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OPERATOR–WEIGHTED COMPOSITION OPERATORS ON VECTOR–VALUED BLOCH SPACES

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Abstract. Let φ be an analytic self-map of $\mathbb{D}$ and ψ be an analytic operator-valued function on $\mathbb{D}$. Then the operator-weighted composition operator $W_{\psi,\varphi}$ is defined by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \quad z \in \mathbb{D},$$

where f is an analytic function $\mathbb{D} \to X$, X is any complex Banach space. In this paper by considering $W_{\psi,\varphi}$ on vector-valued Bloch spaces, some qualitative properties of these operators will be characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and, for any complex Banach space X, $H(\mathbb{D},X)$ denotes the space of all analytic functions $f\colon \mathbb{D} \to X$. Recall a vector-valued function $f\colon \mathbb{D} \to X$ is analytic if for every $x^* \in X^*$ the function $x^* \circ f\colon \mathbb{D} \to \mathbb{C}$ is analytic in the classical sense. Let $L(X,Y)$ be the space of bounded linear operators $X \to Y$ where X and Y are complex Banach spaces. Let $\psi : \mathbb{D} \to L(X,Y)$ be an analytic function and φ an analytic self-map of $\mathbb{D}$. Then we define the operator-weighted composition operator $W_{\psi,\varphi}$ by $f \mapsto \psi(f \circ \varphi)$, that is

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \quad z \in \mathbb{D}, \tag{1.1}$$

for $f \in H(\mathbb{D},X)$. $W_{\psi,\varphi}$ is a linear map $H(\mathbb{D},X) \to H(\mathbb{D},Y)$. We have $W_{\psi,\varphi} = M_\psi C_\varphi$, where M_ψ is the operator-valued multiplier $f \mapsto \psi f$ and C_φ is the composition operator $f \mapsto f \circ \varphi$. Thus the operator-weighted composition operators are a large class of operators contains other classes in the vector-valued or scalar-valued setting. For example, if $X = Y = \mathbb{C}$ then we have weighted composition operators $W_{\psi,\varphi}f(z) = \psi(z)f(\varphi(z))$, which are the generalization of multiplication operators $M_\psi f(z) = \psi(z)f(z)$ and composition operators $C_\varphi f(z) = f(\varphi(z))$. These operators have been studied extensively on several analytic Banach spaces. In the vector-valued setting, weighted composition operators have been studied widely on vector-valued Hardy, Bergman, Dirichlet, Bloch and BMOA spaces, see [2, 6, 8, 9, 10, 11, 12, 13, 18]. Weighted compositions appear naturally: for a large class of Banach spaces X, all

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linear onto isometries between X-valued H^∞ spaces are of the form (1.1) for suitable ψ and φ.

Operator-weighted composition operators are a new subject in the study of operators on analytic function spaces and are defined in [14] and [11]. Laitila and Tylli [11] characterized boundedness and (weak) compactness of these operators on $H_v^\infty(X)$ spaces and the author(s) in [14] and [3] obtained the same results for locally convex spaces of analytic vector-valued functions.

In this paper we are going to investigate operator-weighted composition operators on vector-valued Bloch spaces. Boundedness and (weak) compactness of these operators will be characterized.

Let X be a complex Banach space and $\alpha > 0$. The vector-valued Bloch space $\mathscr{B}^\alpha(X)$ is the set of all analytic functions $f : \mathbb{D} \to X$ such that

$$\|f\|_{\mathscr{B}^\alpha(X)} = \|f(0)\|_X + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \|f'(z)\|_X < \infty.$$

Also the little vector-valued Bloch space $\mathscr{B}_0^\alpha(X)$ is the closed subspace of $\mathscr{B}^\alpha(X)$ consisting of the analytic functions f with the property $\lim_{|z|\to 1}(1-|z|^2)^\alpha \|f'(z)\|_X = 0$. If $\alpha = 1$, we simply write $\mathscr{B}(X)$ and $\mathscr{B}_0(X)$. In the scalar-valued case $X = \mathbb{C}$, $\mathscr{B}^\alpha(X) = \mathscr{B}^\alpha$. It is proved that (see [15])

$$\|f\|_{\mathscr{B}} \approx \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{B_p},$$

where φ_a is the Mobius transformation $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ and B_p is the Bergman space consisting of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

Here $dA(z)$ is the normalized area measure on $\mathbb{D}$ and $p > 0$. Moreover, an analytic function $f : \mathbb{D} \to X$ is in the vector-valued Bergman space if

$$\|f\|_{B_p(X)}^p = \int_{\mathbb{D}} ||f(z)||_X^p \, dA(z) < \infty.$$

The organization of the paper is as follows. In section 2, a norm estimate of the operator is obtained. Section 3 is related to the compactness and weak compactness of the operator-weighted composition operator. Finally, (weak) compactness of the operator T_ψ which is essential for the study of $W_{\psi,\varphi}$ will be discussed. The operator $T_\psi : X \to \mathscr{B}(Y)$ is defined by $x \mapsto \psi(.)x$ which is a new ingredient in the vector-valued context. We also present some examples.

Throughout the paper all constants are denoted by c which may vary from one position to another. For two values A and B, $A \approx B$ means that there are positive constants c_1 and c_2 such that $c_1 B \leqslant A \leqslant c_2 B$. Also $A \lesssim B$ means that there exists a positive constant c such that $A \leqslant cB$.

2. Boundedness

In this section we find norm estimate of $W_{\psi,\varphi} : \mathscr{B}(X) \to \mathscr{B}(Y)$. First we recall the following lemma which is essential for our proof.

