geqslant 1$, since the eigenvector corresponding to $\lambda = 1/n$ is

$$x = \left(1, -(n-1)/\alpha, -(n-1)/\alpha^2, , \ldots, -(n-1)/\alpha^{n-1}, 0, 0, \ldots\right) \in \ell_1,$$

we have $\lambda = 1/n \in \sigma_p(A_\alpha^*, \ell_1)$, i.e;

$$S = \{1/n : n \in \mathbb{N}\} \subset \sigma_p(A_\alpha^*, c_0^* \cong \ell_1).$$

Since

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{\alpha}\left|1 - \frac{1}{\lambda(n+1)}\right| \to \frac{1}{\alpha} > 1, \ \ (n \to \infty),$$

the series $\sum_n |x_n|$ is divergent, if $\lambda \notin \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. So there is no other eigenvalue, i.e; we have

$$\sigma_p(A_\alpha^*, \ell_1) = S = \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}. \quad \square$$

In this section, finally, we compute the spectrum of A_α over c_0.

THEOREM 11. $\sigma(A_\alpha, c_0) = S \cup \{0\}$ *for* $0 < \alpha < 1$.

Proof. Since $\dim c_0 = \infty$, $0 \in \sigma(A_\alpha, c_0)$ and A_α is compact linear operator from Theorem 6, if $\lambda \in \sigma(A_\alpha, c_0)$, then $\lambda \in \sigma_p(A_\alpha, c_0)$. Therefore, we have $\sigma(A_\alpha, c_0) = S \cup \{0\}$. $\square$

4.2. Spectrum of generalized Rhaly-Cesàro operator on c

In this section, we will examine the spectrum of operator A_α over c.

THEOREM 12. $\sigma_p(A_\alpha, c) = S$ *for* $0 < \alpha < 1$.

Proof. It is similar to the proof of the previous Theorem 9. $\square$

The following lemma is useful for finding the adjoint of a linear transformation on the sequence space c.

LEMMA 3. [43, p. 267] *If* $T : c \to c$ *is a linear transformation and* $T^* : \ell_1 \to \ell_1$, $T^* g = g \circ T$, $g \in c^* \cong \ell_1$, *then* T *and* T^* *have matrix representations, also* $T^* : \ell_1 \to \ell_1$

is given by

$$T^* = A^* = \begin{pmatrix} \chi(\lim A) & (\vartheta_n)_{n=0}^{\infty} \\ (a_k)_{k=0}^{\infty} & A^t \end{pmatrix}$$

$$= \begin{pmatrix} \chi(\lim A) & \vartheta_0 & \vartheta_1 & \vartheta_2 & \cdots \\ a_0 & a_{00} & a_{10} & a_{20} & \cdots \\ a_1 & a_{01} & a_{11} & a_{21} & \cdots \\ a_2 & a_{02} & a_{12} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$a_k = \lim_n a_{nk}$$

$$\chi(A) = \lim Ae - \sum_{k=0}^{\infty} \lim Ae_k = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk}$$

$$\vartheta_n = \chi(P_n \circ T) = (P_n \circ T)e - \sum_k a_{nk},$$

$$a_{nk} = P_n(T(e_k)) = (T(e_k))_n.$$

If $\chi(A) \neq 0$, then A is called co-regular matrix and if $\chi(A) = 0$, then A is called co-null matrix.

Let's now find the adjoint over c of the generalized Rhaly Cesáro matrix.

LEMMA 4. *For* $0 < \alpha < 1$*, adjoint of* A_α *on* c *is given by*

$$A_\alpha^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{\alpha}{2} & \frac{\alpha^2}{3} & \frac{\alpha^3}{4} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{\alpha}{3} & \frac{\alpha^2}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{\alpha}{4} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.4}$$

