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F_a–FRAME AND RIESZ SEQUENCES IN $L^2(\mathbb{R}_+)$

YUN-ZHANG LI AND WEI ZHANG

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Abstract. In application, $L^2(\mathbb{R}_+)$ can model casual signal space. This paper addresses the F_a-frame theory in $L^2(\mathbb{R}_+)$. The notion of F_a-frame for $L^2(\mathbb{R}_+)$ is somewhat like but distinct from that of frame. One of its special cases is a dilation-and-modulation frame for $L^2(\mathbb{R}_+)$. By intuition, F_a-frames have properties similar to usual frames. But they are nontrivial. In this paper, we introduce the notions of F_a-Bessel sequence and F_a-frame sequence in $L^2(\mathbb{R}_+)$. We characterize F_a-Bessel sequences, frame sequences and Riesz sequences, establish the links between F_a-Bessel sequences (F_a-frame sequences) and usual Bessel sequences (frame sequences), between F_a-orthonormal sequences and Parseval F_a-frames, and obtain an expansion with respect to Parseval F_a-frame sequences.

1. Introduction

The notions of frame and Riesz basis were first introduced by Duffin and Schaeffer in studying nonharmonic Fourier series ([6]). An at most countable sequence $\{e_i\}_{i\in\mathscr{I}}$ in a separable Hilbert space $\mathscr{H}$ is called a *Riesz sequence* if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\sum_{i\in\mathscr{I}}|c_i|^2 \leqslant \|\sum_{i\in\mathscr{I}}c_ie_i\|^2 \leqslant B\sum_{i\in\mathscr{I}}|c_i|^2 \text{ for } c\in l_0(\mathscr{I}), \tag{1.1}$$

where A, B are called *Riesz bounds*, $l_0(\mathscr{I})$ denotes the set of finitely supported sequences on $\mathscr{I}$; it is called *frame sequence* if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i\in\mathscr{I}}|\langle f, e_i\rangle|^2 \leqslant B\|f\|^2 \text{ for } f\in\overline{\operatorname{span}}\{e_i\}, \tag{1.2}$$

where A, B are called *frame bounds*. In particular, it is called a *Parseval frame sequence* if $A = B = 1$ in (1.2). It is called a *Bessel sequence* in $\mathscr{H}$ if there exists a constant $0 < B < \infty$ such that

$$\sum_{i\in\mathscr{I}}|\langle f, e_i\rangle|^2 \leqslant B\|f\|^2 \text{ for } f\in\mathscr{H}, \tag{1.3}$$

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where B is called a *Bessel bound*. It is easy to check that $\{e_i\}_{i\in\mathscr{I}}$ is a Bessel sequence if (1.3) holds for $f\in\overline{\operatorname{span}}\{e_i\}$. In particular, a frame sequence (Riesz sequence) $\{e_i\}_{i\in\mathscr{I}}$ is called a *frame* (*Riesz basis*) for $\mathscr{H}$ if $\overline{\operatorname{span}}\{e_i\}=\mathscr{H}$. It is well known that a Riesz sequence is a frame sequence with the frame bounds being Riesz bounds, and that a Riesz sequence is an exact frame sequence. The fundamentals of frames can be found in [3, 6, 20, 21, 27]. Throughout this paper, for a sequence $\{h_i\}_{i\in\mathscr{I}}$ in $\mathscr{H}$, $h=\sum\limits_{i\in\mathscr{I}}h_i$ means that the series $\sum\limits_{i\in\mathscr{I}}h_i$ unconditionally converges to h in $\mathscr{H}$. For a measurable set E in $\mathbb{R}$, χ_E denotes the characteristic function of E, two measurable functions on E being equal means that they are equal almost everywhere on E.

Due to its applications in signal denoising, image compression, numerical treatment of operator equations, etc., the theory of wavelet and Gabor frames in $L^2(\mathbb{R})$ has interested many mathematicians, and seen great achievements during past more than twenty years ([2, 4, 5, 7, 8, 13, 14, 17, 18, 24–26]). This paper focuses on the frame theory in $L^2(\mathbb{R}_+)$, where $\mathbb{R}_+=(0,\infty)$. In [16], numerical experiments were made to establish that the nonnegative integer shifts of the Gaussian function formed a Riesz sequence in $L^2(\mathbb{R}_+)$. In [15], a sufficient condition was obtained to determine whether the nonnegative translates form a Riesz sequence on $L^2(\mathbb{R}_+)$. Observe that $\mathbb{R}$ is a group under the usual addition operation, while $\mathbb{R}_+$ is not. Some authors applied Cantor group operation to $\mathbb{R}_+$, and used Walsh series theory to study wavelet frames for $L^2(\mathbb{R}_+)$. The readers can refer to [1, 9–12, 22] and references therein for details. Recently, Hasankhani and Dehghan in [19] introduced the notion of function-value frame in $L^2(\mathbb{R}_+)$. For simplicity, we call a function-value frame as an F_a-frame. We recall some notions and results in [19].

Given $a>1$, a measurable function f on $\mathbb{R}_+$ is said to be *a-dilation periodic* if $f(a\cdot)=f(\cdot)$ a.e. on $\mathbb{R}_+$, and a sequence $\{f_n\}_{n\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}_+$ is said to be *a-dilation periodic* if every f_n is a-dilation periodic. Obviously, an a-dilation periodic function is determined by its values on $[1,a)$. Throughout this paper, we always denote by $\{\psi_m\}_{m\in\mathbb{Z}}$ the a-dilation periodic function sequence satisfying

$$\psi_m(\cdot)=\frac{1}{\sqrt{a-1}}e^{2\pi i\frac{m\cdot}{a-1}}\text{ on }[1,a). \tag{1.4}$$

For measurable functions f and g on $\mathbb{R}_+$, we write $\langle f,g\rangle_a(\cdot)$ for a-dilation periodic function satisfying

$$\langle f,g\rangle_a(\cdot)=\sum_{j\in\mathbb{Z}}a^jf(a^j\cdot)\overline{g(a^j\cdot)}\text{ a.e. on }[1,a) \tag{1.5}$$

if $\sum\limits_{j\in\mathbb{Z}}a^j|f(a^j\cdot)\overline{g(a^j\cdot)}|$ converges a.e. on $[1,a)$. In particular, we write $\|f\|_a^2(\cdot)=\langle f,f\rangle_a(\cdot)$ on $[1,a)$. We call $\langle f,g\rangle_a$ the F_a*-inner product* of f and g, which is so-called function-valued inner product in [19, Definition 2.1]. Herein ψ_m and $\langle f,g\rangle_a$ are slightly different from those in [19]. They are both required to be a-dilation periodic. Such requirement can simplify our expressions later. From [19, Theorem 2.2], we know the F_a-inner product has many properties similar to those of usual inner products.

For a sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$, we define its F_a-span by

$$F_a\text{-span}\{f_n\} = \left\{ \sum_{n,m\in\mathbb{Z}} c_{n,m}\psi_m f_n : c = \{c_{n,m}\}_{n,m\in\mathbb{Z}} \in l_0(\mathbb{Z}^2) \right\}, \tag{1.6}$$

and denote by $\overline{F_a\text{-span}}\{f_n\}$ the closure of F_a-span$\{f_n\}$ in $L^2(\mathbb{R}_+)$. We claim that F_a-span is different from the usual span. Indeed,

$$F_a\text{-span}\{f_n\} = \left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} c_{n,m}\psi_m \right) f_n : c = \{c_{n,m}\}_{n,m\in\mathbb{Z}} \in l_0(\mathbb{Z}^2) \right\},$$

and

$$\text{span}\{f_n\} = \left\{ \sum_{n\in\mathbb{Z}} c_n f_n : c = \{c_n\}_{n\in\mathbb{Z}} \in l_0(\mathbb{Z}) \right\}.$$

The coefficients of f_n are functions in the former, and are scalars in the latter. We say $\{f_n\}_{n\in\mathbb{Z}}$ is F_a*-complete* in $L^2(\mathbb{R}_+)$ if $\overline{F_a\text{-span}}\{f_n\} = L^2(\mathbb{R}_+)$. For $f, g \in L^2(\mathbb{R}_+)$, f and g are called F_a*-orthogonal* (write $f \perp_{F_a} g$) if $\langle f, g\rangle_a(\cdot) = 0$ a.e. on [1, a).

