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THE EXISTENCE AND EXPRESSIONS OF THE INVERSE ALONG OPERATORS B AND C

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Abstract. For given $A,B,C \in \mathscr{B}(\mathscr{H})$, if there exists $X \in \mathscr{B}(\mathscr{H})$ such that $XAB = B$, $CAX = C$, $\mathscr{R}(X) = \mathscr{R}(B)$ and $\mathscr{R}(X^*) = \mathscr{R}(C^*)$, then A is called (B,C)-invertible and X is called the (B,C)-inverse of A. In this paper, we find some explicit properties of the one sided-inverses and (B,C)-inverses for linear bounded operators. Moreover, the solution X of the operator equations $XAB = B$ and $CAX = C$ is expressed in terms of the inner inverses of the operators A, B and C. We also present the equivalent conditions for the existence and expressions of the inverses along operators B and C.

1. Introduction and preliminaries

Let $\mathscr{H}$ and $\mathscr{K}$ be separable, infinite dimensional, complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ by $\mathscr{B}(\mathscr{H},\mathscr{K})$ and by $\mathscr{B}(\mathscr{H})$ when $\mathscr{H} = \mathscr{K}$. For $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, let A^*, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ be the adjoint, the range and the null space of A, respectively. Let $\overline{\mathscr{M}}$ denote the closure of $\mathscr{M} \subseteq \mathscr{H}$. $I_{\mathscr{M}}$ denotes the identity onto $\mathscr{M}$ or I if there is no confusion. By A^- we denote an inner inverse of $A \in \mathscr{B}(\mathscr{H})$, i.e, $AA^-A = A$. In addition, Y is said to be an outer inverse of A, if $YAY = Y$. For two operators $P,Q \in \mathscr{B}(\mathscr{H})$, the commutator of P and Q is the operator $[P,Q] =: PQ - QP$. Commutators arise naturally in many aspects of operator theory, and they play an important role in this theory. It is well known that the set of commutators is dense in the set of all operators [4, page 124] and [5].

Inverses along an operator were introduced in [12], (B,C)-inverses were introduced in [8] and also studied in [16]. In the aforementioned papers both notions were defined in the context of semigroups. Then these classes of inverses were extended to the context of rectangular matrices in [3]. In [12, Theorem 6] it was proved that if an element of a semigroup S has inverse along other element in S then, the inverse is unique. In [8, Theorem 2.1] the uniqueness of the inverse along a pair of elements in a semigroup was also proved. The uniqueness of these kind of inverse is also true in the context of rectangular matrices [3, Proposition 3.5 and Corollary 3.8]. The equivalent conditions for the existence and the formula of the inverse along a regular lower triangular matrix and so on are derived in [3, 8, 13]. Recently, a new concept called

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left (right) g-MP inverse in a $*$-semigroup was introduced in [15] in the context of $*$-monoids. More results on the inverse along some elements can be found in mathematical literature [9] and [14, 16]. Motivated by these papers we investigate the inverse along a pair of operators B and C on a Hilbert space. For given $A,B,C \in \mathscr{B}(\mathscr{H})$, if there exists $X \in \mathscr{B}(\mathscr{H})$ such that

$$XAB = B, \quad CAX = C, \quad \mathscr{R}(X) = \mathscr{R}(B) \quad \text{and} \quad \mathscr{R}(X^*) = \mathscr{R}(C^*),$$

then A is called (B,C)-invertible and X is called the (B,C)-inverse of A (see [8, 9] for the definition on any semigroup). We find some explicit properties of the one sided-inverses and (B,C)-inverses. The solution X of the operator equations $XAB = B$ and $CAX = C$ is expressed in terms of the inner inverses of the operators A, B and C. The equivalent conditions for the existence and expressions of the inverses along operators B and C are obtained.

2. Some lemmas

In this section, we begin with some lemmas which play important roles in the sequel. The following lemma is a standard result.

LEMMA 2.1. ([6, Theorem 3.1]) *Let $A,B,C \in \mathscr{B}(\mathscr{H})$ be such that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed. The equation $AXB = C$ has a solution if and only if $AA^-CB^-B = C$. The general solution is of the form*

$$X = A^-CB^- + U - A^-AUBB^-, \quad \forall U \in \mathscr{B}(\mathscr{H}).$$

LEMMA 2.2. *For an operator $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{R}(T)$ is closed if and only if there exists $X \in \mathscr{B}(\mathscr{H})$ such that $TXT = T$.*

We need the following well-known criteria about ranges. The following item (ii) is from [10, Theorem 2.2].

LEMMA 2.3. ([7, 10, 11]) *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

(i) $\mathscr{R}(A) + \mathscr{R}(B) = \mathscr{R}((AA^* + BB^*)^{\frac{1}{2}})$*;*

(ii) $\mathscr{R}(A)$ *is closed if and only if* $\mathscr{R}(A) = \mathscr{R}(AA^*)$ *if and only if* $\mathscr{R}(A^*)$ *is closed;*

(iii) *If S and T are invertible, then $\mathscr{R}(SAT)$ is closed if and only if $\mathscr{R}(A)$ is closed.*

Throughout this work the next well-known criterion due to Douglas [7] (see also Fillmore-Williams [10]) about range inclusions and factorization of operators will be crucial.

LEMMA 2.4. ([7, 10]) *If $A,B \in \mathscr{B}(\mathscr{H})$, then the following are equivalent:*

(i) $A = BC$ for some operator $C \in \mathscr{B}(\mathscr{H})$;

(ii) $\|A^*x\| \leqslant k\|B^*x\|$ for some $k > 0$ and all $x \in \mathscr{H}$;

(iii) $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.

If one of these conditions holds, then there exists a unique solution $C_0 \in \mathscr{B}(\mathscr{H})$ of the equation $BX = A$ such that $R(C_0) \subset \overline{\mathscr{R}(B^*)}$ and $\mathscr{N}(C_0) = \mathscr{N}(A)$. This solution is called the Douglas reduced solution.

3. The inverse along an element $C \in \mathscr{B}(\mathscr{H})$

In this section we consider the inverse along an element $C \in \mathscr{B}(\mathscr{H})$. As for the operator case, it is defined as follows. One can refer to [3, 8, 9, 12, 13, 14] for more details on the inverse along an element $C \in \mathscr{B}(\mathscr{H})$.

DEFINITION 3.1. ([3, 8, 9, 12, 13, 14]) For given $A, C \in \mathscr{B}(\mathscr{H})$, if there exists $X \in \mathscr{B}(\mathscr{H})$ such that

(i)
$$XAC = C \quad \text{and} \quad \mathscr{R}(X^*) \subseteq \mathscr{R}(C^*),$$
then A is called left invertible along C and X is called a left inverse of A along C, denoted by $X = A_{LC}$.