Proof. By Lemma 3, we have

$$\begin{aligned} \chi(A_\alpha) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} &= \lim Ae - \sum_{k=0}^{\infty} \lim Ae_k \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} = \lim_{n\to\infty} \frac{\alpha^n}{n+1}\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^n}\right) \\ &= \lim_{n\to\infty} \frac{\alpha^n}{n+1} \frac{1-\left(\frac{1}{\alpha}\right)^{n+1}}{1-\frac{1}{\alpha}} = \lim_{n\to\infty} \frac{1-\alpha^{n+1}}{(n+1)(1-\alpha)} \\ &= \frac{1}{1-\alpha} \lim_{n\to\infty} \frac{1-\alpha^{n+1}}{n+1} = 0 \end{aligned}$$

and so A_α is a co-null matrix. Also we get

$$a_k = \lim_n a_{nk} = \lim_{n\to\infty} \frac{\alpha^{n-k}}{n+1} = 0$$

and

$$
\begin{aligned}
\sum_{k=0}^{n} a_{nk} &= \sum_{k=0}^{n} \frac{\alpha^{n-k}}{n+1} = \frac{\alpha^n}{n+1}\sum_{k=0}^{n}\left(\frac{1}{\alpha}\right)^k \\
&= \frac{\alpha^n}{n+1}\left\{1+\frac{1}{\alpha}+\cdots+\left(\frac{1}{\alpha}\right)^n\right\} = \frac{\alpha^n}{n+1}\left\{\frac{1-\frac{1}{\alpha^{n+1}}}{1-\frac{1}{\alpha}}\right\} \\
&= \frac{\alpha^n}{n+1}\frac{\alpha}{\alpha-1}\left\{1-\frac{1}{\alpha^{n+1}}\right\} = \frac{1}{(\alpha-1)(n+1)}\left\{\alpha^{n+1}-1\right\}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
(P_n\circ A_\alpha)e &= \left\{\frac{\alpha^n x_0+\alpha^{n-1}x_1+\cdots+\alpha x_{n-1}+x_n}{n+1}\right\}_{x=e} \\
&= \frac{\alpha^n+\alpha^{n-1}+\cdots+\alpha+1}{n+1} = \frac{1}{n+1}\frac{1-\alpha^{n+1}}{1-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_n &= (P_n\circ A_\alpha)e-\sum_{k=0}^{n}a_{nk} = \frac{1}{n+1}\left\{\frac{1-\alpha^{n+1}}{1-\alpha}-\frac{1}{(\alpha-1)}\left(\alpha^{n+1}-1\right)\right\} \\
&= \frac{1}{(n+1)(1-\alpha)}\left\{1-\alpha^{n+1}+\alpha^{n+1}-1\right\} = 0.
\end{aligned}
$$

This proves Lemma. □

Now we can calculate the point spectrum of the adjoint of A_α on c.

THEOREM 13. *$\sigma_p(A_\alpha^*, c^*\cong\ell_1) = \left\{\frac{1}{n} : n=1,2,\ldots\right\}\cup\{0\}$ for $0<\alpha<1$.*

Proof. Let $x\neq 0$ and $A_\alpha^* x = \lambda x$; i.e;

$$
A_\alpha^* x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{\alpha}{2} & \frac{\alpha^2}{3} & \frac{\alpha^3}{4} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{\alpha}{3} & \frac{\alpha^2}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{\alpha}{4} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \\ \vdots \end{pmatrix} \tag{4.5}
$$

$$
\Longleftrightarrow \begin{cases} \lambda x_0 = 0 \\ \lambda x_n = \frac{1}{n}x_n+\frac{\alpha}{n+1}x_{n+1}+\frac{\alpha^2}{n+2}x_{n+2}+\cdots, \text{ for all n} \geqslant 1 \end{cases}
$$

is valid. If $0=\lambda x_0$ then we have $\lambda=0$ or $x_0=0$. We obtain $\lambda=0\in\sigma_p(A_\alpha^*, c^*\cong\ell_1)$ since the eigenvector corresponding to $\lambda=0$ is $x=(1,0,0,\ldots)$. From the second equation of (4.5), we get

$$\left.\begin{array}{r} x_1+\frac{\alpha}{2}x_2+\frac{\alpha^2}{3}x_3+\cdots=\lambda x_1 \\ \frac{1}{2}x_2+\frac{\alpha}{3}x_3+\frac{\alpha^2}{4}x_4+\cdots=\lambda x_2 \end{array}\right\} \Rightarrow x_2=\frac{\lambda-1}{\alpha\lambda}x_1$$