DEFINITION 1.1. ([19, Definitions 3.2, 3.7]) A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called an F_a-orthonormal sequence if

$$\langle f_m, f_n\rangle_a(\cdot) = \delta_{m,n} \text{ a.e. on } [1,a), \tag{1.7}$$

where the *Kronecker delta* is defined by $\delta_{n,m} = \begin{cases} 1 \text{ if } n = m; \\ 0 \text{ if } n \neq m. \end{cases}$ And it is called an F_a-orthonormal basis for $L^2(\mathbb{R}_+)$ if it is an F_a-orthonormal sequence and F_a-complete in $L^2(\mathbb{R}_+)$.

DEFINITION 1.2. ([19, Definition 4.5]) A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called an F_a-frame sequence in $L^2(\mathbb{R}_+)$ if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|_a^2(\cdot) \leqslant \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \leqslant B\|f\|_a^2(\cdot) \text{ a.e. on } [1,a) \text{ for } f \in \overline{F_a\text{-span}}\{f_n\}, \tag{1.8}$$

where A, B are called frame bounds. It is called an F_a-frame if it is an F_a-frame sequence and F_a-complete in $L^2(\mathbb{R}_+)$.

Observe that (1.7) and (1.8) are two pointwise expressions. This shows that the notions of F_a-orthonormal basis and F_a-frame are very different from the usual ones in $L^2(\mathbb{R}_+)$. So it is mathematically reasonable to establish F_a-frame theory in $L^2(\mathbb{R}_+)$. Hasankhani and Dehghan in [19] proved that an arbitrary F_a-orthonormal sequence admits an orthonormal sequence-like expansion in $L^2(\mathbb{R}_+)$, and gave a link between F_a-frames and usual frames. By [19, Theorems 3.10, 4.8, Proposition 3.5, Corollary 3.8], we have

PROPOSITION 1.1. *For a sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$, we have*

(i) If $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-orthonormal sequence, then $f=\sum_{n\in\mathbb{Z}}\langle f,f_n\rangle_a f_n$ and $\|f\|_a^2(\cdot) = \sum_{n\in\mathbb{Z}}|\langle f,f_n\rangle_a(\cdot)|^2$ a.e. on $[1,a)$ for $f\in\overline{F_a\text{-span}}\{f_n\}$.

(ii) $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-frame (F_a-orthonormal sequence, F_a-orthonormal basis) for $L^2(\mathbb{R}_+)$ if and only if $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is a frame (orthonormal sequence, orthonormal basis) for $L^2(\mathbb{R}_+)$.

The proof of Proposition 1.1 in [19] shows that it is nontrivial to establish F_a-frame theory in $L^2(\mathbb{R}_+)$. We think that it is easy by intuition because we are used to the usual frame theory. In particular, taking $f_n(\cdot)=a^{\frac{n}{2}}\psi(a^n\cdot)$ with $\psi\in L^2(\mathbb{R}_+)$ in Proposition 1.1 (ii), we see that $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-frame for $L^2(\mathbb{R}_+)$ if and only if

$$\{a^{\frac{n}{2}}\psi_m(\cdot)\psi(a^n\cdot)\}_{m,n\in\mathbb{Z}} \tag{1.9}$$

is a frame for $L^2(\mathbb{R}_+)$. This is an interesting fact since (1.9) is a dilation-and-modulation system in $L^2(\mathbb{R}_+)$ generated by ψ. It is well known that translation, dilation and modulation are fundamental operations in wavelet analysis. A translation-and-dilation system is a wavelet system; and a translation-and-modulation system is a Gabor system. They have been extensively studied, while the systems of the form (1.9) have not. Observe that the modulation in (1.9) is somewhat different from the one in Gabor systems. Frames of the form (1.9) in $L^2(\mathbb{R}_+)$ are investigated in [23]. In application, $L^2(\mathbb{R}_+)$ can model casual signal space. Therefore, to establish F_a-frame theory is not only a mathematical problem, but also worth expecting to have potential applications in signal processing and the frame theory of dilation-and-modulation systems in $L^2(\mathbb{R}_+)$. This paper addresses some fundamental problems on F_a-frame theory in $L^2(\mathbb{R}_+)$. For this purpose, we introduce following Definitions 1.3–1.5.

We denote by $L^2(\mathbb{Z}\times[1,a))$ the Hilbert space

$$L^2(\mathbb{Z}\times[1,a))=\left\{f=\{f_n\}_{n\in\mathbb{Z}}:\int_1^a\sum_{n\in\mathbb{Z}}|f_n(x)|^2dx<\infty,\{f_n\}_{n\in\mathbb{Z}}\text{ is }a\text{-dilation periodic}\right\}$$

with the inner product

$$\langle f,g\rangle_{L^2(\mathbb{Z}\times[1,a))}=\int_1^a\sum_{n\in\mathbb{Z}}f_n(x)\overline{g_n(x)}dx \text{ for } f,g\in L^2(\mathbb{Z}\times[1,a)).$$

DEFINITION 1.3. A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$ if there exists a constant $B>0$ such that

$$\sum_{n\in\mathbb{Z}}|\langle f,f_n\rangle_a(\cdot)|^2\leqslant B\|f\|_a^2(\cdot) \text{ a.e. on } [1,a) \text{ for } f\in L^2(\mathbb{R}_+), \tag{1.10}$$

where B is called Bessel bound; it is called a Parseval F_a-frame sequence in $L^2(\mathbb{R}_+)$ if it is an F_a-frame sequence with frame bound 1 ($A=B=1$ in (1.8)).

REMARK 1.1. By Lemma 2.4 below, $f \in L^2(\mathbb{R}_+)$ in (1.10) can be replaced by $f \in \overline{F_a\text{-span}}\{f_n\}$.

DEFINITION 1.4. An F_a-Bessel sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is said to have Riesz property if $\sum\limits_{n\in\mathbb{Z}} g_n f_n = 0$ for some sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{Z}\times[1,a))$ implies that $g_n = 0$ for every $n \in \mathbb{Z}$.

DEFINITION 1.5. A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called an F_a-Riesz sequence if there exist constants $0 < A \leqslant B < \infty$ such that

$$A \sum_{n,m\in\mathbb{Z}} |c_{n,m}|^2 \leqslant \|\sum_{n,m\in\mathbb{Z}} c_{n,m}\psi_m f_n\|^2_{L^2(\mathbb{R}_+)} \leqslant B \sum_{n,m\in\mathbb{Z}} |c_{n,m}|^2 \tag{1.11}$$

for $c \in l_0(\mathbb{Z}^2)$, where A, B are called Riesz bounds; it is called an F_a-Riesz basis for $L^2(\mathbb{R}_+)$ if it is an F_a-Riesz sequence and F_a-complete in $L^2(\mathbb{R}_+)$.

By a careful observation to [3, Lemma 5.5.4], we have

PROPOSITION 1.2. *A sequence $\{e_i\}_{i\in\mathscr{I}}$ in a Hilbert space $\mathscr{H}$ is a frame sequence with frame bounds A and B if and only if $\sum\limits_{i\in\mathscr{I}} c_i e_i$ is well defined for $c \in l^2(\mathscr{I})$ and*

$$A \sum_{i\in\mathscr{I}} |c_i|^2 \leqslant \|\sum_{i\in\mathscr{I}} c_i e_i\|^2 \leqslant B \sum_{i\in\mathscr{I}} |c_i|^2 \text{ for } c \in \mathscr{N}^{\perp},$$

where $\mathscr{N} = \{c \in l^2(\mathscr{I}) : \sum\limits_{i\in\mathscr{I}} c_i e_i = 0\}$.