(ii)
$$CAX = C \quad \text{and} \quad \mathscr{R}(X) \subseteq \mathscr{R}(C),$$
then A is called right invertible along C and X is called a right inverse of A along C, denoted by $X = A_{RC}$.

(iii)
$$XAC = C = CAX, \quad \mathscr{R}(X) = \mathscr{R}(C) \quad \text{and} \quad \mathscr{R}(X^*) = \mathscr{R}(C^*),$$
then A is called invertible along C and X is called the inverse of A along C, denoted by $X = A_C$.

Recall that $asc(T)$ (resp. $des(T)$), the ascent (resp. descent) of $T \in \mathscr{B}(\mathscr{H})$, is the smallest non-negative integer k such that $\mathscr{N}(T^{k+1}) = \mathscr{N}(T^k)$ (resp. $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$). If no such k exists, then $asc(T) = \infty$ (resp. $des(T) = \infty$). It is well known, $des(T) = asc(T)$ if $asc(T)$ and $des(T)$ are finite [17]. For $T \in \mathscr{B}(\mathscr{H})$, if there exists an operator $X \in \mathscr{B}(\mathscr{H})$ satisfying $TX = XT$, $XTX = X$ and $T^{k+1}X = T^k$, where $k = ind(T)$, then X is called a Drazin inverse of T, denoted by $X = T^D$ [2]. Particularly, if $ind(T) = 1$, then X is called the group inverse, denoted by $X = T^\sharp$. An operator $T \in \mathscr{B}(\mathscr{H})$ has its Drazin inverse T^D if and only if it has finite ascent and descent, which is equivalent with that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$, say of order k. In such case $ind(T) = asc(T) = des(T) = k$ [5, 17].

THEOREM 3.1. *For $C \in \mathscr{B}(\mathscr{H})$, there exist $X, Y \in \mathscr{B}(\mathscr{H})$ such that $XC^2 = C = C^2Y$ if and only if C is group invertible. In this case,*
$$X \in \left\{C^\sharp + S(I - CC^\sharp) : S \in \mathscr{B}(\mathscr{H})\right\}, \quad Y \in \left\{C^\sharp + (I - CC^\sharp)T : T \in \mathscr{B}(\mathscr{H})\right\}.$$
In particular, if $X = Y$, $\mathscr{R}(X) = \mathscr{R}(C)$ and $\mathscr{R}(X^*) = \mathscr{R}(C^*)$, then $X = C_C = C^\sharp$.

Proof. $\Leftarrow$ If C is group invertible, then there exist $X = C^\sharp + S(I - CC^\sharp)$ and $Y = C^\sharp + (I - CC^\sharp)T$ for arbitrary $S,T \in \mathscr{B}(\mathscr{H})$ such that $XC^2 = C = C^2Y$.

$\Rightarrow$ If $XC^2 = C = C^2Y$, then $\mathscr{R}(C) = \mathscr{R}(C^2)$ and $\mathscr{N}(C) = \mathscr{N}(C^2)$. It follows $ind(C) = 1$ and C is group invertible. We can write C as 2×2 matrix form $C = C_{11} \oplus 0$ with respect to space decomposition $\mathscr{H} = \mathscr{R}(C) \oplus \mathscr{N}(C)$, where $C_{11} \in \mathscr{B}(\mathscr{R}(C))$ is invertible. Correspondingly, put $X = (X_{ij})_{1\leqslant i,j\leqslant 2}$ and $Y = (Y_{ij})_{1\leqslant i,j\leqslant 2}$. By $C = XC^2 = C^2Y$ we get

$$\begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} C_{11}^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11}^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

So,

$$\begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}C_{11}^2 & 0 \\ X_{21}C_{11}^2 & 0 \end{pmatrix} = \begin{pmatrix} C_{11}^2Y_{11} & C_{11}^2Y_{12} \\ 0 & 0 \end{pmatrix}.$$

Since C_{11} is invertible, we derive that $X_{11} = Y_{11} = C_{11}^{-1}$, $Y_{12} = 0$ and $X_{21} = 0$. Hence,

$$X = \begin{pmatrix} C_{11}^{-1} & X_{12} \\ 0 & X_{22} \end{pmatrix} \in \left\{ C^\sharp + (I - CC^\sharp)S : S \in \mathscr{B}(\mathscr{H}) \right\}$$

and

$$Y = \begin{pmatrix} C_{11}^{-1} & 0 \\ Y_{21} & Y_{22} \end{pmatrix} \in \left\{ C^\sharp + (I - CC^\sharp)T : T \in \mathscr{B}(\mathscr{H}) \right\}.$$

In particular, if $X = Y$, $\mathscr{R}(X) = \mathscr{R}(C)$ and $\mathscr{R}(X^*) = \mathscr{R}(C^*)$, then $X_{12} = 0$, $Y_{21} = 0$ and $X_{22} = Y_{22} = 0$. By Definition 3.1 (iii), $X = C_C = C^\sharp$. $\square$

Next, we consider the case that $[A,C] = 0$.

THEOREM 3.2. *Let $A,B,C \in \mathscr{B}(\mathscr{H})$ be such that $[A,C] = 0$. If A_C and B_C exist, then*

(i) *C_C exists, $[A_C, C_C] = 0$ and $(AC)_C = C_CA_C$;*

(ii) *A_C is unique and $A_C = C(AC)_C = (CA)_CC$;*

(iii) *$[A_C, A] = 0$ and $[A_C, C] = 0$;*

(iv) *$(AB)_C$ and $(BA)_C$ exist with $(AB)_C = B_CA_C$ and $(BA)_C = A_CB_C$.*

Proof. Since A_C exists, by Definition 3.1, there exists $X = A_C \in \mathscr{B}(\mathscr{H})$ such that $XAC = C = CAX$ with $\mathscr{R}(X) \subseteq \mathscr{R}(C)$ and $\mathscr{R}(X^*) \subseteq \mathscr{R}(C^*)$. By Lemma 2.4, there exist $M,N \in \mathscr{B}(\mathscr{H})$ such that $X = MC = CN$. If $[A,C] = 0$, then

$$MAC^2 = C = C^2AN.$$

It implies that $\mathscr{R}(C) = \mathscr{R}(C^2)$ and $\mathscr{N}(C) = \mathscr{N}(C^2)$. Hence, $ind(C) = 1$ and C is group invertible with $C_C = C^\sharp$ by Theorem 3.1. Let $\mathscr{H} = \mathscr{R}(C) \oplus \mathscr{N}(C)$. Then $C \in \mathscr{B}(\mathscr{H})$ has the form $C = C_{11} \oplus 0$, where $C_{11} \in \mathscr{B}(\mathscr{R}(C))$ is invertible. The condition