LEMMA 1. [19, Lemma 2.1] *For $\alpha > 0$ and any complex Banach space X, if $f \in \mathscr{B}^{\alpha}(X)$, then*

1. $||f(z)||_X \leqslant c||f||_{\mathscr{B}^{\alpha}(X)}$ *for any $z \in \mathbb{D}$ and $0 < \alpha < 1$;*
2. $||f(z)||_X \leqslant c\log \frac{2}{1-|z|^2}||f||_{\mathscr{B}^{\alpha}(X)}$ *for any $z \in \mathbb{D}$ and $\alpha = 1$;*
3. $||f(z)||_X \leqslant c\frac{1}{(1-|z|^2)^{\alpha-1}}||f||_{\mathscr{B}^{\alpha}(X)}$ *for any $z \in \mathbb{D}$ and $\alpha > 1$.*

For estimation of $||f'(z)||_X$, $f \in \mathscr{B}^{\alpha}(X)$, we have

$$(1-|z|^2)^{\alpha}||f'(z)||_X \leqslant ||f(0)||_X + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}||f'(z)||_X = ||f||_{\mathscr{B}^{\alpha}(X)}.$$

So

$$||f'(z)||_X \leqslant \frac{||f||_{\mathscr{B}^{\alpha}(X)}}{(1-|z|^2)^{\alpha}}.$$

In the following theorem we need a differentiation of $\psi(z)(f(\varphi(z)))$. We can differentiate $\psi(z)(f(\varphi(z)))$ similarly as in the scalar-valued case. In fact, by the product rule,

$$(\psi(z)(f(\varphi(z))))' = \psi'(z)(f(\varphi(z))) + \psi(z)(f'(\varphi(z)))\cdot\varphi'(z)$$

which is a Y-valued analytic function. Note that although for each z, $\psi(z)$ is a linear operator from X to Y, the function $z \mapsto \psi(z)$ is analytic, so it does have a derivative. Indeed, since $\psi'(z)$ and $f'(z)$ exist, we have

$$\lim_{w\to z}\frac{1}{w-z}(\psi(w)-\psi(z))$$

exists in $L(X,Y)$ and

$$\lim_{w\to z}\frac{1}{w-z}(f(w)-f(z))$$

exists in X. Then one must verify that

$$\lim_{w\to z}\frac{1}{w-z}((\psi f)(w)-(\psi f)(z))$$

exists in X with the limit equal to the usual product form.

First, define

$$q_1(\varphi,\psi,z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)|,$$

$$q_2(\varphi,\psi,z)=(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}.$$

We then set

$$Q_1(\varphi,\psi)=\sup_{z\in\mathbb{D}}q_1(\varphi,\psi,z)\quad\text{and}\quad Q_2(\varphi,\psi)=\sup_{z\in\mathbb{D}}q_2(\varphi,\psi,z).$$

THEOREM 2. *Let X and Y be complex Banach spaces. Then for $W_{\psi,\varphi}:\mathscr{B}(X)\to\mathscr{B}(Y)$, we have $W_{\psi,\varphi}$ is bounded if $Q_1(\varphi,\psi)$ and $Q_2(\varphi,\psi')$ are finite. Moreover,*

$$\max\{Q_1(\varphi,\psi),Q_2(\varphi,\psi')\}\lesssim\|W_{\psi,\varphi}\|\lesssim\max\{q_2(\varphi,\psi,0),Q_1(\varphi,\psi),Q_2(\varphi,\psi')\}$$

Proof. Suppose $f\in\mathscr{B}(X)$. For every $z\in\mathbb{D}$ we have

$$\begin{aligned}\|(W_{\psi,\varphi}(f)'(z))\|_Y&=\|\,[\psi(z)(f(\varphi(z)))]'\,\|_Y\\&=\|\psi'(z)(f(\varphi(z)))+\psi(z)(f'(\varphi(z))\varphi'(z))\|_Y\\&\leqslant\|\psi'(z)\|_{L(X,Y)}\|f(\varphi(z))\|_X+\|\psi(z)\|_{L(X,Y)}\|f'(\varphi(z))\|_X|\varphi'(z)|\\&\leqslant c\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}\|f\|_{\mathscr{B}(X)}+\|\psi(z)\|_{L(X,Y)}\frac{|\varphi'(z)|\|f\|_{\mathscr{B}(X)}}{1-|\varphi(z)|^2}.\end{aligned}$$

So

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathscr{B}(Y)}\leqslant{}&c\log\frac{1}{1-|\varphi(0)|^2}\|\psi(0)\|_{L(X,Y)}\|f\|_{\mathscr{B}(X)}\\&+c\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}\|f\|_{\mathscr{B}(X)}\\&+\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|\;\|\psi(z)\|_{L(X,Y)}\|f\|_{\mathscr{B}(X)}.\end{aligned}$$

This proves the boundedness claim and the upper estimate of the norm. Fix $x_0\in X$. Let $w\in\mathbb{D}$ and define the functions f_w as

$$f_w(z)=\frac{1}{\overline{\varphi(w)}}\left\{\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}-1\right\}x_0.$$

As in the proof of Theorem 2.1 [16], $f_w\in\mathscr{B}(X)$ and $M=\sup\{\|f_w\|_{\mathscr{B}(X)}:w\in\mathbb{D}\}<\infty$. Also $f_w(\varphi(w))=0$ and $f_w'(\varphi(w))=x_0/(1-|\varphi(w)|^2)$, which implies $(W_{\psi,\varphi}f_w)'(w)=\psi(w)f_w'(\varphi(w))\varphi'(w)$. So

$$\begin{aligned}M\|W_{\psi,\varphi}\|\geqslant\|W_{\psi,\varphi}f_w\|_{\mathscr{B}(Y)}&=\sup_{z\in\mathbb{D}}(1-|z|^2)\|(W_{\psi,\varphi}f_w)'(z)\|_Y\\&\geqslant(1-|w|^2)\|(W_{\psi,\varphi}f_w)'(w)\|_Y\\&=(1-|w|^2)\|\psi(w)(f_w'(\varphi(w)))\varphi'(w)\|_Y\\&=(1-|w|^2)\|\psi(w)(x_0)\frac{\varphi'(w)}{1-|\varphi(w)|^2}\|_Y\\&=\frac{(1-|w|^2)|\varphi'(w)|}{1-|\varphi(w)|^2}\|\psi(w)(x_0)\|_Y.\end{aligned}$$