Proposition 1.2 reveals the essential difference between frame sequence and Riesz sequence. If $\mathscr{N} = \{0\}$, it characterizes Riesz sequences. If $\{e_i\}_{i\in\mathscr{N}}$ is complete in $\mathscr{H}$, it characterizes frames for $\mathscr{H}$. And if $\mathscr{N} = \{0\}$ and $\{e_i\}_{i\in\mathscr{N}}$ is complete in $\mathscr{H}$, it characterizes Riesz bases for $\mathscr{H}$. For simplicity, we call Proposition 1.2 a *Riesz characterization* of frame sequence.

This paper focuses on F_a-frames and Riesz sequences in $L^2(\mathbb{R}_+)$. In section 2, we characterize F_a-Bessel sequences and F_a-frame sequences, establish the connections between F_a-Bessel sequences (F_a-frame sequences) and usual Bessel sequences (frame sequences), between F_a-orthonormal sequences and Parseval F_a-frames, and give an expansion with respect to Parseval F_a-frame sequences. In section 3, we derive a Riesz characterization of F_a-frame sequences like Proposition 1.2. Although these results sound correct by intuition, their proofs are nontrivial. Another fundamental problem on F_a-frames is to find F_a-dual frames. And we will study it in a following paper.

2. F_a-Bessel and frame sequences

This section focuses on F_a-Bessel and frame sequences in $L^2(\mathbb{R}_+)$. We begin with some notions and lemmas. B_a denotes the set

$$B_a = \{f \in L^\infty(\mathbb{R}_+) : f \text{ is } a\text{-dilation periodic }\}.$$

And $L_0^2(\mathbb{Z}\times[1,a))$ and $\mathscr{D}$ denote the sets

$$L_0^2(\mathbb{Z}\times[1,a)) = \{f \in L^2(\mathbb{Z}\times[1,a)) : \text{ there exists } N \in \mathbb{N} \text{ such that } f_n = 0 \text{ for } n \text{ with } |n| > N\},$$

and

$$\mathscr{D} = \{f : f \text{ is defined on } \mathbb{R}_+, \text{ compactly supported and infinitely differentiable }\},$$

respectively. It is well known that $\mathscr{D}$ is dense in $L^2(\mathbb{R}_+)$.

Let X be an arbitrary measure space with a positive measure μ. For $1 \leqslant p < \infty$, we denote by $L^p(\mu)$ the Banach space consisting of all complex measurable functions f on X with the norm

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} < \infty.$$

In particular, $L^2(\mu)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mu)} = \int_X f\overline{g}d\mu.$$

In what follows, for any two measurable functions f and g on X (not necessarily in $L^2(\mu)$), we always write

$$\langle f, g\rangle_{L^2(\mu)} = \int_X f\overline{g}d\mu$$

if $f\overline{g} \in L^1(\mu)$. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. Define the F_a*-analysis operator* D and the F_a*-synthesis operator* R by

$$Df = \{\langle f, f_n\rangle_a\}_{n\in\mathbb{Z}} \tag{2.1}$$

for a measurable function f on $\mathbb{R}_+$ if it is well defined, and

$$Rg = \sum_{n\in\mathbb{Z}} g_n f_n \tag{2.2}$$

for $g = \{g_n\}_{n\in\mathbb{Z}} \in L^2(\mathbb{Z}\times[1,a))$ if it is well defined.

By a standard argument, we have the following two lemmas:

LEMMA 2.1. *For an arbitrary sequence* $\{f_n\}_{n\in\mathbb{Z}}$ *in* $L^2(\mathbb{R}_+)$*, we have*

$$\langle f, Rg\rangle_{L^2(\mathbb{R}_+)} = \langle Df, g\rangle_{L^2(\mathbb{Z}\times[1,a))}$$

for $f \in \mathscr{D}$ *and* $g \in L_0^2(\mathbb{Z}\times[1,a))$.

LEMMA 2.2. *Let* X *be a measure space with a positive measure* μ*, and* Ω *a dense linear subspace of* $L^2(\mu)$*. Suppose* g *is a measurable function on* X*, and there exists a constant* C *such that*

$$|\langle f, g\rangle_{L^2(\mu)}| \leqslant C\|f\|_{L^2(\mu)}$$

for $f \in \Omega$*. Then* $g \in L^2(\mu)$ *and* $\|g\|_{L^2(\mu)} \leqslant C$.

LEMMA 2.3. *(i)* $\int_{[1,a)} |f(x)|^2 dx = \sum_{m\in\mathbb{Z}} \left|\langle f, \psi_m\rangle_{L^2[1,a)}\right|^2$ *for* $f \in L^1[1,a)$.

(ii) For $f, g \in L^2(\mathbb{R}_+)$ *and* $\varphi \in B_a$, *we have*

$$\langle f, g\rangle_a \in L^1[1,a),\ \langle f, \varphi g\rangle_a = \overline{\varphi}\langle f, g\rangle_a, \tag{2.3}$$

$$\langle f, g\rangle_{L^2(\mathbb{R}_+)} = \int_1^a \langle f, g\rangle_a(x)dx. \tag{2.4}$$

$$\|f+g\|_a^2(\cdot) = \|f\|_a^2(\cdot) + \|g\|_a^2(\cdot) \text{ a.e. on } [1,a) \text{ if } f \perp_{F_a} g. \tag{2.5}$$

(iii) $\sum_{m\in\mathbb{Z}} \left|\langle f, \psi_m g\rangle_{L^2(\mathbb{R}_+)}\right|^2 = \int_1^a |\langle f, g\rangle_a(x)|^2 dx$ *for* $f, g \in L^2(\mathbb{R}_+)$.

(iv) For $f, g \in L^2(\mathbb{R}_+)$, $f \perp_{F_a} g$ *if and only if* $f \perp \psi_m g$ *for each* $m \in \mathbb{Z}$.

(v) For $f, g \in L^2(\mathbb{R}_+)$, *if* $f \perp_{F_a} g$, *then* $f \perp \varphi\psi_m g$ *for* $m \in \mathbb{Z}$ *and* $\varphi \in B_a$.

Proof. It is obvious that (iii) implies (iv). And by a standard argument, we have (ii). Next we prove (i), (iii) and (v).

(i) Since $\{\psi_m\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2[1,a)$ when restricted on $[1,a)$,

$$\int_{[1,a)} |f(x)|^2 dx = \sum_{m\in\mathbb{Z}} \left|\langle f, \psi_m\rangle_{L^2[1,a)}\right|^2 \tag{2.6}$$

if $f \in L^2[1,a)$. Now we prove (2.6) for $f \in L^1[1,a) \setminus L^2[1,a)$. Suppose $f \in L^1[1,a) \setminus L^2[1,a)$. Then the left-hand side of (2.6) is infinity. Now we prove by contradiction that the right-hand side of (2.6) is also infinity. Suppose it is finite. Since $\{\psi_m\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2[1,a)$ when restricted on $[1,a)$,

$$g = \sum_{m\in\mathbb{Z}} \left(\int_{[1,a)} f(x)\overline{\psi_m(x)}dx\right)\psi_m$$

belongs to $L^2[1,a)$, and thus to $L^1[1,a)$. It has the same Fourier coefficients as f. So $f = g$ by the uniqueness theorem of Fourier coefficients, and thus $f \in L^2[1,a)$. This is a contradiction.

(iii) By (2.3) and (2.4), we have

$$\langle f, \psi_m g\rangle_{L^2(\mathbb{R}_+)} = \int_1^a \langle f, g\rangle_a(x)\overline{\psi_m(x)}dx.$$

This leads to (iii) by applying (i) to $\langle f, g\rangle_a$.

(v) By (2.3), we have $f \perp_{F_a} \overline{\varphi} g$ and thus (v) by applying (iv). The proof is completed. □

As an immediate consequence of (2.5) and Lemma 2.3 (iv), we have

LEMMA 2.4. *Let* $\{f_n\}_{n\in\mathbb{Z}}$ *be a sequence in* $L^2(\mathbb{R}_+)$, *and* P *and* Q *be the orthogonal projections of* $L^2(\mathbb{R}_+)$ *onto* $\overline{F_a\text{-span}}\{f_n\}$ *and its orthogonal complement, respectively. Then* $Qf \perp_{F_a} Pf$ *and*

$$\|f\|_a^2(\cdot) = \|Pf\|_a^2(\cdot) + \|Qf\|_a^2(\cdot) \text{ a.e. on } [1,a) \tag{2.7}$$

for $f \in L^2(\mathbb{R}_+)$.