$[A,C]=0$ implies that $A=A_{11}\oplus A_{22}$ with $A_{11}\in\mathscr{B}(\mathscr{R}(C))$, $A_{22}\in\mathscr{B}(\mathscr{N}(C))$ and $A_{11}C_{11}=C_{11}A_{11}$. From $\mathscr{R}(X)=\mathscr{R}(C)$ one has

$$X=\begin{pmatrix} X_{11} & X_{12}\\ 0 & 0\end{pmatrix}.$$

From $XAC=C=CAX$ one gets

$$\begin{pmatrix} C_{11} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} X_{11} & X_{12}\\ 0 & 0\end{pmatrix}\begin{pmatrix} A_{11} & 0\\ 0 & A_{22}\end{pmatrix}\begin{pmatrix} C_{11} & 0\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix} C_{11} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} A_{11} & 0\\ 0 & A_{22}\end{pmatrix}\begin{pmatrix} X_{11} & X_{12}\\ 0 & 0\end{pmatrix}.$$

So

$$C_{11}=X_{11}A_{11}C_{11}=C_{11}A_{11}X_{11} \text{ and } C_{11}A_{11}X_{12}=0.$$

The invertibility of C_{11} implies that A_{11} is invertible, $X_{11}=A_{11}^{-1}$ and $X_{12}=0$. Hence $AC=A_{11}C_{11}\oplus 0$ and $X=A_C=A_{11}^{-1}\oplus 0$. Similarly, we have $(AC)_C=C_{11}^{-1}A_{11}^{-1}\oplus 0$. So we have the following results.

(i) C_C exists, $[A_C,C_C]=0$ and $(AC)_C=C_CA_C$;

(ii) A_C is unique and $A_C=C(AC)_C=(CA)_CC$;

(iii) $[A_C,A]=0$ and $[A_C,C]=0$;

(iv) Since B_C exists, there exist $Y=B_C,M',N'\in\mathscr{B}(\mathscr{H})$ such that $YBC=C=CBY$ and $Y=M'C=CN'$. So

$$YXABC=YAXBC=M'CAXBC=M'CBC=YBC=C,$$

$CABYX=ACBYX=ACX=C$ and $YX=YMC=CN'MC$, i.e.,

$$\mathscr{R}(YX)\subseteq\mathscr{R}(C)=\mathscr{R}(YXABC)\subseteq\mathscr{R}(YX)$$

and

$$\mathscr{R}((YX)^*)\subseteq\mathscr{R}(C^*)=\mathscr{R}((CABYX)^*)\subseteq\mathscr{R}((YX)^*).$$

Hence $YX=B_CA_C=(AB)_C$. Similarly we have $(BA)_C=A_CB_C$. □

Theorem 3.2 shows that, if $[A,C]=0$ and A_C exists, then C must be group invertible. Specially, if $C=A$ (resp. $C=A^k$, where $k=ind(A)$) in above theorem, one has the following corollary.

COROLLARY 3.1. ([12, Theorem 11]) *Let $A\in\mathscr{B}(\mathscr{H})$.*

(i) *A is invertible if and only if A is invertible along I. In this case, $A^{-1}=A_I$.*

(ii) *A is group invertible if and only if A is invertible along A. In this case, $A^{\#}=A_A$.*

(iii) *A is Drazin invertible if and only if A is invertible along A^k. In this case, $A^D=A_{A^k}$.*

As we know, AC and CA have the same Drazin invertibility and $C(AC)^D=C[(AC)^D]^2AC=(CA)^DC$ [2].

THEOREM 3.3. *Let $A, C \in \mathscr{B}(\mathscr{H})$ such that AC is Drazin invertible. Then A_{LC} exists if and only if A_{RC} exists. More precisely, if A_{LC} exists, then $A_{LC} = A_{RC} = A_C = C(AC)^D$. If A_{RC} exists, then $A_{RC} = A_{LC} = A_C = (CA)^D C$.*

Proof. $\Rightarrow$ Let $X = A_{LC}$ and $k = ind(AC)$. Then there exists $M \in \mathscr{B}(\mathscr{H})$ such that $X = MC$ and

$$\begin{aligned} C = XAC = MCAC = M^2C(AC)^2 = \cdots = M^kC(AC)^k \\ = M^kC(AC)^k(AC)(AC)^D = C(AC)(AC)^D = CA\left[C(AC)^D\right]. \end{aligned}$$

Then A_{RC} exists with $A_{LC} = A_{RC} = A_C = C(AC)^D$.

$\Leftarrow$ Dually, it follows that $C = \left[(CA)^D C\right] AC$ and $A_{RC} = A_{LC} = A_C = (CA)^D C$. □

In Theorem 3.3, if AC is Drazin invertible, left C-invertibility and right C-invertibility both coincide with C-invertibility. Note that an operator T is Moore-Penrose invertible if and only if $\mathscr{R}(T)$ is closed. The Moore-Penrose inverse of T is denoted by $T^\dagger$ [2].

DEFINITION 3.2. For given $A \in \mathscr{B}(\mathscr{H})$, we call A is left (resp. right) g-MP invertible if there exist $X, Y \in \mathscr{B}(\mathscr{H})$ such that $A = XA^2 = YAA^*A$ (resp. $A = A^2X = AA^*AY$).

Let $X = YA$. Then $\mathscr{R}(X^*) \subseteq \mathscr{R}(A^*)$ by Lemma 2.4. It is obvious that there exists $Y \in \mathscr{B}(\mathscr{H})$ such that $A = YAA^*A$ (resp. $A = AA^*AY$) if and only if $(A^*)_{LA}$ (resp. $(A^*)_{RA}$) exists. For the left g-MP invertible operator, we get the following results.