$$\left.\begin{array}{r} \frac{1}{2}x_2+\frac{\alpha}{3}x_3+\frac{\alpha^2}{4}x_4+\cdots=\lambda x_2 \\ \frac{1}{3}x_3+\frac{\alpha}{4}x_4+\cdots=\lambda x_3 \end{array}\right\} \Longrightarrow x_3=\frac{(\lambda-\frac{1}{2})(\lambda-1)}{(\alpha\lambda)^2}x_1 \tag{4.6}$$

$$\vdots$$

where $x_1 \neq 0$. From (4.6), we have

$$x_n=\frac{1}{\alpha^{n-1}}\frac{\left(\lambda-\frac{1}{n-1}\right)\ldots(\lambda-1)}{\lambda^{n-1}}x_1=\frac{1}{\alpha^{n-1}}\prod_{k=1}^{n-1}\left(1-\frac{1}{k\lambda}\right)x_1 \text{ for all } n>1 \tag{4.7}$$

where $x_1 \neq 0$. We obtain $\lambda=1\in\sigma_p\left(A_\alpha^*,c^*\cong\ell_1\right)$ since the eigenvector corresponding to $\lambda=1$ is $x=(x_0,x_1,0,0,\ldots)\in\ell_1$ where $x_1\neq 0$. From (4.7), for all $m\in\mathbb{N}$, $\lambda=\frac{1}{m}\in\sigma_p\left(A_\alpha^*,c^*\cong\ell_1\right)$, because $x_1\neq 0$ $x=(x_0,x_1,\ldots,x_m,0,0,\ldots)$, is response to $\lambda=\frac{1}{m}$ which is eigenvector where $x_k\neq 0$ for all $k=1,\ldots,m$. So, we obtain $S=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\subset\sigma_p\left(A_\alpha^*,c^*\cong\ell_1\right)$.

Do you have another eigenvalue? If $\lambda\neq 0$ and $\lambda\neq\frac{1}{m}$ for all $m\in\mathbb{N}$, then we have

$$\left|\frac{x_{n+1}}{x_n}\right|\overset{\lambda\neq\frac{1}{k}}{=}\frac{1}{\alpha}\left|1-\frac{1}{\lambda n}\right|\to\frac{1}{\alpha}>1,\ (n\to\infty)$$

i.e; there is no other $\lambda\in\mathbb{C}$ that makes $\sum_n|x_n|<\infty$. Therefore, we get

$$\sigma_p\left(A_\alpha^*,\ell_1\right)=S\cup\{0\}=\left\{\frac{1}{n}:n=1,2,\ldots\right\}\cup\{0\}. \qquad \square$$

THEOREM 14. $\sigma\left(A_\alpha,c\right)=S\cup\{0\}$ *for* $0<\alpha<1$.

Proof. The proof is done as the proof of Theorem 11. $\square$

THEOREM 15. [10] *If* $A\in B(c)$, *then* $\sigma(A,c)=\sigma(A,\ell_\infty)$.

COROLLARY 1. $\sigma\left(A_\alpha,\ell_\infty\right)=S\cup\{0\}$ *for* $0<\alpha<1$.

4.3. An application to the summability

In this section, let us prove a Mercerian theorem with the help of spectrum.

The convergence domain c_A of $A=(a_{nk})$ is defined by $c_A=\{x:Ax\in c\}$. If $a_{nk}=0$ for $n>k$, then A is called a triangle matrix. If $c_A=c$, then A is called a Mercerian matrix and if $c_A\subset c$, then A is called a conservative matrix.

THEOREM 16. [43] *A conservative triangle* A *is Mercerian iff* A^{-1} *is conservative.*

THEOREM 17. *Let* $\lambda \neq 0$ *and* $\lambda \neq \frac{1}{1-m} \in \mathbb{R}$, $(m = 0,1,2,\ldots)$. *If* $A := \lambda I + (1-\lambda)A_\alpha$, *then* $c_A = c$.