Since x_0 is arbitrary, then

$$\|W_{\psi,\varphi}\| \geqslant \frac{1}{M}\frac{(1-|w|^2)|\varphi'(w)|}{1-|\varphi(w)|^2}\|\psi(w)\|_{L(X,Y)}.$$

Again for $w\in\mathbb{D}$ being arbitrary, we have

$$\|W_{\psi,\varphi}\| \geqslant \frac{1}{M}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}. \tag{2.1}$$

Now we define other functions

$$g_w(z) = 2\log\frac{1}{1-\overline{\varphi(w)}z} - \left(\log\frac{1}{1-\overline{\varphi(w)}z}\right)^2 \Big/ \log\frac{1}{1-|\varphi(w)|^2},$$

$$h_w(z) = g_w(z)x_0.$$

By using the same method in [16], $h_w\in\mathscr{B}(X)$ and $L=\sup\{\|h_w\|_{\mathscr{B}(X)}:\ w\in\mathbb{D}\}<\infty$. Also $h_w'(\varphi(w))=0$ and $h_w(\varphi(w))=x_0\log(1/(1-|\varphi(w)|^2))$, which implies $(W_{\psi,\varphi}f_w)'(w)=\psi'(w)(h_w(\varphi(w)))$. Then

$$\begin{aligned}
L\|W_{\psi,\varphi}\| &\geqslant \|W_{\psi,\varphi}h_w\|_{\mathscr{B}(Y)}\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\|(W_{\psi,\varphi}h_w)'(z)\|_Y\\
&\geqslant (1-|w|^2)\|(W_{\psi,\varphi}h_w)'(w)\|_Y\\
&= (1-|w|^2)\|\psi'(w)(h_w(\varphi(w)))\|_Y\\
&= (1-|w|^2)\log\frac{1}{1-|\varphi(w)|^2}\|\psi'(w)(x_0)\|_Y.
\end{aligned}$$

So

$$\|W_{\psi,\varphi}\| \geqslant \frac{1}{L}\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}. \tag{2.2}$$

The equations (2.1) and (2.2) imply that

$$\begin{aligned}
\|W_{\psi,\varphi}\| \gtrsim \max\Big\{&\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)|\\
&+\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}\Big\}. \quad \square
\end{aligned}$$

The above result can be applied to other cases $0<\alpha<1$ and $\alpha>1$ between different Bloch spaces $W_{\psi,\varphi}:\mathscr{B}^{\alpha}(X)\to\mathscr{B}^{\beta}(Y)$, where $\beta>0$. The proofs are similar.

3. (Weak) compactness

A bounded linear operator between two Banach spaces X and Y is called compact, weakly compact, if it maps the closed unit ball of X onto a relatively compact, a relatively weakly compact set in Y. The class of all (weakly) compact operators between X and Y is denoted by $K(X,Y)$ $(W(X,Y))$. The essential and weak essential norm of an operator $T : X \to Y$ are defined by

$$\|T\|_e = dist(T, K(X,Y)), \quad \|T\|_w = dist(T, W(X,Y)).$$

The operator T is compact if and only if $\|T\|_e = 0$ and is weakly compact if and only if $\|T\|_w = 0$.

Here we use the linear operator $(C_r f)(z) = f(rz)$ for $f : \mathbb{D} \to X$ analytic and $0 < r < 1$.

LEMMA 3. *The operators $C_r : \mathscr{B}^\alpha(X) \to \mathscr{B}^\alpha(X)$ satisfy the following properties.*

1. $||C_r|| \leqslant 1$ *for any* $0 < r < 1$.
2. *For every* $R \in (0,1)$,

$$\lim_{r\to 1} \sup_{||f||_{\mathscr{B}^\alpha(X)} \leqslant 1} \sup_{|z|\leqslant R} \max\{||(f - C_r f)'(z)||_X, ||(f - C_r f)(z)||_X\} = 0.$$

3. *Suppose that $W_{\psi,\varphi} : \mathscr{B}^\alpha(X) \to \mathscr{B}^\alpha(Y)$ is bounded. If $T_\psi : X \to \mathscr{B}^\alpha(Y)$ is a (weakly) compact operator, then $W_{\psi,\varphi} \circ C_r : \mathscr{B}^\alpha(X) \to \mathscr{B}^\alpha(Y)$ is (weakly) compact operator.*

Proof. For (1), we have

$$\begin{aligned}||C_r|| &= \sup_{||f||_{\mathscr{B}^\alpha(X)} \leqslant 1} \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha ||rf'(rz)||_X \\ &\leqslant \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|rz|^2)^\alpha} \leqslant \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|z|^2)^\alpha} = 1.\end{aligned}$$

Let $0 < r, R < 1$ and $f : \mathbb{D} \to X$ be an analytic function and $z \in \mathbb{D}$. By setting $\rho = (|z|+1)/2$, we have

$$\begin{aligned}||f'(z) - rf'(rz)||_X &= \left\| \int_0^{2\pi} \left(\frac{\rho f'(\rho e^{i\theta})}{\rho - ze^{-i\theta}} - \frac{\rho r f'(\rho e^{i\theta})}{\rho - rze^{-i\theta}} \right) \frac{d\theta}{2\pi} \right\|_X \\ &\leqslant \sup_{\theta\in[0,2\pi)} \frac{(1-r)||f'(\rho e^{i\theta})||_X}{|\rho - ze^{-i\theta}||\rho - rze^{-i\theta}|} \\ &\leqslant c(1-r) \frac{||f||_{\mathscr{B}^\alpha(X)}}{(1-|z|)^{2+\alpha}},\end{aligned}$$

where c is a positive constant. Moreover, since

$$(f-C_rf)(z)=e^{i\theta}\int_0^{|z|}(f-C_rf)'(te^{i\theta})\,dt,$$

where $z=|z|e^{i\theta}$, we get

$$\begin{aligned}||(f-C_rf)(z)||_X &\leqslant c(1-r)||f||_{\mathscr{B}^\alpha(X)}\int_0^{|z|}\frac{1}{(1-|te^{i\theta}|)^{2+\alpha}}\,dt\\ &= c(1-r)||f||_{\mathscr{B}^\alpha(X)}\int_0^{|z|}\frac{1}{(1-t)^{2+\alpha}}\,dt\\ &= \frac{c}{2+\alpha-1}(1-r)||f||_{\mathscr{B}^\alpha(X)}\left(\frac{1}{(1-|z|)^{2+\alpha-1}}-1\right)\end{aligned}$$

Now, the proof of (2) is complete.