THEOREM 3.4. *For given $A \in \mathscr{B}(\mathscr{H})$, there exist $X, Y \in \mathscr{B}(\mathscr{H})$ such that*

$$A = XA^2 = YAA^*A$$

if and only if there exists $N \in \mathscr{B}(\mathscr{H})$ such that

$$A = NA^2A^*A.$$

In this case, $\mathscr{R}(A)$ is closed, $[NA^2, AA^*] = 0$,

$$XA \in \left\{AA^\dagger + SA(I - AA^\dagger) : S \in \mathscr{B}(\mathscr{H})\right\}$$

and

$$Y \in \left\{(AA^*)^\dagger + T(I - AA^\dagger) : T \in \mathscr{B}(\mathscr{H})\right\}.$$

Proof. $\Leftarrow$ Since $A = NA^2A^*A$,

$$A^*A = (NA^2A^*A)^*(NA^2A^*A) = A^*A\left[(NA^2)^*NA^2\right]A^*A.$$

By Lemmas 2.2 and 2.3, $\mathscr{R}(A^*) = \mathscr{R}(A^*A)$ and $\mathscr{R}(A)$ are closed. Then $\mathscr{H} = \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} A_{11}^*\Delta^{-1} & 0 \\ A_{12}^*\Delta^{-1} & 0 \end{pmatrix},$$

where $\Delta = A_{11}A_{11}^* + A_{12}A_{12}^*$. Put $N = (N_{ij})_{1\leqslant i,j\leqslant 2}$. By $A = NA^2A^*A$ we get

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} A_{11}^2 & A_{11}A_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11}^* & 0 \\ A_{12}^* & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} N_{11}A_{11}\Delta A_{11} & N_{11}A_{11}\Delta A_{12} \\ N_{21}A_{11}\Delta A_{11} & N_{21}A_{11}\Delta A_{12} \end{pmatrix}.$$

By the relations $A_{11} = N_{11}A_{11}\Delta A_{11}$ and $A_{12} = N_{11}A_{11}\Delta A_{12}$ one derives that $N_{11}A_{11} = \Delta^{-1}$. By the relations $N_{21}A_{11}\Delta A_{11} = 0$ and $N_{21}A_{11}\Delta A_{12} = 0$ we derive that $N_{21}A_{11} = 0$. So

$$NA^2 = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} A_{11}^2 & A_{11}A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Delta^{-1}A_{11} & \Delta^{-1}A_{12} \\ 0 & 0 \end{pmatrix} = (A^\dagger)^*$$

and

$$NA^2A^*A = A = AA^*(A^\dagger)^* = AA^*NA^2, \quad i.e., \ [NA^2, AA^*] = 0.$$

Hence, there exist $X = AA^*N$ and $Y = NA$ such that $A = XA^2$ and $A = YAA^*A$.

$\Rightarrow$ Since there exist X and Y such that

$$A = XA^2 = YAA^*A = (YXA)AA^*A,$$

$\mathscr{R}(A)$ is closed and there exists $N = YX$ such that $A = NA^2A^*A$. Now, put $X = (X_{ij})_{1\leqslant i,j\leqslant 2}$ and $Y = (Y_{ij})_{1\leqslant i,j\leqslant 2}$. By $A = XA^2 = YAA^*A$ one gets

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{11}^2 & A_{11}A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} \Delta A_{11} & \Delta A_{12} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} X_{11}A_{11}^2 & X_{11}A_{11}A_{12} \\ X_{21}A_{11}^2 & X_{21}A_{11}A_{12} \end{pmatrix} = \begin{pmatrix} Y_{11}\Delta A_{11} & Y_{11}\Delta A_{12} \\ Y_{21}\Delta A_{11} & Y_{21}\Delta A_{12} \end{pmatrix}.$$

By the relations $A_{11} = X_{11}A_{11}^2 = Y_{11}\Delta A_{11}$ and $A_{12} = X_{11}A_{11}A_{12} = Y_{11}\Delta A_{12}$ one derives that $Y_{11} = \Delta^{-1}$ and $X_{11}A_{11} = I$. By the relations $X_{21}A_{11}^2 = 0$, $X_{21}A_{11}A_{12} = 0$, $Y_{21}\Delta A_{11} = 0$ and $Y_{21}\Delta A_{12} = 0$ one derives that $Y_{21} = 0$ and $X_{21}A_{11} = 0$. Hence

$$Y = \begin{pmatrix} \Delta^{-1} & Y_{12} \\ 0 & Y_{22} \end{pmatrix} \in \left\{ (AA^*)^\dagger + T(I - AA^\dagger) : T \in \mathscr{B}(\mathscr{H}) \right\}$$

and

$$XA = \begin{pmatrix} I & X_{11}A_{12} \\ 0 & X_{21}A_{12} \end{pmatrix} \in \left\{ AA^\dagger + SA(I - AA^\dagger) : S \in \mathscr{B}(\mathscr{H}) \right\}. \quad \square$$

According to the proof of Theorem 3.4, the relations among left inverse, right inverse and the Moore-Penrose inverse are given below.

COROLLARY 3.2. *For given $A \in \mathscr{B}(\mathscr{H})$, $\mathscr{R}(A)$ is closed if and only if there exists $Y_1 \in \mathscr{B}(\mathscr{H})$ such that $A = Y_1AA^*A$ if and only if there exists $Y_2 \in \mathscr{B}(\mathscr{H})$ such that $A = AA^*AY_2$ if and only if A_{LA}^* exists if and only if A_{RA}^* exists if and only if A is MP-invertible. In this case, $(A^\dagger)^* = Y_1A = AY_2$, where*

$$Y_1 \in \left\{ (AA^*)^\dagger + T(I - AA^\dagger) : T \in \mathscr{B}(\mathscr{H}) \right\}$$

and

$$Y_2 \in \left\{ (A^*A)^\dagger + (I - A^\dagger A)S : S \in \mathscr{B}(\mathscr{H}) \right\}.$$

We have the following main result.

THEOREM 3.5. *Let $A,C \in \mathscr{B}(\mathscr{H})$. Then A_C exists if and only if there exists $X \in \mathscr{B}(\mathscr{H})$ such that $X = XAX$,*

$$\mathscr{N}(C) = \mathscr{N}(X) = \mathscr{N}(AX) = \mathscr{N}(AC)$$

and

$$\mathscr{R}(C) = \mathscr{R}(X) = \mathscr{R}(XA) = \mathscr{R}(CA).$$

Moreover, for given A,C, if X exists, then X is unique.

Proof. $\Rightarrow$ If $X = A_C$ exists, then $XAC = C = CAX$, $\mathscr{R}(X) \subseteq \mathscr{R}(C)$ and $\mathscr{R}(X^*) \subseteq \mathscr{R}(C^*)$. There exist $M,N \in \mathscr{B}(\mathscr{H})$ such that $X = MC = CN$. By Lemma 2.4,

$$\mathscr{N}(X) \subseteq \mathscr{N}(AX) \subseteq \mathscr{N}(C) \subseteq \mathscr{N}(X), \quad \mathscr{N}(C) \subseteq \mathscr{N}(AC) \subseteq \mathscr{N}(C)$$

and

$$\mathscr{R}(C) \subseteq \mathscr{R}(XA) \subseteq \mathscr{R}(X) \subseteq \mathscr{R}(C), \quad \mathscr{R}(C) \subseteq \mathscr{R}(CA) \subseteq \mathscr{R}(C).$$

One gets

$$\mathscr{N}(C) = \mathscr{N}(X) = \mathscr{N}(AX) = \mathscr{N}(AC)$$

and

$$\mathscr{R}(C) = \mathscr{R}(X) = \mathscr{R}(XA) = \mathscr{R}(CA).$$

Moreover, $(I - XA)C = 0$ and $\mathscr{R}(X) = \mathscr{R}(C)$ imply that $(I - XA)X = 0$.