Proof. By hypothesis, we have $\frac{\lambda}{\lambda-1} \neq 0$ and $\frac{\lambda}{\lambda-1} \neq \frac{1}{1-m}$ $(m = 0,1,2,\ldots)$. If $\lambda = 1$, then it is clear that $c_A = c$. Let $\lambda \neq 1$. Since

$$A := \lambda I + (1-\lambda)A_\alpha = (\lambda-1)\left[\frac{\lambda}{\lambda-1}I - A_\alpha\right]$$

by Theorem 14, $\frac{\lambda}{\lambda-1} \notin \sigma(A_\alpha, c)$ and thus $\frac{\lambda}{\lambda-1} \in \rho(A_\alpha, c)$. Therefore, $\left[\frac{\lambda}{\lambda-1}I - A_\alpha\right]^{-1} \in B(c)$. Hence

$$A^{-1} = (\lambda-1)^{-1}\left[\frac{\lambda}{\lambda-1}I - A_\alpha\right]^{-1} = [\lambda I + (1-\lambda)A_\alpha]^{-1} \in B(c).$$

By Theorem 16, $c_A = c$. □

5. Fine spectrum

If X is a Banach space and $B(X)$ denotes the collection of all bounded linear operators on X and $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. For example, if an operator is in state III_2, then $\overline{R(T)} \neq X$ and T^{-1} exists and is discontinuous. From the closed graph theorem, I_2 is empty (see [20]).

Applying Goldberg's classification to the operator $T_\lambda := \lambda I - T$, where $\lambda \in \sigma(T,X)$ the spectrum of T, considered as an operator in $B(X)$ where $X = c_0$ or $X = c$, we have

(I) $T_\lambda = \lambda I - T$ is surjective

(II) $\overline{R(T_\lambda)} = X$, but $R(T_\lambda) \neq X$,

(III) $\overline{R(T_\lambda)} \neq X$

and three possibilities for T_λ^{-1}:

(1) $T_\lambda = \lambda I - T$ is injective and $T_\lambda^{-1} =: R_\lambda(T)$ is bounded,

(2) $T_\lambda = \lambda I - T$ is injective and T_λ^{-1} is unbounded, and

(3) $T_\lambda = \lambda I - T$ is not injective.

If λ is a complex number such that $T_\lambda = \lambda I - T \in I_1$ or $T_\lambda = \lambda I - T \in II_1$, then $\lambda \in \rho(T,X)$. All scalar values of λ not in $\rho(T,X)$ comprise the spectrum of T. The further classification of $\sigma(T,X)$ gives rise to the fine spectrum of T. That is, $\sigma(T,X)$ can be divided into the subsets $I_2\sigma(T,X)$, $I_3\sigma(T,X)$, $II_2\sigma(T,X)$, $II_3\sigma(T,X)$, $III_1\sigma(T,X)$, $III_2\sigma(T,X)$, $III_3\sigma(T,X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state III_2 (say), then we write $\lambda \in III_2\sigma(T,X)$.

We can summarize the above situation in a table as follows:

		1	2	3
		$R_\lambda(T)$ exists and is bounded	$R_\lambda(T)$ exists and is unbounded	$R_\lambda(T)$ does not exists
I	$R(\lambda I - T) = X$	$\lambda \in \rho(T)$	–	$\lambda \in \sigma_p(T)$
II	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T)$	$\lambda \in \sigma_c(T)$	$\lambda \in \sigma_p(T)$
III	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_p(T)$

Table 1: *Goldberg's decomposition of the spectrum*

This classification of the spectrum is called the Goldberg Classification. Let's give the theorems that will help the Goldberg Classification.

THEOREM 18. [20, p. 58] *If T^* has a bounded inverse, then $R(T^*)$ is closed.*

THEOREM 19. [20, p. 59] *T has a dense range if and only if T^* is 1-1.*

THEOREM 20. [20, p. 60] *$R(T^*) = X^*$ if and only if T has a bounded inverse.*

THEOREM 21. [20, p. 60] *$\overline{R(T)} = X$ and T has a bounded inverse if and only if $R(T^*) = X^*$ and T^* has a bounded inverse.*

The relationship between the fine spectrum of bounded linear operator and fine spectrum of its adjoint is given by Fig. 1.

The fine spectrum of the operators on some sequence spaces was first discussed in [11], [21], [30], [31], [38] and [42]. Later, many authors [2], [4], [5], [6], [8], [9], [12], [13], [19], [22], [23], [24], [26], [33], [35], [36], [37], etc. have made a fine division of the spectrum and the work on this subject is still ongoing.