Finally, for $f\in\mathscr{B}^\alpha(X)$ with $f(z)=\sum_{k=0}^{\infty}x_kz^k$, put $P_n(f)=\sum_{k=0}^{n}x_kz^k$, $n\geqslant 0$ and $q_k(f)=x_k$, $k\in\mathbb{N}$. Here q_k are operators $\mathscr{B}^\alpha(X)\to X$. Since $z^kx_k=\int_0^{2\pi}f(ze^{i\theta})e^{-ik\theta}\frac{d\theta}{2\pi}$, then

$$\begin{aligned}&||z^kx_k||_X\leqslant c||f||_{\mathscr{B}^\alpha(X)}\quad 0<\alpha<1,\\ &||z^kx_k||_X\leqslant c\log\frac{2}{1-|z|^2}||f||_{\mathscr{B}^\alpha(X)}\quad \alpha=1,\\ &||z^kx_k||_X\leqslant c\frac{||f||_{\mathscr{B}^\alpha(X)}}{(1-|z|^2)^{\alpha-1}}\quad \alpha>1,\end{aligned}$$

where $c>0$ is a constant. The above relations hold for every $z\in\mathbb{D}$. Let $z\in\mathbb{D}$ and $|z|=1/2$. So

$$\begin{aligned}&||x_k||_X\leqslant c2^k||f||_{\mathscr{B}^\alpha(X)}\quad 0<\alpha<1,\\ &||x_k||_X\leqslant c2^k\log 8/3||f||_{\mathscr{B}^\alpha(X)}\quad \alpha=1,\\ &||x_k||_X\leqslant \frac{c2^k4^{\alpha-1}}{3^{\alpha-1}}||f||_{\mathscr{B}^\alpha(X)}\quad \alpha>1.\end{aligned}$$

Therefore q_k are bounded on $\mathscr{B}^\alpha(X)$ in each case. So $T_\psi\circ q_k:\mathscr{B}^\alpha(X)\to\mathscr{B}^\alpha(Y)$ are compact operators. Since

$$(W_{\psi,\varphi}P_nf)(z)=\sum_{k=0}^{n}\varphi(z)^k.\psi(z)x_k=\sum_{k=0}^{n}\varphi(z)^k.(T_\psi q_kf)(z),$$

it follows that $W_{\psi,\varphi}P_n$ are compact operators $\mathscr{B}^\alpha(X)\to\mathscr{B}^\alpha(Y)$ for all n. Fix $0<r<1$, let $\varepsilon>0$ and fix n_0 so that $\sum_{k=n_0+1}^{\infty}kr^k<\varepsilon$. Then, for any $f\in\mathscr{B}^\alpha(X)$ with

$f(z)=\sum_{k=0}^{\infty} x_k z^k$,

$$\begin{aligned}||(C_r-P_{n_0}C_r)f||_{\mathscr{B}^\alpha(X)} &= ||((C_r-P_{n_0}C_r)f)(0)||_X \\ &\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha ||((C_r-P_{n_0}C_r)f)'(z)||_X \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha ||((C_r-P_{n_0}C_r)f)'(z)||_X \\ &\leqslant ||r\sum_{k=n_0+1}^{\infty} kr^k x_k z^{k-1}||_X < c\varepsilon ||f||_{\mathscr{B}^\alpha(X)}\end{aligned}$$

The above relation holds for every $n > n_0$. It follows that $||C_r - P_nC_r|| \to 0$, as $n\to\infty$. Then $\|W_{\psi,\varphi}C_r - W_{\psi,\varphi}P_nC_r\| \to 0$ as $n\to\infty$. Thus $W_{\psi,\varphi}C_r$ is compact operator. The proof in the case of weakly compact is similar. □

To characterize compactness, the two conditions we use are

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)| = 0 \tag{3.1}$$

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)} = 0. \tag{3.2}$$

Weak compactness and compactness of the operator $W_{\psi,\varphi}$ are related to the operator T_ψ.

THEOREM 4. *Let X and Y be complex Banach spaces and $W_{\psi,\varphi}:\mathscr{B}(X)\to\mathscr{B}(Y)$ be bounded. Then $W_{\psi,\varphi}:\mathscr{B}(X)\to\mathscr{B}(Y)$ is (weakly) compact if and only if*

1. *$T_\psi : X\to\mathscr{B}(Y)$ is (weakly) compact, and*
2. *The conditions (3.1) and (3.2) hold.*

Since $T_\psi = W_{\psi,\varphi}A$ where $A: X\to\mathscr{B}(X)$ is defined by $A(x)=f_x$, $f_x(z)=x$, the (weak compactness) compactness of $W_{\psi,\varphi}$ implies T_ψ is (weakly) compact.