LEMMA 2.5. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$ and $\varphi\in B_a$. Then*

$$\overline{span}\{\varphi\psi_m f_n\}_{m,n\in\mathbb{Z}}\subset \overline{F_a\text{-span}}\{f_n\}.$$

Proof. To prove the lemma, we only need to prove that, for $f\in L^2(\mathbb{R}_+)$, $f\perp\psi_m f_n$ for $m,n\in\mathbb{Z}$ implies that $f\perp\varphi\psi_m f_n$ for $m,n\in\mathbb{Z}$ and $\varphi\in B_a$. Suppose $f\perp\psi_m f_n$ for $m,n\in\mathbb{Z}$. Then $f\perp_{F_a} f_n$ for each $n\in\mathbb{Z}$ by Lemma 2.3 (iv), and thus $f\perp\varphi\psi_m f_n$ for $m,n\in\mathbb{Z}$ and $\varphi\in B_a$ by Lemma 2.3 (v). The proof is completed. □

LEMMA 2.6. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. The following are equivalent:*

(*i*) *$\{f_n\}_{n\in\mathbb{Z}}$ is F_a-complete in $L^2(\mathbb{R}_+)$.*

(*ii*) *For $f\in L^2(\mathbb{R}_+)$, we have $f=0$ whenever $\langle f,f_n\rangle_a(\cdot)=0$ a.e. on $[1,a)$.*

Proof. By the definition of F_a-completeness, $\{f_n\}_{n\in\mathbb{Z}}$ is F_a-complete in $L^2(\mathbb{R}_+)$ if and only if $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is complete in $L^2(\mathbb{R}_+)$, equivalently, $f=0$ is a unique solution to

$$\langle f,\psi_m f_n\rangle_{L^2(\mathbb{R}_+)}=0 \text{ for } m,n\in\mathbb{Z} \tag{2.8}$$

in $L^2(\mathbb{R}_+)$. By Lemma 2.3 (iii), we have

$$\sum_{m\in\mathbb{Z}}|\langle f,\psi_m f_n\rangle_{L^2(\mathbb{R}_+)}|^2=\int_1^a|\langle f,f_n\rangle_a(x)|^2dx.$$

Thus (2.8) is equivalent to

$$\langle f,f_n\rangle_a(\cdot)=0 \text{ a.e. on } [1,a) \text{ for } n\in\mathbb{Z}.$$

The lemma therefore follows. □

Next we turn to the main results of this section. The first one gives a characterization of F_a-Bessel sequences in $L^2(\mathbb{R}_+)$ similar to the one of usual Bessel sequences.

THEOREM 2.1. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. Define the operators D and R as in (2.1) and (2.2) respectively. Then the following are equivalent:*

(*i*) *$\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$ with F_a-Bessel bound B.*

(*ii*) *$\sum\limits_{n\in\mathbb{Z}}|\langle f,f_n\rangle_a(\cdot)|^2\leqslant B\|f\|_a^2(\cdot)$ a.e. on $[1,a)$ for $f\in\mathscr{D}$.*

(*iii*) *$\|Rg\|_{L^2(\mathbb{R}_+)}\leqslant\sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))}$ for $g\in L_0^2(\mathbb{Z}\times[1,a))$.*

(*iv*) *Rg is well defined and $\|Rg\|_{L^2(\mathbb{R}_+)}\leqslant\sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))}$ for $g\in L^2(\mathbb{Z}\times[1,a))$.*

$D^*=R$ and $R^*=D$ if one of (i)–(iv) holds.

Proof. Obviously, (i) implies (ii). Next we prove that (ii)$\Rightarrow$(iii)$\Rightarrow$ (iv)$\Rightarrow$(i).

(ii)$\Rightarrow$(iii). Arbitrarily fix $g\in L_0^2(\mathbb{Z}\times[1,a))$. By (ii), we have

$$\|Df\|^2_{L^2(\mathbb{Z}\times[1,a))}\leqslant B\int_1^a\|f\|_a^2(x)dx=B\|f\|^2_{L^2(\mathbb{R}_+)} \text{ for } f\in\mathscr{D}. \tag{2.9}$$

And by Lemma 2.1, we have

$$\langle f, Rg\rangle_{L^2(\mathbb{R}_+)} = \langle Df, g\rangle_{L^2(\mathbb{Z}\times[1,a))} \text{ for } f \in \mathscr{D}.$$

This leads to

$$\left|\langle f, Rg\rangle_{L^2(\mathbb{R}_+)}\right| \leqslant \sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))}\|f\|_{L^2(\mathbb{R}_+)} \text{ for } f \in \mathscr{D}$$

by (2.9). Thus

$$\|Rg\|_{L^2(\mathbb{R}_+)} \leqslant \sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g \in L^2_0(\mathbb{Z}\times[1,a)) \tag{2.10}$$

by Lemma 2.2.

(iii)$\Rightarrow$(iv). Fix $g \in L^2(\mathbb{Z}\times[1,a))$. For an arbitrary increasing sequence $\{F_l\}_{l=1}^\infty$ of finite subsets of $\mathbb{Z}$ satisfying $\bigcup\limits_{l=1}^{\infty} F_l = \mathbb{Z}$, define $\{g^{(l)}\}_{l=1}^\infty$ by

$$g_n^{(l)} = \begin{cases} g_n & \text{if } n \in F_l; \\ 0 & \text{if } n \notin F_l. \end{cases}$$

Then

$$\lim_{l\to\infty}\|g^{(l)} - g\|_{L^2(\mathbb{Z}\times[1,a))} = 0. \tag{2.11}$$

Again using (2.10) to $g^{(l)}$, we have

$$\|Rg^{(l)}\|_{L^2(\mathbb{R}_+)} \leqslant \sqrt{B}\|g^{(l)}\|_{L^2(\mathbb{Z}\times[1,a))} \tag{2.12}$$

for $l \in \mathbb{N}$. Letting $l \to \infty$ in (2.12), we obtain that

$$\|Rg\|_{L^2(\mathbb{R}_+)} \leqslant \sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))}$$

by (2.11).

(iv)$\Rightarrow$(i). Obviously, (iv) implies (iii). So we only need to prove (iii) implies (i). Suppose (iii) holds, that is

$$\|Rg\|_{L^2(\mathbb{R}_+)} \leqslant \sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g \in L^2_0(\mathbb{Z}\times[1,a)). \tag{2.13}$$

By Lemma 2.1, we have

$$\langle Rg, f\rangle_{L^2(\mathbb{R}_+)} = \langle g, Df\rangle_{L^2(\mathbb{Z}\times[1,a))}$$

for $f \in \mathscr{D}$ and $g \in L^2_0(\mathbb{Z}\times[1,a))$. It follows that

$$\begin{aligned}\left|\langle g, Df\rangle_{L^2(\mathbb{Z}\times[1,a))}\right| &\leqslant \|Rg\|_{L^2(\mathbb{R}_+)}\|f\|_{L^2(\mathbb{R}_+)} \\ &\leqslant \sqrt{B}\|f\|_{L^2(\mathbb{R}_+)}\|g\|_{L^2(\mathbb{Z}\times[1,a))}\end{aligned}$$

for $f \in \mathscr{D}$ and $g \in L^2_0(\mathbb{Z}\times[1,a))$ by (2.13), and thus

$$\|Df\|_{L^2(\mathbb{Z}\times[1,a))} \leqslant \sqrt{B}\|f\|_{L^2(\mathbb{R}_+)} \tag{2.14}$$

for $f \in \mathscr{D}$ by Lemma 2.2. For a general $f \in L^2(\mathbb{R}_+)$, there exists a sequence $\{f^{(k)}\}$ in $\mathscr{D}$ such that