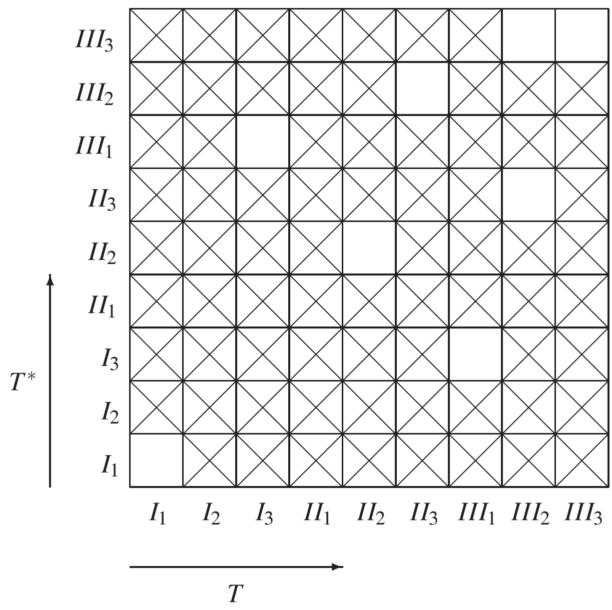

Figure 1. *State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X.*

5.1. The fine spectrum of generalized Rhaly-Cesàro matrices on c_0

We will examine the fine spectrum of the generalized Rhaly Cesàro operator on c_0, which is compact in this section.

THEOREM 22. $0 \in II_2\sigma(A_\alpha, c_0)$ *for* $0 < \alpha < 1$.

Proof. Since $\sigma_p(A_\alpha, c_0) = S$, we have $0 \notin \sigma_p(A_\alpha, c_0)$. Thus, there exists $(A_\alpha)^{-1}$. Therefore, $A_\alpha \in (1) \cup (2)$. Let us now show that $A_\alpha \in II$, that is, $\overline{R(A_\alpha)} = c_0$ and $R(A_\alpha) \neq c_0$. Since $\sigma_p(A_\alpha^*, c_0^* \cong \ell_1) = S$ and therefore since $0 \notin \sigma_p(A_\alpha^*, \ell_1)$, the operator A_α^* is 1-1. Thus, from Theorem 19, we get $\overline{R(A_\alpha)} = c_0$. Let us now show that $R(A_\alpha) \neq c_0$. If $A_\alpha x = y$ is solved, then we get

$$y_n = \frac{1}{n+1}\sum_{k=0}^{n} \alpha^{n-k} x_k.$$

Thus, we obtain

$$x_0 = y_0 \text{ and } x_n = (n+1)y_n - \alpha n y_{n-1}$$

from the equations

$$\begin{aligned}(n+1)y_n &= \alpha^n x_0 + \alpha^{n-1}x_1 + \cdots + \alpha x_{n-1} + x_n \\ \alpha n y_{n-1} &= \alpha\left(\alpha^{n-1}x_0 + \alpha^{n-2}x_1 + \cdots + x_{n-1}\right).\end{aligned}$$

Hence, $A_\alpha^{-1} = (b_{nk})$ matrix is given by

$$b_{nk} = \begin{cases} n+1 \ , & k=n \\ -\alpha n \ , & k=n-1 \\ 0 \ , & \text{otherwise} \end{cases} .$$

If we take $y = (y_n) = \left(\frac{(-1)^n}{n+1}\right) \in c_0$, then for all n, we get

$$(x_n) = \left((n+1)\frac{(-1)^n}{n+1} - (-1)^{n-1}\frac{n\alpha}{n} \right) = ((-1)^n(1+\alpha)).$$

Therefore, $x \notin c_0$. Thus, since $y = (y_n) \in c_0$, but $x = (x_n) \notin c_0$, A_α is not onto, that is, $R(A_\alpha) \neq c_0$. Hence $A_\alpha \in II$. As a result, $A_\alpha \in II_1$ or $A_\alpha \in II_2$. Since $0 \in \sigma(A_\alpha, c_0)$, we have $A_\alpha \notin II_1$. Then we get $A_\alpha \in II_2$, that is, $0 \in II_2\sigma(A_\alpha, c_0)$. □

THEOREM 23. *For all* $\lambda = \frac{1}{m}$, $m = (1,2,\ldots)$, $\lambda \in III_3\sigma(A_\alpha, c_0)$ *where* $0 < \alpha < 1$.