$\Leftarrow$ If $X = XAX$, $\mathscr{N}(X) = \mathscr{N}(C)$ and $\mathscr{R}(X) = \mathscr{R}(C)$, then $\mathscr{R}(X)$ and $\mathscr{R}(C)$ are closed by Lemma 2.2. Hence, $\mathscr{R}(X^*)$ and $\mathscr{R}(C^*)$ are closed by Lemma 2.3 and

$$\mathscr{R}(X^*) = \mathscr{N}(X)^\perp = \mathscr{N}(C)^\perp = \mathscr{R}(C^*).$$

From $(I - XA)X = 0$ one gets

$$\mathscr{R}(C) = \mathscr{R}(X) \subseteq \mathscr{N}(I - XA),$$

which implies that $C = XAC$. From $X(I - AX) = 0$ one gets

$$\mathscr{R}(I - AX) \subseteq \mathscr{N}(X) = \mathscr{N}(C),$$

which implies that $C = CAX$.

Finally, for given A,C, if X,X' are two inverses of A along C, then

$$\mathscr{N}(X) = \mathscr{N}(C) = \mathscr{N}(X')$$

and

$$\mathscr{R}(X) = \mathscr{R}(C) = \mathscr{R}(X'),$$

which imply

$$(I - XA)X' = (I - XA)X = 0, \quad X(I - AX') = X'(I - AX') = 0,$$

i.e., $X' = XAX' = X$. $\square$

Theorem 3.5 implies that the solution of $CAX = C = XAC$ is an out inverse of an operator A over complex field with prescribed range space $\mathscr{R}(C)$ and null space $\mathscr{N}(C)$ (see [1, 2]).

REMARK 3.1. Theorem 3.5 shows that, if A_C exists then:

(i) A_C is the out inverse of A. $\mathscr{R}(A_C) = \mathscr{R}(C)$ is closed and $\mathscr{N}(A_C) = \mathscr{N}(C)$.

(ii) AA_C and A_CA are idempotents with

$$\mathscr{R}(AA_C) = \mathscr{R}(AC), \quad \mathscr{R}(A_CA) = \mathscr{R}(CA),$$
$$\mathscr{N}(AA_C) = \mathscr{N}(AC), \quad \mathscr{N}(A_CA) = \mathscr{N}(CA).$$

In fact, by Theorem 3.5, $\mathscr{N}(AA_C) = \mathscr{N}(AC)$ and $\mathscr{R}(A_CA) = \mathscr{R}(CA)$. If $X = A_C$ exists, then there exist $X, M, N \in \mathscr{B}(\mathscr{H})$ such that $XAC = C = CAX$, $X = CN = MC$. Multiplying by A on the left, one has $AC = AXAC$ and $AX = ACN$, which implies that $\mathscr{R}(AC) = \mathscr{R}(AX)$. Multiplying by A on the right, one has $CA = CAXA$ and $XA = MCA$, which implies that $\mathscr{N}(XA) = \mathscr{N}(CA)$.

(iii) By the conjugate transformation we have $X^* = (A_C)^* = (A^*)_{C^*}$,

$$\mathscr{N}(C^*) = \mathscr{N}(X^*) = \mathscr{N}(A^*X^*) = \mathscr{N}(A^*C^*),$$
$$\mathscr{R}(C^*) = \mathscr{R}(X^*) = \mathscr{R}(X^*A^*) = \mathscr{R}(C^*A^*).$$

If A is invertible along C, then A and C have the following operator structure.

THEOREM 3.6. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$.*

(i) *A_C exists if and only if the following conditions (a) and (b) hold:*

(a) C is a closed range operator, i.e., there exists an invertible operator $C_{11} \in \mathscr{B}(\mathscr{R}(C^*), \mathscr{R}(C))$ such that $C = C_{11} \oplus 0$.

(b) $A = (A_{ij})_{1 \leqslant i,j \leqslant 2}$ as an operator from $\mathscr{R}(C) \oplus \mathscr{N}(C^*)$ into $\mathscr{R}(C^*) \oplus \mathscr{N}(C)$ has the property that A_{11} is invertible.

In this case, A_C is unique and $A_C = A_{11}^{-1} \oplus 0$.

(ii) *If A_C exists, then $(ACC^*)_C$ and $(C^*CA)_C$ exist. In this case,*

$$(C^*CA)_C = A_C(C^*C)^\dagger = A_CC^\dagger(C^*)^\dagger$$

and

$$(ACC^*)_C = (CC^*)^\dagger A_C = (C^*)^\dagger C^\dagger A_C.$$

(iii) *If A_C and B_C exist, then $(ACB)_C$ and $(BCA)_C$ exist. In this case,*

$$(ACB)_C = B_CC^-A_C \quad \text{and} \quad (BCA)_C = A_CC^-B_C.$$

Proof. (i) $\Leftarrow$ It is obvious.

$\Rightarrow$ Theorem 3.5 and Remark 3.1 (iii) had proved that the solution X is unique and $\mathscr{R}(C)=\mathscr{R}(X)$ is closed with

$$\mathscr{N}(C)=\mathscr{N}(X), \quad \mathscr{N}(C^*)=\mathscr{N}(X^*), \quad \mathscr{R}(C)=\mathscr{R}(X), \quad \mathscr{R}(C^*)=\mathscr{R}(X^*).$$

We know there exists an invertible operator $C_{11}, X_{11} \in \mathscr{B}(\mathscr{R}(C^*),\mathscr{R}(C))$ such that $C = C_{11}\oplus 0$ and $X = X_{11}\oplus 0$. Let $A=(A_{ij})_{1\leqslant i,j\leqslant 2}$. From $XAC=C=CAX$ we get

$$X_{11}A_{11}C_{11}=C_{11}=C_{11}A_{11}X_{11}.$$

So $X_{11}=A_{11}^{-1}$ and $A_C=A_{11}^{-1}\oplus 0$.