$$\lim_{k\to\infty} \|f^{(k)} - f\|_{L^2(\mathbb{R}_+)} = 0. \tag{2.15}$$

It follows that

$$\|Df^{(k)} - Df^{(l)}\|_{L^2(\mathbb{Z}\times[1,a))} \leqslant \sqrt{B}\|f^{(k)} - f^{(l)}\|_{L^2(\mathbb{R}_+)} \to 0$$

as $k, l \to \infty$ by (2.14), and thus

$$\lim_{k\to\infty} \|Df^{(k)} - h\|_{L^2(\mathbb{Z}\times[1,a))} = 0 \tag{2.16}$$

for some $h \in L^2(\mathbb{Z}\times[1,a))$. Next we show $h = Df$. By (2.16), there exists a subsequence $\{f^{(k_l)}\}$ of $\{f^{(k)}\}$ such that

$$\lim_{l\to\infty} \langle f^{(k_l)}, f_n\rangle_a(\cdot) = h_n(\cdot) \text{ a.e. on } [1,a) \tag{2.17}$$

for each $n \in \mathbb{Z}$. Observe that

$$\int_1^a \sum_{j\in\mathbb{Z}} \left| a^{\frac{j}{2}} f^{(k_l)}(a^j x) - a^{\frac{j}{2}} f(a^j x)\right|^2 dx = \|f^{(k_l)} - f\|^2_{L^2(\mathbb{R}_+)} \to 0$$

as $l \to \infty$ by (2.4) and (2.15). So there exists a subsequence $\{f^{(k_{l_m})}\}$ of $\{f^{(k_l)}\}$ such that

$$\lim_{m\to\infty} \sum_{j\in\mathbb{Z}} \left| a^{\frac{j}{2}} f^{(k_{l_m})}(a^j\cdot) - a^{\frac{j}{2}} f(a^j\cdot)\right|^2 = 0$$

a.e. on $[1,a)$. This leads to

$$\begin{aligned}
\langle f^{(k_{l_m})}, f_n\rangle_a(\cdot) &= \left\langle \{a^{\frac{j}{2}} f^{(k_{l_m})}(a^j\cdot)\}, \{a^{\frac{j}{2}} f_n(a^j\cdot)\}\right\rangle_{l^2(\mathbb{Z})} \\
&\to \left\langle \{a^{\frac{j}{2}} f(a^j\cdot)\}, \{a^{\frac{j}{2}} f_n(a^j\cdot)\}\right\rangle_{l^2(\mathbb{Z})} \\
&= \langle f, f_n\rangle_a(\cdot)
\end{aligned}$$

a.e. on $[1,a)$ for each n as $m \to \infty$, and thus $h = Df$ by (2.17). Therefore,

$$\lim_{k\to\infty} \|Df^{(k)} - Df\|_{L^2(\mathbb{Z}\times[1,a))} = 0$$

by (2.16). Also

$$\|Df^{(k)}\|_{L^2(\mathbb{Z}\times[1,a))} \leqslant \sqrt{B}\|f^{(k)}\|_{L^2(\mathbb{R}_+)} \tag{2.18}$$

by (2.14). Letting $k \to \infty$ in (2.18), we obtain that

$$\|Df\|_{L^2(\mathbb{Z}\times[1,a))} \leqslant \sqrt{B}\|f\|_{L^2(\mathbb{R}_+)} \tag{2.19}$$

for $f \in L^2(\mathbb{R}_+)$. Replacing f by hf in (2.19) with h being an arbitrary element in B_a, we have

$$\int_1^a |h(x)|^2 \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(x)|^2 dx \leqslant B \int_1^a |h(x)|^2 \|f\|^2_a(x) dx \tag{2.20}$$

for $f \in L^2(\mathbb{R}_+)$ by (2.3) and (2.4). This implies (i). Indeed, if (i) does not hold, then $\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 > B\|f\|_a^2(\cdot)$ on some $E \subset [1,a)$ with $|E| > 0$. Then

$$\int_1^a |h(x)|^2 \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(x)|^2 dx > B\int_1^a |h(x)|^2 \|f\|_a^2(x) dx$$

for $h \in B_a$ satisfying $h = \chi_{\bigcup_{j\in\mathbb{Z}} a^j E}$ on $[1,a)$. It contradicts (2.20).

Suppose one of (i)–(iv) holds. Then they all hold since we have proved their mutual equivalence. Arbitrarily fix $f \in L^2(\mathbb{R}_+)$ and $g = \{g_n\}_{n\in\mathbb{Z}} \in L^2(\mathbb{Z}\times[1,a))$. Choose sequences $\{f^{(k)}\}$ in $\mathscr{D}$ and $\{g^{(k)}\}$ in $L_0^2(\mathbb{Z}\times[1,a))$ such that

$$\lim_{k\to\infty} \|f^{(k)} - f\|_{L^2(\mathbb{R}_+)} = \lim_{k\to\infty} \|g^{(k)} - g\|_{L^2(\mathbb{Z}\times[1,a))} = 0. \tag{2.21}$$

Then

$$\lim_{k\to\infty} \|Df^{(k)} - Df\|_{L^2(\mathbb{Z}\times[1,a))} = \lim_{k\to\infty} \|Rg^{(k)} - Rg\|_{L^2(\mathbb{R}_+)} = 0 \tag{2.22}$$

by (2.19) and (iv). By Lemma 2.1, we have

$$\langle f^{(k)}, Rg^{(k)}\rangle_{L^2(\mathbb{R}_+)} = \langle Df^{(k)}, g^{(k)}\rangle_{L^2(\mathbb{Z}\times[1,a))}$$

for every k. Letting $k \to \infty$ leads to

$$\langle f, Rg\rangle_{L^2(\mathbb{R}_+)} = \langle Df, g\rangle_{L^2(\mathbb{Z}\times[1,a))}$$

by (2.21) and (2.22). By the arbitrariness of f and g, we have $D^* = R$ and $R^* = D$. The proof is completed. $\square$

The following theorem establishes the connection between F_a-Bessel sequences (F_a-frame sequences) and usual Bessel sequences (frame sequences).

THEOREM 2.2. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. Then the following are equivalent:*

(i) $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Bessel sequence (F_a-frame sequence) in $L^2(\mathbb{R}_+)$.

(ii) $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence (frame sequence) in $L^2(\mathbb{R}_+)$.

In this case, they have the same bounds.

Proof. By Remark 1.1, $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Bessel sequence with Bessel bound B if and only if

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \leqslant B\|f\|_a^2(\cdot) \text{ a.e. on } [1,a) \text{ for } f \in \overline{F_a\text{-span}}\{f_n\}. \tag{2.23}$$

So to prove the theorem, we only need to prove that, for positive constants C, and A, B with $A \leqslant B$,

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \leqslant C\|f\|_a^2(\cdot) \left(A\|f\|_a^2(\cdot) \leqslant \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \leqslant B\|f\|_a^2(\cdot)\right) \tag{2.24}$$

a.e. on $[1,a)$ for $f\in\overline{F_a\text{-span}}\{f_n\}$ if and only if

$$\sum_{m,n\in\mathbb{Z}}|\langle f,\,\psi_m f_n\rangle_{L^2(\mathbb{R}_+)}|^2\leqslant C\|f\|^2_{L^2(\mathbb{R}_+)}$$

$$\left(A\|f\|^2_{L^2(\mathbb{R}_+)}\leqslant\sum_{m,n\in\mathbb{Z}}|\langle f,\,\psi_m f_n\rangle_{L^2(\mathbb{R}_+)}|^2\leqslant B\|f\|^2_{L^2(\mathbb{R}_+)}\right) \tag{2.25}$$

for $f\in\overline{F_a\text{-span}}\{f_n\}$. By Lemma 2.3 (ii) and (iii), we have

$$\|f\|^2_{L^2(\mathbb{R}_+)}=\int_1^a\|f\|^2_a(x)dx,$$

$$\sum_{m,n\in\mathbb{Z}}|\langle f,\,\psi_m f_n\rangle_{L^2(\mathbb{R}_+)}|^2=\int_1^a\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x)|^2dx.$$