Proof of sufficiency in Theorem 4. Suppose that $T_\psi : X\to\mathscr{B}(Y)$ is weakly compact and (3.1) and (3.2) hold. It will be enough to prove that

$$\begin{aligned}\|W_{\psi,\varphi}\|_w &\leqslant 2\lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)| \\ &\quad + 2\lim_{s\to 1}\sup_{|\varphi(z)|>s}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}.\end{aligned}$$

By Lemma 3(3), the operators $W_{\psi,\varphi}C_r$ are weakly compact too. So

$$\|W_{\psi,\varphi}\|_w \leqslant \|W_{\psi,\varphi} - W_{\psi,\varphi}C_r\|.$$

For every $f \in \mathscr{B}(X)$ and $z \in \mathbb{D}$ we have

$$\begin{aligned}(1-|z|^2)\|(W_{\psi,\varphi}f - W_{\psi,\varphi}C_r f)'(z)\|_Y \\ &\leqslant \max\{\sup_{|\varphi(z)|>s}(1-|z|^2)\|(W_{\psi,\varphi}f - W_{\psi,\varphi}C_r f)'(z)\|_Y, \\ &\qquad \sup_{|\varphi(z)|\leqslant s}(1-|z|^2)\|(W_{\psi,\varphi}f - W_{\psi,\varphi}C_r f)'(z)\|_Y\} \\ &= \max\{I,J\}.\end{aligned}$$

For I, using Lemma 1 we have

$$\begin{aligned}I \leqslant &\sup_{|\varphi(z)|>s}(1-|z|^2)\|\psi'(z)(f(\varphi(z))) - \psi'(z)(C_r(f(\varphi(z))))\|_Y \\ &+ \sup_{|\varphi(z)|>s}\|\psi(z)(f'(\varphi(z)))\varphi'(z) - \psi(z)((C_r f)'(\varphi(z)))\varphi'(z)\|_Y \\ \leqslant &\sup_{|\varphi(z)|>s}(1-|z|^2)\|\psi'(z)\|\ \|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ &+ \sup_{|\varphi(z)|>s}(1-|z|^2)\|\psi(z)\|\ \|f'(\varphi(z))\varphi'(z) - (C_r f)'(\varphi(z))\varphi'(z)\|_X \\ \leqslant &\, 2\sup_{|\varphi(z)|>s}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)}\|f\|_{\mathscr{B}(X)} \\ &+ 2\sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)|\ \|f\|_{\mathscr{B}(X)}.\end{aligned}$$

Put

$$E = \sup_{z\in\mathbb{D}}(1-|z|^2)\|\psi'(z)\|_{L(X,Y)} \quad \textit{and} \quad F = \sup_{z\in\mathbb{D}}(1-|z|^2)\|\psi(z)\|_{L(X,Y)}\ |\varphi'(z)|.$$

To see that E and F are finite, take $f_1(z) = x_0$ and $f_2(z) = x_0 z$, where $x_0 \in X$ is arbitrary with $||x_0||_X = 1$. Then using the boundedness of $W_{\psi,\varphi}$ implies the results. For J,

$$\begin{aligned}J \leqslant &\sup_{\|f\|_{\mathscr{B}(X)}\leqslant 1}\sup_{|\varphi(z)|\leqslant s}(1-|z|^2)\|\psi'(z)\|_{L(X,Y)}\ \|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ &+ \sup_{\|f\|_{\mathscr{B}(X)}\leqslant 1}\sup_{|\varphi(z)|\leqslant s}(1-|z|^2)\|\psi(z)\|_{L(X,Y)}\ \|f'(\varphi(z))\varphi'(z) - (C_r f)'(\varphi(z))\varphi'(z)\|_X \\ \leqslant &\, E\sup_{\|f\|_{\mathscr{B}(X)}\leqslant 1}\sup_{|\varphi(z)|\leqslant s}\|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ &+ F\sup_{\|f\|_{\mathscr{B}(X)}\leqslant 1}\sup_{|\varphi(z)|\leqslant s}\|f'(\varphi(z)) - (C_r f)'(\varphi(z))\|_X.\end{aligned}$$

Letting $r \to 1$, according to Lemma 3(2), we have

$$\lim_{r\to 1} J = 0. \quad \square$$

The idea for the proof of the necessity part is to use a Leibov-type argument similar to the one in [10].

LEMMA 5. *Let* $\{f_n\}$ *be a sequence in* $\mathscr{B}_0(X)$, $\|f_n\|_{\mathscr{B}(X)} = 1$ *and* $\|f_n\|_{B_2(X)} \to 0$ *as* $n \to \infty$. *Then there exists a subsequence* $\{f_{n_k}\}$ *such that it is equivalent to the natural basis of* c_0, *that is, the map* $(\lambda_k) \mapsto \sum_k \lambda_k f_{n_k}$ *is an isomorphism from* c_0 *into* $\mathscr{B}_0(X)$.

Proof. It can be proved that $\|f_n\|_{\mathscr{B}(X)} \leqslant \sup_{\|x^*\|\leqslant 1} \|x^* \circ f_n\|_{\mathscr{B}}$. So there exists an $x^* \in X^*$ such that $\|f_n\|_{\mathscr{B}(X)} \leqslant \|x^* \circ f_n\|_{\mathscr{B}}$. On the other hand $\|x^* \circ f_n\|_{\mathscr{B}} \leqslant \|f_n\|_{\mathscr{B}(X)}$. Now we define $g_n = x^* \circ f_n$. Then $g_n \in \mathscr{B}_0$, $\|g_n\|_{\mathscr{B}} = 1$ and $\|g_n\|_{B_2} \to 0$ as $n \to \infty$. Set

$$\gamma(g_n, a) = \|g_n \circ \varphi_a - g_n(a)\|_{B_2}.$$

The change of variable implies that $\gamma(g_n, a) \leqslant \|g_n \circ \varphi_a\|_{B_2} \leqslant c_a \|g_n\|_{B_2}$ where c_a is an increasing function of $|a|$. Fix any $0 < r < 1$ and $|a| \leqslant r$, $c_a \leqslant c_r$. It follows from $\|g_n\|_{B_2} \to 0$ that

$$\sup_{|a|\leqslant r} \gamma(g_n, a) \to 0$$

as $n \to \infty$. Since $\{g_n\} \in \mathscr{B}_0$, $\gamma(g_n, a) \to 0$ as $|a| \to 1$ for each n. These properties imply that there exist increasing sequences of positive integers $\{n_k\}$ and numbers $0 < r_k < 1$ such that for each k, $\|g_{n_k}\|_{B_2} < 2^{-k-1}$ and

$$\sup_{|a|\leqslant r_k} \gamma(g_{n_k}, a) < 2^{-k-1}, \qquad \sup_{|a|\geqslant r_{k+1}} \gamma(g_{n_k}, a) < 2^{-k-1}.$$