Proof. Since $\sigma_p(A_\alpha, c_0) = S$, $\lambda = \frac{1}{m} \in \sigma_p(A_\alpha, c_0) = S$ for all m. Therefore, $T_\lambda = (\lambda I - A_\alpha)$ has no inverse, i.e; we have $T \in (3)$. The adjoint operator $T^* = \lambda I - A_\alpha^*$ is not 1-1 for $\lambda = \frac{1}{m}$, because $\lambda = \frac{1}{m} \in \sigma_p(A_\alpha^*, c_0)$. From Theorem 19, $T_\lambda = \lambda I - A_\alpha$ does not have a dense image. Therefore, $\overline{R(T)} \neq c_0$; that is, $T_\lambda \in III$. Accordingly, $T_{\frac{1}{m}} = \frac{1}{m}I - A_\alpha \in III_3$ and $\lambda = \frac{1}{m} \in III_3\sigma(A_\alpha, c_0)$ are obtained. □

5.2. The fine spectrum of generalized Rhaly-Cesàro matrices on c

We will examine the fine spectrum of the generalized Rhaly Cesàro operator on c, which is compact in this section.

THEOREM 24. $0 \in III_2\sigma(A_\alpha, c)$ *for* $0 < \alpha < 1$.

Proof. For $0 < \alpha < 1$, adjoint of A_α on c is given by

$$A_\alpha^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{\alpha}{2} & \frac{\alpha^2}{3} & \frac{\alpha^3}{4} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{\alpha}{3} & \frac{\alpha^2}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{\alpha}{4} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

from Lemma 4. Thus, for $y = (1,0,0,\ldots)$, there is no $x \in \ell_1$ satisfying $A_\alpha^* x = y$; that is, A_α^* is not onto. Therefore, from Theorem 20, A_α has not a bounded inverse and thus $A_\alpha \in (2)$. On the other hand, the operator A_α^* is not 1-1, because $0 \in \sigma_p(A_\alpha^*, \ell_1)$. Thus, A_α does not have a dense range from Theorem 19, that is, $A_\alpha \in III$, and consequently $A_\alpha \in III_2$, and so $0 \in III_2\sigma(M, c)$. □

THEOREM 25. *For all* $\lambda = \frac{1}{m}$, $m = (1,2,\ldots)$, $\lambda \in III_3\sigma(A_\alpha, c)$ *where* $0 < \alpha < 1$.

Proof. The proof can be made in the way of Theorem 23. □

6. Subdivision of the spectrum of A_α

A bounded linear operator T in a Banach space X is given. Then, if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$ as $k \to \infty$, sequence $(x_k)_k$ in X is called a Weyl sequence for T.

In what follows, we call the set

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{K} : \text{there exists aWeyl sequence for } \lambda I - T\} \tag{6.1}$$

as the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T) := \{\lambda \in \sigma(T) : \lambda I - T \text{ is not surjective}\} \tag{6.2}$$

is called defect spectrum of T.

The two subspectra (6.1) and (6.2) form a (not necessarily disjoint) subdivision

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_\delta(T) \tag{6.3}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T) = \{\lambda \in \mathbb{K} : \overline{R(\lambda I - T)} \neq X\} \tag{6.4}$$

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{co}(T) \tag{6.5}$$

of the spectrum. Clearly, $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma_{co}(T) \subseteq \sigma_\delta(T)$. Moreover, we note that

$$\sigma_r(T) = \sigma_{co}(T) \backslash \sigma_p(T) \tag{6.6}$$

and

$$\sigma_c(T) = \sigma(T) \backslash [\sigma_p(T) \cup \sigma_{co}(T)] \tag{6.7}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint.

PROPOSITION 1. [7, Proposition 1.3] *The spectra and subspectra of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations:*

(a) $\sigma(T^*) = \sigma(T)$.
(b) $\sigma_c(T^*) \subseteq \sigma_{ap}(T)$.
(c) $\sigma_{ap}(T^*) = \sigma_\delta(T)$.
(d) $\sigma_\delta(T^*) = \sigma_{ap}(T)$.
(e) $\sigma_p(T^*) = \sigma_{co}(T)$.
(f) $\sigma_{co}(T^*) \supseteq \sigma_p(T)$.
(g) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*) = \sigma_p(T) \cup \sigma_{ap}(T^*)$.