Thus (2.25) is equivalent to

$$\int_1^a\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x)|^2dx\leqslant C\int_1^a\|f\|^2_a(x)dx$$

$$\left(A\int_1^a\|f\|^2_a(x)dx\leqslant\int_1^a\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x)|^2dx\leqslant B\int_1^a\|f\|^2_a(x)dx\right) \tag{2.26}$$

for $f\in\overline{F_a\text{-span}}\{f_n\}$. Therefore, we only need to prove the equivalence between (2.24) and (2.26) to finish the proof. It is obvious (2.24) implies (2.26). Next we prove the converse implication. We only deal with the "$\leqslant$" part outside the bracket, the other part can be proved similarly. Suppose

$$\int_1^a\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x)|^2dx\leqslant C\int_1^a\|f\|^2_a(x)dx \tag{2.27}$$

for $f\in\overline{F_a\text{-span}}\{f_n\}$. Since $\int_1^a\|f\|^2_a(x)=\int_0^\infty|f(x)|^2dx<\infty$, almost every point in $(1,a)$ is a Lebesgue point of $\|f\|^2_a(\cdot)$ and $\sum\limits_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(\cdot)|^2$. Arbitrarily fix such a point x_0. Replace f in (2.27) by $\frac{1}{\sqrt{2\varepsilon}}f\chi_{\bigcup\limits_{j\in\mathbb{Z}}a^j(x_0-\varepsilon,x_0+\varepsilon)}$ with ε small enough that $(x_0-\varepsilon,\,x_0+\varepsilon)\subset[1,a)$ (this causes no trouble by Lemma 2.5). Then we have

$$\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x)|^2dx\leqslant\frac{C}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\|f\|^2_a(x)dx,$$

and thus

$$\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(x_0)|^2\leqslant C\|f\|^2_a(x_0)$$

by letting $\varepsilon\to0$. By the arbitrariness of x_0, we have

$$\sum_{n\in\mathbb{Z}}|\langle f,\,f_n\rangle_a(\cdot)|^2\leqslant C\|f\|^2_a(\cdot)\text{ a.e. on }[1,a).$$

The proof is completed. $\square$

As an immediate consequence of Theorem 2.2, we have

COROLLARY 2.1. *([19, Theorem 4.8]) For a sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$, the following are equivalent:*

(i) $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-frame for $L^2(\mathbb{R}_+)$ with frame bounds A and B.

(ii) $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R}_+)$ with frame bounds A and B.

REMARK 2.1. In [19], Corollary 2.1 is proved by operator method which does not work for frame sequences.

It is well known that a usual orthonormal sequence is a Parseval frame sequence with every element having norm 1. The following theorem shows that an F_a-orthonormal sequence for $L^2(\mathbb{R}_+)$ is a Parseval F_a-frame sequence for $L^2(\mathbb{R}_+)$ with "pointwise norm" 1.

THEOREM 2.3. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-orthonormal sequence for $L^2(\mathbb{R}_+)$ if and only if $\{f_n\}_{n\in\mathbb{Z}}$ is a Parseval F_a-frame sequence for $L^2(\mathbb{R}_+)$ and $\|f_n\|_a(\cdot)=1$ a.e. on $[1,a)$ for $n\in\mathbb{Z}$.*

Proof. Necessity. Suppose that $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-orthonormal sequence for $L^2(\mathbb{R}_+)$. Then $\|f_n\|_a(\cdot)=1$ a.e. on $[1,a)$ by the definition of F_a-orthonormal sequence, and $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R}_+)$ by Proposition 1.1 (ii). It follows that $\{\psi_m f_n\}_{m,n\in\mathbb{Z}}$ is a Parseval frame sequence for $L^2(\mathbb{R}_+)$, and thus $\{f_n\}_{n\in\mathbb{Z}}$ is a Parseval F_a-frame sequence for $L^2(\mathbb{R}_+)$ by Theorem 2.2.

Sufficiency. Suppose that $\{f_n\}_{n\in\mathbb{Z}}$ is a Parseval F_a-frame sequence for $L^2(\mathbb{R}_+)$ and $\|f_n\|_a^2(\cdot)=1$ a.e. on $[1,a)$ for $n\in\mathbb{Z}$. Then

$$\sum_{n\in\mathbb{Z}}|\langle f,f_n\rangle_a(\cdot)|^2=\|f\|_a^2(\cdot) \text{ a.e. on } [1,a) \text{ for } f\in\overline{F_a\text{-span}}\{f_n\}. \tag{2.28}$$

It follows that

$$1=\|f_m\|_a^2(\cdot)=\sum_{n\in\mathbb{Z}}|\langle f_m,f_n\rangle_a(\cdot)|^2=1+\sum_{m\neq n\in\mathbb{Z}}|\langle f_m,f_n\rangle_a(\cdot)|^2.$$

a.e. on $[1,a)$ for $m\in\mathbb{Z}$, and thus $\langle f_m,f_n\rangle_a(\cdot)=\delta_{m,n}$ a.e. on $[1,a)$. Therefore, $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-orthonormal sequence for $L^2(\mathbb{R}_+)$. The proof is completed. □

Next we show that a Parseval F_a-frame sequence admits an expansion as a usual Parseval frame. For this purpose, we need the following lemma:

LEMMA 2.7. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$. Then*

$$\langle Rg,h\rangle_a(\cdot)=\sum_{n\in\mathbb{Z}}g_n(\cdot)\langle f_n,h\rangle_a(\cdot)$$

for $g\in L^2(\mathbb{Z}\times[1,a))$ and $h\in L^2(\mathbb{R}_+)$.

Proof. By Theorem 2.1, for $g \in L^2(\mathbb{Z}\times[1,a))$ and $h \in L^2(\mathbb{R}_+)$, Rg is well-defined, and

$$\int_0^\infty Rg(x)\overline{h(x)}dx = \sum_{n\in\mathbb{Z}} \int_0^\infty g_n(x) f_n(x)\overline{h(x)}dx,$$

equivalently,

$$\int_1^a \langle Rg, h\rangle_a(x)dx = \sum_{n\in\mathbb{Z}} \int_1^a g_n(x)\langle f_n, h\rangle_a(x)dx. \tag{2.29}$$

Observe that

$$\begin{aligned}\int_1^a \sum_{n\in\mathbb{Z}} |g_n(x)\langle f_n, h\rangle_a(x)|dx &\leqslant \int_1^a \left(\sum_{n\in\mathbb{Z}} |g_n(x)|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |\langle h, f_n\rangle_a(x)|^2\right)^{\frac{1}{2}} dx \\ &\leqslant \left(\int_1^a \sum_{n\in\mathbb{Z}} |g_n(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_1^a \sum_{n\in\mathbb{Z}} |\langle h, f_n\rangle_a(x)|^2 dx\right)^{\frac{1}{2}} \\ &\leqslant \sqrt{B}\|g\|_{L^2(\mathbb{Z}\times[1,a))}\|h\|_{L^2(\mathbb{R}_+)} \\ &< \infty\end{aligned}$$

where B is an F_a-Bessel bound of $\{f_n\}_{n\in\mathbb{Z}}$. It follows that $\sum_{n\in\mathbb{Z}} g_n\langle f_n, h\rangle_a \in L^1[1,a)$, and (2.29) can be rewritten as

$$\int_1^a \langle Rg, h\rangle_a(x)dx = \int_1^a \sum_{n\in\mathbb{Z}} g_n(x)\langle f_n, h\rangle_a(x)dx \tag{2.30}$$

for $g \in L^2(\mathbb{Z}\times[1,a))$ and $h \in L^2(\mathbb{R}_+)$. So almost every point of $(1,a)$ is a Lebesgue point of both $\langle Rg, h\rangle_a$ and $\sum_{n\in\mathbb{Z}} g_n\langle f_n, h\rangle_a$ (obviously $\langle Rg, h\rangle_a \in L^1[1,a)$). Arbitrarily fix such a x_0. Replacing h in (2.30) by $\frac{1}{2\varepsilon}h\chi_{\bigcup\limits_{j\in\mathbb{Z}} a^j(x_0-\varepsilon, x_0+\varepsilon)}$ with $\varepsilon > 0$ satisfying $(x_0-\varepsilon, x_0+\varepsilon) \subset (1,a)$, we have

$$\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} \langle Rg, h\rangle_a(x)dx = \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} \sum_{n\in\mathbb{Z}} g_n(x)\langle f_n, h\rangle_a(x)dx.$$