For every $a \in \mathbb{D}$ we then have $\gamma(g_{n_k}, a) < 2^{-k-1}$ for all but possibly one index k, for which $\gamma(g_{n_k}, a) \leqslant 1$. Hence $\sum_k \gamma(g_{n_k}, a) < 1 + \frac{1}{2} = \frac{3}{2}$. Now we define the map $S : c_0 \to \mathscr{B}_0$ by

$$S\lambda = \sum_{k=1}^{\infty} \lambda_k g_{n_k}$$

for $\lambda = (\lambda_k) \in c_0$. Since $\|\sum_{k=1}^{\infty} \lambda_k g_{n_k}\|_{B_2} \leqslant \sum_{k=1}^{\infty} |\lambda_k| \|g_{n_k}\|_{B_2}$, the series converges in B_p. It can be seen from the inequality

$$\gamma(S\lambda, a) \leqslant \sum_{k=1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \leqslant \frac{3}{2} \|\lambda\|_\infty$$

that $\|S\lambda\|_{\mathscr{B}} \leqslant \frac{3}{2} c \|\lambda\|_\infty$. Since $\lambda \in c_0$, we have $\lambda_k \to 0$. So there exists an integer K such that $|\lambda_k| < \varepsilon$ for $k > K$. Then

$$\begin{aligned}
\gamma(S\lambda, a) &\leqslant \sum_{k=1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \\
&= \sum_{k=1}^{K} |\lambda_k| \gamma(g_{n_k}, a) + \sum_{k=K+1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \\
&\leqslant \|\lambda\|_\infty \sum_{k=1}^{K} |\lambda_k| \gamma(g_{n_k}, a) + \frac{3}{2}\varepsilon.
\end{aligned}$$

Since $\gamma(g_{n_k},a)\to 0$ as $|a|\to 1$ for each k, and $\varepsilon>0$ was arbitrary, this implies that $S\lambda\in\mathscr{B}_0$. Therefore S is a bounded linear operator from c_0 into $\mathscr{B}_0$.

To check that S is one to one, we prove that S is bounded below. For $\lambda=(\lambda_k)\in c_0$, there exists an index K for which $|\lambda_K|=\|\lambda\|_\infty$. We know that $|g_{n_K}(0)|\leqslant\|g_{n_K}\|_{B_2}<\frac{1}{4}$ and $\|g_{n_K}\|_{\mathscr{B}}=1$. So there exists a positive constant c such that $\sup_{a\in\mathbb{D}}\gamma(g_{n_K},a)\approx 1-|g_{n_K}(0)|>1-\frac{1}{4}$. Hence there is a point $a\in\mathbb{D}$ such that $\gamma(g_{n_K},a)>1-\frac{1}{4}$. Note that for $k\neq K$, we have $\gamma(g_{n_k},a)<2^{-k-1}$. Therefore

$$\begin{aligned}\|S\lambda\|_{\mathscr{B}}&\geqslant c\gamma(S\lambda,a)\gtrsim|\lambda_K|\gamma(g_{n_K},a)-\sum_{k\neq K}|\lambda_k|\gamma(g_{n_k},a)\\&\geqslant\left(1-\frac{1}{4}\right)\|\lambda\|_\infty-\frac{1}{2}\|\lambda\|_\infty=\frac{1}{4}\|\lambda\|_\infty.\end{aligned}$$

We have proved that S is an isomorphism from c_0 into $\mathscr{B}_0$. An easy calculation shows that $S\lambda=x^*(T\lambda)$ where $T\lambda=\sum_{k=1}^\infty\lambda_k f_{n_k}$. Then T is an isomorphism from c_0 into $\mathscr{B}_0(X)$ and we are done. □

Proof of necessity in Theorem 4. Suppose that the conditions (3.1) and (3.2) fail. We will complete the proof by proving that $W_{\psi,\varphi}:\mathscr{B}(X)\to\mathscr{B}(Y)$ fixes a copy of c_0 and therefore it is not weakly compact.

If the condition (3.1) fails, then there exist $c>0$ and a sequence $\{a_n\}\in\mathbb{D}$ such that $|\varphi(a_n)|\to 1$ as $n\to\infty$ and

$$\frac{1-|a_n|^2}{1-|\varphi(a_n)|^2}\|\psi(a_n)\|_{L(X,Y)}|\varphi'(a_n)|>c.$$

Let $x\in X$ with $\|x\|_X=1$. Define the functions f_n by $f_n(z)=g_n(z)x$ where

$$g_n(z)=\frac{(1-|\varphi(a_n)|^2)^2}{(1-\overline{\varphi(a_n)}z)^2}-\frac{1-|\varphi(a_n)|^2}{1-\overline{\varphi(a_n)}z}.$$

Then $M=||f_n||_{\mathscr{B}(X)}<\infty$, $f_n(\varphi(a_n))=0$ and $f_n'(\varphi(a_n))=\overline{\varphi(a_n)}/(1-|\varphi(a_n)|^2)x$. Furthermore $f_n\in\mathscr{B}_0(X)$ and

$$\begin{aligned}\|f_n\|_{B_2(X)}^2&=\int_{\mathbb{D}}\left\|\frac{(1-|\varphi(a_n)|^2)^2}{(1-\overline{\varphi(a_n)}z)^2}-\frac{1-|\varphi(a_n)|^2}{1-\overline{\varphi(a_n)}z}x\right\|_X^2 dA(z)\\&\leqslant 4\int_{\mathbb{D}}\frac{(1-|\varphi(a_n)|^2)^4}{|1-\overline{\varphi(a_n)}z|^4}+\frac{(1-|\varphi(a_n)|^2)^2}{|1-\overline{\varphi(a_n)}z|^2}\,dA(z)\\&=4(1-|\varphi(a_n)|^2)^4\int_{\mathbb{D}}\frac{(1}{|1-\overline{\varphi(a_n)}z|^4}dA(z)\\&\quad+4(1-|\varphi(a_n)|^2)^2\int_{\mathbb{D}}\frac{1}{|1-\overline{\varphi(a_n)}z|^2}\,dA(z)\\&\leqslant 4c(1-|\varphi(a_n)|^2)^2+4c(1-|\varphi(a_n)|^2)^2\log\frac{1}{1-|\varphi(a_n)|^2}.\end{aligned}$$