We can write the above definition as the following table

- A lot of separation of the spectrum is possible. The non-discrete spectrum (Aporoximate point spectrum, defect spectrum and compression spectrum) can be found in the book entitled "Nonlinear Spectral Theory", published by J. Appell et al.
- This separation of an operator for the first time in the literature was handled in 2011 by Kh. Amirov and Nuh Durna, Mustafa Yıldırım [1].
- After these studies, this separation has been studied by various authors in [14], [15], [16], [17], [18], [19], [41] and is still being studied.

		(1) $R_\lambda(T)$ exists and is bounded	(2) $R_\lambda(T)$ exists and is unbounded	(3) $R_\lambda(T)$ does not exists
(I)	$R(\lambda I-T)=X$	$\lambda\in\rho(T)$	–	$\lambda\in\sigma_p(T)$ $\lambda\in\sigma_{ap}(T)$
(II)	$R(\lambda I-T)\neq X$ $\overline{R(\lambda I-T)}=X$	$\lambda\in\rho(T)$	$\lambda\in\sigma_c(T)$ $\lambda\in\sigma_{ap}(T)$ $\lambda\in\sigma_\delta(T)$	$\lambda\in\sigma_p(T)$ $\lambda\in\sigma_{ap}(T)$ $\lambda\in\sigma_\delta(T)$
(III)	$\overline{R(\lambda I-T)}\neq X$	$\lambda\in\sigma_r(T)$ $\lambda\in\sigma_\delta(T)$ $\lambda\in\sigma_{co}(T)$	$\lambda\in\sigma_r(T)$ $\lambda\in\sigma_{ap}(T)$ $\lambda\in\sigma_\delta(T)$ $\lambda\in\sigma_{co}(T)$	$\lambda\in\sigma_p(T)$ $\lambda\in\sigma_{ap}(T)$ $\lambda\in\sigma_\delta(T)$ $\lambda\in\sigma_{co}(T)$

Table 2: *Separations of the spectrum* ([1])

6.1. Subdivision of the spectrum of A_α on c_0

In this section, We will examine subdivision of the spectrum of the generalized Rhaly Cesàro operator on c_0.

THEOREM 26. *For* $0<\alpha<1$,
a) $\sigma_{ap}(A_\alpha,c_0)=S\cup\{0\}$
b) $\sigma_\delta(A_\alpha,c_0)=S\cup\{0\}$
c) $\sigma_{co}(A_\alpha,c_0)=S$.

Proof. a) Since $\sigma(A_\alpha, c_0) = S \cup \{0\}$ from Theorem 11, $III_3\sigma(A_\alpha, c_0) = S$ from Theorem 23 and $II_2\sigma(A_\alpha, c_0) = \{0\}$ from Theorem 22, we get $III_1\sigma(A_\alpha, c_0) = \emptyset$ from Table 2. Hence, we have

$$\sigma_{ap}(A_\alpha, c_0) = \sigma(A_\alpha, c_0) \diagdown III_1\sigma(A_\alpha, c_0) = S \cup \{0\}$$

from Table 2.

b) We have $I_3\sigma(A_\alpha, c_0) = \emptyset$ from Table 2, because $\sigma(A_\alpha, c_0) = S \cup \{0\}$, $III_3\sigma(A_\alpha, c_0) = S$ and $II_2\sigma(A_\alpha, c_0) = \{0\}$ from respectively Theorem 11, 23 and 22. Hence, we get

$$\sigma_\delta(A_\alpha, c_0) = \sigma(A_\alpha, c_0) \diagdown I_3\sigma(A_\alpha, c_0) = S \cup \{0\}$$

from Table 2.

c) Since $\sigma(A_\alpha, c_0) = S \cup \{0\}$, $III_3\sigma(A_\alpha, c_0) = S$ and $II_2\sigma(A_\alpha, c_0) = \{0\}$ from respectively Theorem 11, 23 and 22, we have $III_1\sigma(A_\alpha, c_0) = \emptyset$ from Table 2. Consequently,

$$\sigma_{co}(A_\alpha, c_0) = III_1\sigma(A_\alpha, c_0) \cup III_2\sigma(A_\alpha, c_0) \cup III_3\sigma(A_\alpha, c_0) = S$$

from Table 2. □

LEMMA 5. *For* $0 < \alpha < 1$,
a) $\sigma_{ap}\left(A_\alpha^*, \ell^1\right) = S \cup \{0\}$
b) $\sigma_\delta\left(A_\alpha^*, \ell^1\right) = S \cup \{0\}$.