Letting $\varepsilon \to 0$, we obtain that

$$\langle Rg, h\rangle_a(x_0) = \sum_{n\in\mathbb{Z}} g_n(x_0)\langle f_n, h\rangle_a(x_0).$$

By the arbitrariness of x_0, the lemma follows. □

THEOREM 2.4. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Parseval F_a-frame sequence in $L^2(\mathbb{R}_+)$. Then $f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n$ for $f \in \overline{F_a\text{-span}}\{f_n\}$.*

Proof. Since $\{f_n\}_{n\in\mathbb{Z}}$ is a Parseval F_a-frame sequence in $L^2(\mathbb{R}_+)$,

$$\|f\|_a^2(\cdot) = \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \text{ a.e. on } [1,a)$$

for $f \in \overline{F_a\text{-span}}\{f_n\}$. So, using the polarization identity in $l^2(\mathbb{Z})$ we have

$$\langle f, h\rangle_a(\cdot) = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a(\cdot)\langle f_n, h\rangle_a(\cdot) \text{ a.e. on } [1,a)$$

for $f, h \in \overline{F_a\text{-span}}\{f_n\}$. This leads to

$$\langle f, h\rangle_a(\cdot) = \left\langle \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n, h \right\rangle_a (\cdot) \text{ a.e. on } [1,a)$$

by Lemma 2.7, equivalently

$$\left\langle f - \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n, h \right\rangle_a (\cdot) = 0 \text{ a.e. on } [1,a)$$

for $f, h \in \overline{F_a\text{-span}}\{f_n\}$. It follows that

$$\left\langle f - \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n, h \right\rangle_{L^2(\mathbb{R}_+)} = 0$$

for $f, h \in \overline{F_a\text{-span}}\{f_n\}$ by (2.4), and thus

$$f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n$$

for $f \in \overline{F_a\text{-span}}\{f_n\}$. The proof is completed. □

3. The Riesz characterization of F_a-frame sequences

This section is devoted to a Riesz characterization of F_a-frame sequences. As its consequences, we obtain characterizations of F_a-frames and F_a-Riesz bases. In addition, we prove that an F_a-Riesz sequence (basis) is an F_a-frame sequence (F_a-frame) with Riesz property. For this purpose, we introduce a notion and a lemma.

By Theorem 2.1, for an arbitrary F_a-Bessel sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$, we can define associated *F_a-frame operator* S_a by $S_a = RD$, that is

$$S_a f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle_a f_n, \text{ for } f \in L^2(\mathbb{R}_+).$$

Then S_a is well defined, and is a bounded operator on $L^2(\mathbb{R}_+)$.

LEMMA 3.1. ([3, Theorem A.6.5]) *Let U_1, U_2, U_3 be self-adjoint operators on a Hilbert space $\mathscr{H}$. If $U_1 \leqslant U_2, U_3 \geqslant 0$, and U_3 commutes with U_1 and U_2, then $U_1U_3 \leqslant U_2U_3$.*

THEOREM 3.1. *A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is an F_a-frame sequence with bounds A, B if and only if the operator R as in (2.2) is well defined on $L^2(\mathbb{Z}\times[1,a))$ and*

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g=\{g_n\}_{n\in\mathbb{Z}} \in (\ker R)^\perp. \quad (3.1)$$

Proof. Observing that $L^2(\mathbb{Z}\times[1,a)) = \ker(R)+\ker(R)^\perp$, we see that

$$\|Rg\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g\in L^2(\mathbb{Z}\times[1,a))$$

if R is well defined on $L^2(\mathbb{Z}\times[1,a))$, and the right-hand side inequality of (3.1) holds. So we may as well assume that $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B by Theorem 2.1. Next, under this assumption, we prove the equivalence between

$$A\|f\|_a^2(\cdot) \leqslant \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \text{ a.e. on } [1,a) \text{ for } f\in \overline{F_a\text{-span}}\{f_n\} \quad (3.2)$$

and

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \text{ for } g\in(\ker \mathrm{R})^\perp. \quad (3.3)$$

First, we suppose (3.2) holds. Then

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2\right)^2 &= (\langle S_a f, f\rangle_a(\cdot))^2\\ &\leqslant \|S_a f\|_a^2(\cdot)\|f\|_a^2(\cdot)\\ &\leqslant \frac{1}{A}\|S_a f\|_a^2(\cdot)\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2\end{aligned}$$

by Lemma 2.7, and thus

$$A\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(\cdot)|^2 \leqslant \|S_a f\|_a^2(\cdot) \text{ a.e. on } [1,a)$$

for $f\in\overline{F_a\text{-span}}\{f_n\}$. Integrating two sides of the inequality on $[1,a)$, we obtain that

$$A\|Df\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|RDf\|^2_{L^2(\mathbb{R}_+)} \text{ for } f\in\overline{F_a\text{-span}}\{f_n\}. \quad (3.4)$$

Also observe that $Df=0$ for $f\in L^2(\mathbb{R}_+)$ satisfying $f\perp(F_a\text{-span}\{f_n\})$ by Lemma 2.3 (iv). It follows that $\mathrm{range}(D)=\{Df : f\in\overline{F_a\text{-span}} f_n\}$, and (3.4) can be rewritten as

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \text{ for } g\in\mathrm{range}(D).$$

Thus

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g\in\mathrm{range}(D). \quad (3.5)$$

Since $D^* = R$ by Theorem 2.1, we have $\overline{\text{range}(D)} = \ker(R)^{\perp}$. Therefore, (3.3) holds by (3.5).

Now we turn to the converse implication. Suppose (3.3) holds. Then it is easy to check that $\text{range}(R)$ is closed, and thus $\text{range}(D)$ is closed due to $R^* = D$ by Theorem 2.1. Also observe that $\ker(R)^{\perp} = \overline{\text{range}(D)}$ and $\text{range}(D) = \{Df : f \in \overline{F_a\text{-span}} f_n\}$. It follows from (3.3) that

$$A\|Df\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|RDf\|^2_{L^2(\mathbb{R}_+)} \text{ for } f \in \overline{F_a\text{-span}}\{f_n\}. \tag{3.6}$$

Since $D^* = R$, we have

$$\|RDf\|^2_{L^2(\mathbb{R}_+)} = \langle RDf, RDf\rangle_{L^2(\mathbb{R}_+)} = \langle (RD)^*RDf, f\rangle_{L^2(\mathbb{R}_+)} = \langle S_a^2 f, f\rangle_{L^2(\mathbb{R}_+)},$$

and

$$\|Df\|^2_{L^2(\mathbb{Z}\times[1,a))} = \langle Df, Df\rangle_{L^2(\mathbb{Z}\times[1,a))} = \langle RDf, f\rangle_{L^2(\mathbb{R}_+)} = \langle S_a f, f\rangle_{L^2(\mathbb{R}_+)}.$$

So (3.6) can be rewritten as

$$A\langle S_a f, f\rangle_{L^2(\mathbb{R}_+)} \leqslant \langle S_a^2 f, f\rangle_{L^2(\mathbb{R}_+)} \text{ for } f \in \overline{F_a\text{-span}}\{f_n\}.$$

It follows that

$$AS_a \leqslant (S_a)^2 \text{ on } \overline{F_a\text{-span}}\{f_n\}. \tag{3.7}$$