(ii) By item (i), A_C exists if and only if there exists an invertible operator C_{11} and A_{11} such that $C=C_{11}\oplus 0$ and $A=(A_{ij})_{1\leqslant i,j\leqslant 2}$. Since ACC^* and C^*CA have the invertible $(1,1)$-entries $A_{11}C_{11}C_{11}^*$ and $C_{11}^*C_{11}A_{11}$, respectively. By item (i), $(ACC^*)_C$ and $(C^*CA)_C$ exist. Note that $C^\dagger=C_{11}^{-1}\oplus 0$.

$$(C^*CA)_C=(C_{11}^*C_{11}A_{11})^{-1}\oplus 0=A_C(C^*C)^\dagger=A_CC^\dagger(C^*)^\dagger$$

and

$$(ACC^*)_C=(A_{11}C_{11}C_{11}^*)^{-1}\oplus 0=(CC^*)^\dagger A_C=(C^*)^\dagger C^\dagger A_C.$$

(iii) Let $B=(B_{ij})_{1\leqslant i,j\leqslant 2}$. By item (i), B_C exists if and only if B_{11} is invertible. Since B_{11} is invertible if and only if $B_{11}C_{11}A_{11}$ is invertible if and only if $A_{11}C_{11}B_{11}$ is invertible, which are the $(1,1)$-entries of B, BCA and ACB, respectively. Note that $C^-=(C_{ij}^0)_{1\leqslant i,j\leqslant 2}$, where $C_{11}^0=C_{11}^{-1}$ is invertible. C_{12}^0, C_{21}^0 and C_{22}^0 are arbitrary operators on corresponding subspaces. By above item (i) we know

$$(ACB)_C=(A_{11}C_{11}B_{11})^{-1}\oplus 0=B_CC^-A_C$$

and

$$(BCA)_C=(B_{11}C_{11}A_{11})^{-1}\oplus 0=A_CC^-B_C. \quad \square$$

Theorem 3.6 shows that A is Moore-Penrose invertible if and only if A is invertible along A^*. In this case, $A^\dagger=A_{A^*}$. The closedness of $\mathscr{R}(C)$ implies that C has an inner inverse C^-. If A_C exists, then

$$AC+I-C^-C \quad \text{and} \quad CA+I-CC^-$$

are invertible (see [14, Corollary 2.5]). If A_C and B_C exist, then

$$CACB+I-CC^-, \quad CBCA+I-CC^-, \quad ACBC+I-C^-C, \quad BCAC+I-C^-C$$

are invertible. But $(AC)_C$, $(CA)_C$, $(ABC)_C$ and $(CAB)_C$ may not exist.

As for the operator matrix $T=(T_{ij})_{1\leqslant i,j\leqslant 2}$ on the Hilbert space $\mathscr{H}_1\oplus\mathscr{H}_2$, where operator T_{ij} acts from $\mathscr{H}_j$ into $\mathscr{H}_i$, $i,j=1,2$. Denote by $T_{11}=T|_{\mathscr{H}_1}$. We have the following results.

THEOREM 3.7. *Let $A,C\in\mathscr{B}(\mathscr{H})$. Denote by $P=CC^-$ and $Q=C^-C$.*
(i) *If A_C exists, then $(CA)_P=A_CC^-$ and $(AC)_Q=C^-A_C$.*
(ii) *If $C|_{\mathscr{R}(C)}$ is invertible, then*

$$A_C \text{ exists} \Longleftrightarrow (AC)_C \text{ exists} \Longleftrightarrow (CA)_C \text{ exists}.$$

In this case, $(CA)_C=A_CC_C$ and $(AC)_C=C_CA_C$.

Proof. If A_C exists, then C,A and A_C have the matrix forms as in Theorem 3.6 item (i). Then C^- has the matrix representation as $C^-=(C^0_{ij})_{1\leqslant i,j\leqslant 2}$, where $C^0_{11}=C^{-1}_{11}$. Then

$$P=CC^-=\begin{pmatrix} I & C_{11}C^0_{12}\\ 0 & 0\end{pmatrix},\quad Q=C^-C=\begin{pmatrix} I & 0\\ C^0_{21}C_{11} & 0\end{pmatrix}$$

and

$$A_CC^-=\begin{pmatrix} A^{-1}_{11}C^{-1}_{11} & A^{-1}_{11}C^0_{12}\\ 0 & 0\end{pmatrix},\quad C^-A_C=\begin{pmatrix} C^{-1}_{11}A^{-1}_{11} & 0\\ C^0_{21}A^{-1}_{11} & 0\end{pmatrix}.$$

Note that $\mathscr{R}(A_CC^-)=\mathscr{R}(P)$, $\mathscr{R}((A_CC^-)^*)=\mathscr{R}(P^*)$ and

$$\left[A_CC^-\right]CAP=P=PCA\left[A_CC^-\right].$$

By uniqueness of $(CA)_P$, we obtain $(CA)_P=A_CC^-$. Similarly we have $(AC)_Q=C^-A_C$.

(ii) If $C|_{\mathscr{R}(C)}$ is invertible, then there exist $C_{12}\in\mathscr{B}(\mathscr{R}(C)^\perp,\mathscr{R}(C))$, an invertible operator $C|_{\mathscr{R}(C)}=C_{11}\in\mathscr{B}(\mathscr{R}(C))$ and invertible operator $S\in\mathscr{B}(\mathscr{H})$ such that

$$C=\begin{pmatrix} C_{11} & C_{12}\\ 0 & 0\end{pmatrix},\quad S=\begin{pmatrix} I & C^{-1}_{11}C_{12}\\ 0 & I\end{pmatrix},\quad SCS^{-1}=\begin{pmatrix} C_{11} & 0\\ 0 & 0\end{pmatrix}.$$

In the following we divide the proof into three steps.

Step 1. By Definition 3.1, $X=A_C$ exists if and only if $XAC=C=CAX$, $\mathscr{R}(X)=\mathscr{R}(C)$ and $\mathscr{R}(X^*)=\mathscr{R}(C^*)$. By Lemma 2.4, $\mathscr{R}(X)=\mathscr{R}(C)$ if and only if $\mathscr{R}(SXS^{-1})=\mathscr{R}(SCS^{-1})$. And $\mathscr{R}(X^*)=\mathscr{R}(C^*)$ if and only if $\mathscr{R}\left[(SXS^{-1})^*\right]=\mathscr{R}\left[(SCS^{-1})^*\right]$. Furthermore,

$$SXS^{-1}\cdot SAS^{-1}\cdot SCS^{-1}=SCS^{-1}=SCS^{-1}\cdot SAS^{-1}\cdot SXS^{-1}.$$

Hence, A_C exists if and only if $(SAS^{-1})_{SCS^{-1}}$ exists.