The last line of the above relation is due to Theorem 1.12 of [20]. So $\|f_n\|_{B_2(X)} \to 0$ as $n \to \infty$. Also

$$\begin{aligned}\|W_{\psi,\varphi} f_n\|_{\mathscr{B}(X)} &\geqslant (1-|a_n|^2)\|W_{\psi,\varphi}(f_n)'(a_n)\|_X \\ &= (1-|a_n|^2)\|\psi(a_n)(f_n'(\varphi(a_n)))\varphi'(a_n)\|_X \\ &= \frac{1-|a_n|^2}{1-|\varphi(a_n)|^2}\|\psi(a_n)(x)\|_X|\varphi'(a_n)||\varphi(a_n)|,\end{aligned}$$

for every $x \in X$. So

$$\|W_{\psi,\varphi} f_n\|_{\mathscr{B}(X)} \geqslant \frac{1-|a_n|^2}{1-|\varphi(a_n)|^2}\|\psi(a_n)\|_{L(X,Y)}|\varphi'(a_n)||\varphi(a_n)| \geqslant \frac{c}{2}.$$

For using Lemma 5, we should have $||f_n||_{\mathscr{B}(X)} = 1$. This can be done by normalizing the norm of the sequence $\{f_n\}$. Without loss of generality we use $\{f_n\}$ again. So we can find a subsequence $\{f_{n_k}\}$ which is equivalent to the natural basis of c_0 which implies that $\{W_{\psi,\varphi} f_{n_k}\}$ is a weak-null sequence in $\mathscr{B}(X)$. Using the Bessaga-Polczynski selection principle (see [1, 1.3.10]) to $\{W_{\psi,\varphi} f_{n_k}\}$, there exists a subsequence, say $\{f_{n_k}\}$ again, such that $\{W_{\psi,\varphi} f_{n_k}\}$ is a semi-normalized basic sequence in $\mathscr{B}(X)$. Hence there are constants $A, B > 0$ such that

$$\begin{aligned}A.\|\lambda\|_\infty \leqslant \|\sum_{k=1}^{\infty}\lambda_k W_{\psi,\varphi} f_{n_k}\|_{\mathscr{B}(X)} &\leqslant \|W_{\psi,\varphi}\|.\|\sum_{k=1}^{\infty}\lambda_k f_{n_k}\|_{\mathscr{B}(X)} \\ &\leqslant B.\|W_{\psi,\varphi}\|\ \|\lambda\|_\infty,\end{aligned}$$

for every $\lambda = (\lambda_k) \in c_0$. These estimates state that the restriction of $W_{\psi,\varphi}$ to the closed subspace of $\mathscr{B}(X)$ spanned by the sequence $\{f_{n_k}\}$ is an isomorphism onto a linearly isomorphic copy of c_0, and we are done.

If condition (3.2) fails, then we have the same result by using the functions

$$g_n(z) = \frac{-1}{\log(1-|\varphi(z_n)|^2)}\left(3\left(\log\frac{1}{1-\overline{\varphi(z_n)}z}\right)^2 - 2\left(\log\frac{1}{1-\overline{\varphi(z_n)}z}\right)^3\right). \quad \square$$

4. Compactness properties of T_ψ

EXAMPLE 6. There is an analytic operator-valued map $\psi \in H^\infty(L(\ell^1))$, $\psi(z) \in K(\ell^1)$, but $T_\psi : \ell^1 \to \mathscr{B}(\ell^1)$ is not even weakly conditionally compact.

Proof. Define the bounded operator-valued analytic map $\psi : \mathbb{D} \to L(\ell^1)$ by $\psi(z) = \sum_{k=1}^{\infty} z^k e_k^* \otimes e_k$, where (e_k) denotes the standard unit vector basis of ℓ^1 and $(e_k^*) \subset c_0$ its biorthogonal sequence. In other words,

$$\psi(z)x = \sum_{k=1}^{\infty} z^k x_k e_k, \quad x = (x_k) \in \ell^1,\ z \in \mathbb{D}.$$

The claim is that T_ψ is not weakly conditionally compact as an operator $\ell^1 \to \mathscr{B}(\ell^1)$. Suppose to the contrary that T_ψ is weakly conditionally compact. So there exists a weakly Cauchy subsequence $(T_\psi(e_{n_j}))$ such that the difference sequence $(T_\psi(e_{n_{2j+1}} - e_{n_{2j}}))$ is weak-null in $\mathscr{B}(\ell^1)$. By Mazur's theorem,

$$\|\sum_{j=1}^{s} c_j T_\psi(e_{n_{2j+1}} - e_{n_{2j}})\|_{\mathscr{B}(\ell^1)} < \frac{1}{2}$$

for a suitable convex combination, where $\sum_{j=1}^{s} c_j = 1$ and $c_j \geqslant 0$ for $j = 1, \cdots, s$. On the other hand, we have

$$\begin{aligned}
&\|\sum_{j=1}^{s} c_j T_\psi(e_{n_{2j+1}} - e_{n_{2j}})\|_{\mathscr{B}(\ell^1)} \\
&\quad = \sup_{z\in\mathbb{D}}(1-|z|^2)\|\sum_{j=1}^{s} c_j(n_{2j+1}z^{n_{2j+1}-1}e_{n_{2j+1}} - n_{2j}z^{n_{2j}-1}e_{n_{2j}})\|_{\ell^1} \\
&\quad = \sup_{z\in\mathbb{D}}(1-|z|^2)\sum_{j=1}^{s} c_j(n_{2j+1}|z|^{n_{2j+1}-1} + n_{2j}|z|^{n_{2j}-1}) \\
&\quad \geqslant \sum_{j=1}^{s} c_j(n_{2j+1} + n_{2j}) \geqslant 1,
\end{aligned}$$

which is a contradiction. □

LEMMA 7. *Let X be a complex Banach spaces, $X_0 \subset X$ be a closed subspace and $f \in \mathscr{B}(X)$. Then $f \in \mathscr{B}_0(X_0)$ if and only if $f \in \mathscr{B}_0(X)$ and $f(\mathbb{D}) \subset X_0$.*