Since $F_a\text{-span}\{f_n\} \subset \text{range}(R)$ and $\text{range}(R)$ is closed, we derive that $\overline{F_a\text{-span}}\{f_n\} \subset \text{range}(R)$. Also observing that $\text{range}(R) \subset \overline{F_a\text{-span}}\{f_n\}$, we have

$$\text{range}(R) = \overline{F_a\text{-span}}\{f_n\}.$$

Also since $D^* = R$ and both $\text{range}(D)$ and $\text{range}(R)$ are closed, we see that S_a is a bounded and invertible operator when restricted on $\text{range}(R)$. So we have

$$AI \leqslant S_a \text{ on } \overline{F_a\text{-span}}\{f_n\}$$

by (3.7) and Lemma 3.1, equivalently,

$$A\int_1^a \|f\|_a^2(x)dx \leqslant \int_1^a \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(x)|^2 dx \text{ for } f \in \overline{F_a\text{-span}}\{f_n\} \tag{3.8}$$

by (2.4) and Lemma 2.7. Fix $f \in \overline{F_a\text{-span}}\{f_n\}$. Also observing that

$$\int_1^a \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a(x)|^2 dx \leqslant B\int_1^a \|f\|_a^2(x)dx = B\|f\|^2_{L^2(\mathbb{R}_+)} < \infty,$$

we claim that almost every point of $(1,a)$ is a Lebesgue point of $\|f\|_a^2$ and $\sum\limits_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a|^2$. Arbitrarily fix such a point x_0. For $\varepsilon > 0$ with $(x_0-\varepsilon, x_0+\varepsilon) \subset [1,a)$, we have $\frac{1}{\sqrt{2\varepsilon}} f\chi_{\bigcup\limits_{j\in\mathbb{Z}} a^j(x_0-\varepsilon,x_0+\varepsilon)} \in \overline{F_a\text{-span}}\{f_n\}$ by Lemma 2.5.

Replace f in (3.8) by $\frac{1}{\sqrt{2\varepsilon}} f\chi_{\bigcup\limits_{j\in\mathbb{Z}} a^j(x_0-\varepsilon,x_0+\varepsilon)}$, we have

$$\frac{A}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\|f\|_a^2(x)dx \leqslant \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\sum_{n\in\mathbb{Z}}|\langle f, f_n\rangle_a(x)|^2dx,$$

and thus

$$A\|f\|_a^2(x_0) \leqslant \sum_{n\in\mathbb{Z}}|\langle f, f_n\rangle_a(x_0)|^2$$

by letting $\varepsilon\to 0$. So (3.2) holds by the arbitrariness of x_0. The proof is completed. $\square$

As an immediate consequence of Theorem 3.1, we have

COROLLARY 3.1. *A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is an F_a-frame if and only if the following two conditions are satisfied:*
(i) $\{f_n\}_{n\in\mathbb{Z}}$ is F_a-complete in $L^2(\mathbb{R}_+)$.
(ii) The operator R as in (2.2) is well defined on $L^2(\mathbb{Z}\times[1,a))$ and

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \textit{ for } g=\{g_n\}_{n\in\mathbb{Z}}\in(\ker R)^{\perp}.$$

The following theorem gives a characterization of F_a-Riesz sequences in $L^2(\mathbb{R}_+)$.

THEOREM 3.2. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. The following are equivalent:*
(i) $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Riesz sequence in $L^2(\mathbb{R}_+)$ with Riesz bounds A,B.
(ii) The operator R as in (2.2) is well defined on $L^2(\mathbb{Z}\times[1,a))$ and

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \|Rg\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \textit{ for } g=\{g_n\}\in L^2(\mathbb{Z}\times[1,a)). \tag{3.9}$$

(iii) $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-frame sequence in $L^2(\mathbb{R}_+)$ with frame bounds A,B, and has Riesz property.

Proof. (i)$\Rightarrow$(ii). Suppose (i) holds. Then

$$A\sum_{n,m\in\mathbb{Z}}|c_{n,m}|^2 \leqslant \left\|\sum_{n,m\in\mathbb{Z}}c_{n,m}\psi_m f_n\right\|^2_{L^2(\mathbb{R}_+)} \leqslant B\sum_{n,m\in\mathbb{Z}}|c_{n,m}|^2 \tag{3.10}$$

for $c=\{c_{n,m}\}_{n,m\in\mathbb{Z}}\in l_0(\mathbb{Z}^2)$. Since $l_0(\mathbb{Z}^2)$ is dense in $l^2(\mathbb{Z}^2)$, $\sum\limits_{n,m\in\mathbb{Z}}c_{n,m}\psi_m f_n$ converges unconditionally in $L^2(\mathbb{R}_+)$ and (3.10) holds for $c\in l^2(\mathbb{Z}^2)$. For an arbitrary $g\in L^2(\mathbb{Z}\times[1,a))$ with $g_n=\sum\limits_{m\in\mathbb{Z}}c_{n,m}\psi_m$ for each $n\in\mathbb{Z}$, we have

$$\|c\|_{l^2(\mathbb{Z}^2)}=\|g\|_{L^2(\mathbb{Z}\times[1,a))}<\infty.$$

It follows that $\sum\limits_{n\in\mathbb{Z}}g_nf_n$ converges unconditionally in $L^2(\mathbb{R}_+)$, and

$$\sum_{n\in\mathbb{Z}}g_nf_n=\sum_{n,m\in\mathbb{Z}}c_{n,m}\psi_m f_n.$$

Also observing that (3.10) holds for $c \in l^2(\mathbb{Z}^2)$, we obtain (ii).

(ii)$\Rightarrow$(iii). Suppose (ii) holds. Then $\ker R = \{0\}$, and thus $(\ker R)^\perp = L^2(\mathbb{Z} \times [1,a))$. Applying Theorem 3.1, $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-frame sequence with bounds A and B, and has Riesz property.

(iii)$\Rightarrow$(i). Suppose (iii) holds. Then

$$A\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \leqslant \left\| \sum_{n\in\mathbb{Z}} g_n f_n \right\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|g\|^2_{L^2(\mathbb{Z}\times[1,a))} \text{ for } g = \{g_n\} \in L^2(\mathbb{Z}\times[1,a)) \tag{3.11}$$

by Theorem 3.1 and Definition 1.4. For an arbitrary $c = \{c_{n,m}\}_{n,m\in\mathbb{Z}} \in l_0(\mathbb{Z}^2)$, we have

$$\sum_{n,m\in\mathbb{Z}} c_{n,m}\psi_m f_n = \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} c_{n,m}\psi_m \right) f_n = \sum_{n\in\mathbb{Z}} g_n f_n,$$

$$\|c\|^2_{l^2(\mathbb{Z}^2)} = \|g\|^2_{L^2(\mathbb{Z}\times[1,a))},$$

where $g = \{g_n\}_{n\in\mathbb{Z}}$ with $g_n = \sum\limits_{m\in\mathbb{Z}} c_{n,m}\psi_m$. Applying (3.11) to such g, we obtain that

$$A\|c\|^2_{l^2(\mathbb{Z}^2)} \leqslant \left\| \sum_{n,m\in\mathbb{Z}} c_{n,m}\psi_m f_n \right\|^2_{L^2(\mathbb{R}_+)} \leqslant B\|c\|^2_{l^2(\mathbb{Z}^2)}.$$

This gives (i) by the arbitrariness of $c \in l_0(\mathbb{Z}^2)$. The proof is completed. □

As an immediate consequence of Theorem 3.2, we have

COROLLARY 3.2. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L^2(\mathbb{R}_+)$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is an F_a-Riesz basis for $L^2(\mathbb{R}_+)$ if and only if it is an F_a-frame for $L^2(\mathbb{R}_+)$ with Riesz property.*

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Yun-Zhang Li
College of Applied Sciences
Beijing University of Technology
Beijing 100124, P. R. China
e-mail: yzlee@bjut.edu.cn

Wei Zhang
College of Applied Sciences
Beijing University of Technology
Beijing 100124, P. R. China
and
School of Mathematics and Information Sciences
Henan University of Economics and Law
Zhengzhou 450046, P. R. China
e-mail: zwfylhappy@126.com