Step 2. By Theorem 3.6 (i), $(SAS^{-1})_{SCS^{-1}}$ exists if and only if there exists an invertible operator $A^0_{11}\in\mathscr{B}(\mathscr{R}(C))$ such that

$$SAS^{-1}=(A^0_{ij})_{1\leqslant i,j\leqslant 2}.$$

In this case, $(SAS^{-1})_{SCS^{-1}}$ is unique with

$$(SAS^{-1})_{SCS^{-1}}=SXS^{-1}=(A^0_{11})^{-1}\oplus 0.$$

Hence, $A = S^{-1}(A_{ij}^0)_{1\leqslant i,j\leqslant 2}S$ and

$$A_C = X = S^{-1}\begin{pmatrix} (A_{11}^0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S = \begin{pmatrix} (A_{11}^0)^{-1} & (A_{11}^0)^{-1}C_{11}^{-1}C_{12} \\ 0 & 0 \end{pmatrix}.$$

Step 3. Note that the $(1,1)$-entries of SAS^{-1}, $SACS^{-1}$ and $SCAS^{-1}$ have the same invertibility. So

$$(SAS^{-1})_{SCS^{-1}} \text{ exists} \Longleftrightarrow (SACS^{-1})_{SCS^{-1}} \text{ exists} \Longleftrightarrow (SCAS^{-1})_{SCS^{-1}} \text{ exists.}$$

In this case,

$$(SACS^{-1})_{SCS^{-1}} = C_{11}^{-1}(A_{11}^0)^{-1} \oplus 0 = (SCS^{-1})_{SCS^{-1}}(SAS^{-1})_{SCS^{-1}}$$

and

$$(SCAS^{-1})_{SCS^{-1}} = (A_{11}^0)^{-1}C_{11}^{-1} \oplus 0 = (SAS^{-1})_{SCS^{-1}}(SCS^{-1})_{SCS^{-1}}.$$

Step 4. By Step 2, one has that

$$A_C = S^{-1}\left[(SAS^{-1})_{SCS^{-1}}\right]S.$$

Hence, by Step 3, we get $(AC)_C = C_C A_C$ and $(CA)_C = A_C C_C$. $\square$

4. The (B,C)-inverses

In this section, we investigate the inverse along a pair of operators B and C. We find some explicit properties of the one sided-inverses and (B,C)-inverses. One can refer to [3, 8, 9, 12, 13, 14] for more details on the inverse along a pair of B and C.

DEFINITION 4.1. ([3, 8, 9, 12, 13, 14]) For given $A,B,C \in \mathscr{B}(\mathscr{H})$, if there exists $X \in \mathscr{B}(\mathscr{H})$ such that

(i)

$$XAB = B \quad \text{and} \quad \mathscr{R}(X^*) \subseteq \mathscr{R}(C^*),$$

then A is called left (B,C)-invertible and X is called a left (B,C)-inverse of A, denoted by $X = A_{LBC}$.

(ii)

$$CAX = C \quad \text{and} \quad \mathscr{R}(X) \subseteq \mathscr{R}(B),$$

then A is called right (B,C)-invertible and X is called a right (B,C)-inverse of A, denoted by $X = A_{RBC}$.

(iii)

$$XAB = B, \quad CAX = C, \quad \mathscr{R}(X) = \mathscr{R}(B) \quad \text{and} \quad \mathscr{R}(X^*) = \mathscr{R}(C^*),$$

then A is called (B,C)-invertible and X is called the (B,C)-inverse of A, denoted by $X = A_{BC}$.

By Definition 4.1, we observe that

$$(A_{LBC})^* = A^*_{RC^*B^*}, \quad (A_{RBC})^* = A^*_{LC^*B^*} \quad \text{and} \quad (A_{BC})^* = A^*_{C^*B^*}.$$

If $X = A_{BC}$ exists, then the relations $XAB = B$ and $\mathscr{R}(X) = \mathscr{R}(B)$ imply that $0 = (I - XA)B = (I - XA)X$. Hence, A_{BC} is an outer inverse. It is obvious that A is both left and right (B,C)-invertible if and only if A is (B,C)-invertible. And in this case, $A_{LBC} = A_{RBC} = A_{BC}$. In the following theorem, the representation parts of A_{LBC} and A_{RBC} relating the finite matrix by using the Moore-Penrose inverse can be found in [3, Theorem 3.19]

THEOREM 4.1. *Let* $A,B,C \in \mathscr{B}(\mathscr{H})$.

(i) A_{LBC} *exists if and only if* $\mathscr{R}(B^*) = \mathscr{R}(B^*A^*C^*)$. *In addition, if* $\mathscr{R}(B)$ *is closed, then*

$$A_{LBC} \in \left\{ [B(CAB)^- + S(I - CAB(CAB)^-)]C : \quad S \in \mathscr{B}(\mathscr{H}) \right\}.$$

(ii) A_{RBC} *exists if and only if* $\mathscr{R}(C) = \mathscr{R}(CAB)$. *In addition, if* $\mathscr{R}(C)$ *is closed, then*

$$A_{RBC} \in \left\{ B[(CAB)^- C + (I - (CAB)^- CAB)T] : \quad T \in \mathscr{B}(\mathscr{H}) \right\}.$$

(iii) A_{BC} *exists if and only if*

$$\mathscr{R}(B^*) = \mathscr{R}(B^*A^*C^*), \quad \mathscr{R}(C) = \mathscr{R}(CAB).$$

In addition, if $\mathscr{R}(C)$ *or* $\mathscr{R}(B)$ *is closed,* A_{BC} *is unique with* $A_{BC} = B(CAB)^- C$.

Proof. If B and $C \in \mathscr{B}(\mathscr{H})$, then there exist injective dense defined operators $B_{11} \in \mathscr{B}(\overline{\mathscr{R}(B^*)}, \overline{\mathscr{R}(B)})$ and $C_{11} \in \mathscr{B}(\overline{\mathscr{R}(C^*)}, \overline{\mathscr{R}(C)})$ such that $B = B_{11} \oplus 0$ and $C = C_{11} \oplus 0$. Correspondingly, A and X^*, as operators from $\overline{\mathscr{R}(B)} \oplus \mathscr{N}(B^*)$ into $\overline{\mathscr{R}(C^*)} \oplus \mathscr{N}(C)$, are denoted by $A = (A_{ij})_{1 \leqslant i,j \leqslant 2}$ and $X = (X_{ij})_{1 \leqslant i,j \leqslant 2}$, respectively.

(i) Note that $X = A_{LBC}$ if and only if $XAB = B$ and $\mathscr{R}(X^*) \subseteq \mathscr{R}(C^*)$ if and only if there exists $M \in \mathscr{B}(\mathscr{H})$ such that $MCAB = B$, where $X = MC$ if and only if $\mathscr{R}(B^*) = \mathscr{R}(B^*A^*C^*)$.

Now, let A,B,C and X have the matrices forms as above. From $\mathscr{R}(X^*) \subseteq \mathscr{R}(C^*)$ one has $X_{12} = 0$ and $X_{22} = 0$. From $XAB = B$ one gets

$$XAB = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}A_{11}B_{11} & 0 \\ X_{21}A_{11}B_{11} & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since B_{11} is dense defined operator, one gets $X_{11}A_{11} = I$ and $X_{21}A_{11} = 0$.