Proof. Since $\sigma_{ap}\left(A_\alpha^*, \ell^1\right) = \sigma_\delta(A_\alpha, c_0)$ and $\sigma_\delta\left(A_\alpha^*, \ell^1\right) = \sigma_{ap}(A_\alpha, c_0)$ from Theorem 1, proof is clear. □

6.2. Subdivision of the spectrum of $\mathbf{A}_\alpha$ on c

In this section, We will examine subdivision of the spectrum of the generalized Rhaly Cesàro operator on c.

THEOREM 27. *For* $0 < \alpha < 1$,
a) $\sigma_{ap}(A_\alpha, c) = S \cup \{0\}$
b) $\sigma_\delta(A_\alpha, c) = S \cup \{0\}$
c) $\sigma_{co}(A_\alpha, c) = S \cup \{0\}$.

Proof. a) We have $III_1\sigma(A_\alpha, c) = \emptyset$ from Table 2, because $\sigma(A_\alpha, c) = S \cup \{0\}$, $III_3\sigma(A_\alpha, c) = S$ and $III_2\sigma(A_\alpha, c) = \{0\}$ from respectively Theorem 14, 25 and 24. Hence, we get

$$\sigma_{ap}(A_\alpha, c) = \sigma(A_\alpha, c) \diagdown III_1\sigma(A_\alpha, c) = S \cup \{0\}$$

from Table 2.

b) Since $\sigma(A_\alpha, c) = S \cup \{0\}$ from Theorem 14, $III_2\sigma(A_\alpha, c) = \{0\}$ from Theorem 24 and $III_3\sigma(A_\alpha, c) = S$ from Theorem 25, we get $I_3\sigma(A_\alpha, c) = \emptyset$. Hence,

$$\sigma_\delta(A_\alpha, c) = \sigma(A_\alpha, c) \setminus I_3\sigma(A_\alpha, c) = S \cup \{0\}$$

from Table 2.

c) Since $\sigma(A_\alpha, c) = S \cup \{0\}$, $III_2\sigma(A_\alpha, c) = \{0\}$ and $III_3\sigma(A_\alpha, c) = S$ from respectively Theorem 14, 24 and 25, we get $III_1\sigma(A_\alpha, c) = \emptyset$ from Table 2.

$$\sigma_{co}(A_\alpha, c) = III_1\sigma(A_\alpha, c) \cup III_2\sigma(A_\alpha, c) \cup III_3\sigma(A_\alpha, c) = S \cup \{0\}$$

from Table 2. As a result, $\sigma_{co}(A_\alpha, c) = S \cup \{0\}$ from Table 2. □

LEMMA 6. *For* $0 < \alpha < 1$,
a) $\sigma_{ap}\left(A_\alpha^*, c^* \simeq \ell^1\right) = S \cup \{0\}$
b) $\sigma_\delta\left(A_\alpha^*, c^* \simeq \ell^1\right) = S \cup \{0\}$.

Proof. Since $\sigma_{ap}\left(A_\alpha^*, c^* \simeq \ell^1\right) = \sigma_\delta(A_\alpha, c)$ and $\sigma_\delta\left(A_\alpha^*, c^* \simeq \ell^1\right) = \sigma_{ap}(A_\alpha, c)$ from Theorem 1, proof is clear. □

7. Conclusions

The spectra of summability methods and the Goldberg classification of the spectrum and the non-discrete spectral separation of these summability methods were discussed by various authors earlier. Still, a lot of mathematicians work on this subject. Discrete generalized Cesàro operators's spectrum on Hilbert space ℓ_2 was calculated by Rhaly [29] in 1982. In this article, we have obtained the spectra and various spectral separations of this operator over the sequence spaces c_0 and c. In another our paper, we gave the spectral and spectral division of this operator over the sequence spaces ℓ_p, where $1 < p < \infty$. The spectral and spectral separation of this operator over the other sequence spaces are left clear problems.

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