In addition, if $\mathscr{R}(B)$ is closed, then $\mathscr{R}(B^*) = \mathscr{R}(B^*A^*C^*)$ is closed. By Lemma 2.3, $\mathscr{R}(CAB)$ is closed and the inner inverse $(CAB)^-$ exists with $B(CAB)^- CAB = B$. By Lemma 2.1, $MCAB = B$ has the general solution of the form

$$M = B(CAB)^- + S[I - CAB(CAB)^-], \quad \forall S \in \mathscr{B}(\mathscr{H}).$$

Hence, the general representation of A_{LBC} is of the form

$$X = MC = \left[B(CAB)^- + S(I - CAB(CAB)^-)\right]C, \quad \forall S \in \mathscr{B}(\mathscr{H}).$$

(ii) Similar to (i). We can prove that $X = A_{RBC}$ exists if and only if $\mathscr{R}(C) = \mathscr{R}(CAB)$. If $\mathscr{R}(C)$ is closed, then

$$X = B\left[(CAB)^- C + (I - (CAB)^- CAB)T\right], \quad \forall T \in \mathscr{B}(\mathscr{H}).$$

Moreover, if A, B, C, X have matrix forms as above, one has $X_{21} = 0$, $X_{22} = 0$, $A_{11}X_{11} = I$ and $A_{11}X_{12} = 0$.

(iii) Since A is (B,C)-invertible if and only if A is both left and right (B,C)-invertible, we get A is (B,C)-invertible if and only if

$$\mathscr{R}(B^*) = \mathscr{R}(B^*A^*C^*), \quad \mathscr{R}(C) = \mathscr{R}(CAB).$$

By (i) and (ii) we get $X = A_{BC} = X_{11} \oplus 0$ with $X_{11}A_{11} = I$ and $A_{11}X_{11} = I$. So, if $\mathscr{R}(C)$ or $\mathscr{R}(B)$ is closed, A_{BC} is unique with

$$A_{BC} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = B(CAB)^- C. \quad \square$$

By Theorem 4.1 (iii), if A_{BC} exists, from $A_{BC} = A_{11}^{-1} \oplus 0$ and $A = (A_{ij})_{1 \leqslant i,j \leqslant 2}$ we get two idempotents

$$A_{BC}A = \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix}$$

and

$$AA_{BC} = \begin{pmatrix} I & 0 \\ A_{21}A_{11}^{-1} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(C^*) \\ \mathscr{N}(C) \end{pmatrix}.$$

Hence, one has

$$\mathscr{R}(A_{BC}A) = \mathscr{R}(B), \quad \mathscr{N}[(A_{BC}A)^*] = \mathscr{N}(B^*)$$

and

$$\mathscr{R}((AA_{BC})^*) = \mathscr{R}(C^*), \quad \mathscr{N}(AA_{BC}) = \mathscr{N}(C).$$

In fact, there are more range and kernel relations can be obtained by using the matrix representations of A, B, C and A_{BC}. In particular, if $B = C$ we get the following corollary.

COROLLARY 4.1. *Let $A, C \in \mathscr{B}(\mathscr{H})$.*

(i) *A_{LC} exists if and only if $\mathscr{R}(C^*) = \mathscr{R}(C^*A^*C^*)$. In addition, if $\mathscr{R}(C)$ is closed, then*

$$A_{LC} \in \left\{[C(CAC)^- + S(I - CAC(CAC)^-)]C : \quad S \in \mathscr{B}(\mathscr{H})\right\}.$$

(ii) A_{RC} *exists if and only if* $\mathscr{R}(C)=\mathscr{R}(CAC)$. *In addition, if* $\mathscr{R}(C)$ *is closed, then*

$$A_{RC}\in\left\{C\left[(CAC)^{-}C+(I-(CAC)^{-}CAC)T\right]:\quad T\in\mathscr{B}(\mathscr{H})\right\}.$$

(iii) A_C *exists if and only if*

$$\mathscr{R}(C^*)=\mathscr{R}(C^*A^*C^*),\quad \mathscr{R}(C)=\mathscr{R}(CAC).$$

In addition, if $\mathscr{R}(C)$ *is closed, then* A_C *is unique with* $A_C=C(CAC)^{-}C$.

If A is (B,C)-invertible, then $\mathscr{R}(A_{BC})$, $\mathscr{R}(B)$ and $\mathscr{R}(C)$ must be closed. And B_{11},C_{11} in Theorem 4.1 are invertible. In fact, $\mathscr{R}(X)=\mathscr{R}(B)$ implies there exists $N\in\mathscr{B}(\mathscr{H})$ such that $X=BN$. So $B=XAB=BNAB$, which follows that $\mathscr{R}(B)$ is closed. Similarly, $\mathscr{R}(C)$ is also closed. Note that $\mathscr{R}(A_{LBC})$ (resp. $\mathscr{R}(A_{RBC})$) may not be closed. If $B=C=I$, then

$$\mathscr{R}(A_{LBC})=\mathscr{R}(A_{RBC}^*)=\mathscr{H}.$$

As we know, if $A_1,A_2\in\mathscr{B}(\mathscr{H})$ are invertible, then $BA_1=A_2B\Longleftrightarrow A_2^{-1}B=BA_1^{-1}$. If A_1 has right invertible A_R^{-1} and A_2 has left invertible A_L^{-1}, i.e., $A_1A_R^{-1}=I=A_L^{-1}A_2$, then

$$BA_1=A_2B\Longrightarrow A_L^{-1}B=A_L^{-1}BA_1A_R^{-1}=A_L^{-1}A_2BA_R^{-1}=BA_R^{-1}.$$

This was generalized in [8, Theorem 5.1] to the case in the ring. More generally, for right and left (B,C)-inverses of operators, we have the following extension.

THEOREM 4.2. *Let* $A,B,C,A',B',C'\in\mathscr{B}(\mathscr{H})$ *be such that* $DA=A'D$, $DB=B'D$ *and* $DC=C'D$. *If* A_{RBC} *and* $A'_{LB'C'}$ *exist, then* $A'_{LB'C'}D=DA_{RBC}$.

Proof. The argument runs parallel to that for [8, Theorem 5.1]. If $X=:A_{RBC}$ and $Y=:A'_{LB'C'}$ exist, then there exist $M,N\in\mathscr{B}(\mathscr{H})$ such that $X=BM$ and $Y=NC'$. So,

$$A'_{LB'C'}D=NC'D=NDC=NDCAX=NC'A'DX=YA'DX$$

and

$$DA_{RBC}=DBM=B'DM=YA'B'DM=YA'DBM=YA'DX.$$

Hence, $A'_{LB'C'}D=DA_{RBC}$. □

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