Proof. It is obvious that if $f \in \mathscr{B}_0(X_0)$ then $f \in \mathscr{B}_0(X)$ and $f(\mathbb{D}) \subset X_0$. Suppose that $f \in \mathscr{B}_0(X)$ and $f(\mathbb{D}) \subset X_0$. Define $f_r(z) = f(rz)$, $0 < r < 1$. So

$$\begin{aligned}
\|f_r\|_{\mathscr{B}(X_0)} &= ||f(0)||_{X_0} + \sup_{z\in\mathbb{D}}(1-|z|^2)\|rf'(rz)\|_{X_0} \\
&< ||f(0)||_{X_0} + \sup_{z\in\mathbb{D}}(1-|z|^2)\|f'(rz)\|_{X_0} \\
&< ||f(0)||_{X_0} + \sup_{z\in\mathbb{D}}(1-|rz|^2)\|f'(rz)\|_{X} \leqslant \|f\|_{\mathscr{B}(X)}.
\end{aligned}$$

It means that $f_r \in \mathscr{B}(X_0)$. Then

$$\begin{aligned}
\lim_{|z|\to 1}(1-|z|^2)\|f_r'(z)\|_{X_0} &= \lim_{|z|\to 1}(1-|z|^2)\|rf'(rz)\|_{X_0} \\
&< \lim_{|z|\to 1}(1-|z|^2)\|f'(rz)\|_{X_0} \\
&< \lim_{|z|\to 1}(1-|rz|^2)\|f'(rz)\|_{X} = 0.
\end{aligned}$$

Hence $f_r \in \mathscr{B}_0(X_0)$. There are polynomials $p_n(z) = \sum_{j=0}^{N_n} z^j x_j^{(n)}$ such that $p_n \to f$ in $\mathscr{B}(X)$ as $n \to \infty$. By using the same way we have $(p_n)_r \to f_r$ as $n \to \infty$. Also

$$\begin{aligned} \|p_n - (p_n)_r\|_{\mathscr{B}(X)} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\|\sum_{j=1}^{N_n} jz^{j-1}x_j^{(n)}(1-r^j)\|_X \\ &< \sup_{z\in\mathbb{D}}(1-|z|^2)(\sup_{1\leqslant j\leqslant N_n}\|x_j^{(n)}\|_X)N_n\sum_{j=1}^{N_n}(1-r^j) \to 0, \end{aligned}$$

as $r \to 1$. Since

$$\begin{aligned} \|f - f_r\|_{\mathscr{B}(X_0)} &= \|f - f_r\|_{\mathscr{B}(X)} \\ &\leqslant \|f - p_n\|_{\mathscr{B}(X)} + \|p_n - (p_n)_r\|_{\mathscr{B}(X)} + \|(p_n)_r - f_r\|_{\mathscr{B}(X)}, \end{aligned}$$

we deduce that $f_r \to f$ in $\mathscr{B}_0(X_0)$ as $r \to 1$. □

THEOREM 8. *Let X and Y be complex Banach spaces and $\psi \in \mathscr{B}_0(L(X,Y))$. Then $T_\psi : X \to \mathscr{B}(Y)$ is compact (weakly compact) if and only if $\psi(\mathbb{D}) \subset K(X,Y)$ $(\psi(\mathbb{D}) \subset W(X,Y))$.*

Proof. Suppose that $\psi(\mathbb{D}) \subset K(X,Y)$. The previous lemma implies that $\psi \in \mathscr{B}_0(K(X,Y))$. Find $K(X,Y)$-valued polynomials $\psi_n(z) = \sum_{k=0}^{n} z^k U_k^{(n)}$ such that $\psi_n \to \psi$ in $\mathscr{B}(K(X,Y))$. Since $\|T_\psi\| = \|\psi\|_{\mathscr{B}(L(X,Y))}$, we have $\|T_\psi - T_{\psi_n}\| \to 0$ as $n \to \infty$. So it will be sufficient to prove that $T_{\psi_n} : X \to \mathscr{B}(Y)$ is compact. Define the maps $\theta_k : Y \to \mathscr{B}(Y)$ by $(\theta_k y)(z) = z^k y$. Each θ_k is bounded, then $\theta_k \circ U_k^{(n)}$ is compact and so $T_{\psi_n} = \sum_{k=0}^{n} \theta_k \circ U_k^{(n)}$.

Now, suppose that $T_\psi : X \to \mathscr{B}(Y)$ is compact. Fix $z \in \mathbb{D}$ and define $\gamma : \mathscr{B}(Y) \to Y$ by $\gamma(f) = f(z)$. Then γ is a bounded linear operator and $\gamma \circ T_\psi = \psi(z)$. So $\psi(z) : X \to Y$ is a compact operator. □

EXAMPLE 9. (1) Let X be any Banach space, $\psi(z) \equiv U$ and $\varphi(z) = \frac{z+1}{2}$ for $z \in \mathbb{D}$, where $U \in K(X)$ is a fixed operator. Then $W_{\psi,\varphi}$ is compact $\mathscr{B}(X) \to \mathscr{B}(X)$. Indeed

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}\|\psi(z)\|_{L(X,Y)}|\varphi'(z)| \leqslant \frac{1}{2}\|U\|_{L(X,Y)}\lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{1-|z|^2}{-|z|^2} = 0,$$

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}\|\psi'(z)\|_{L(X,Y)} = 0.$$

Also T_ψ is compact by Theorem 8 since $\psi(z) \equiv U \in K(X)$. So Theorem 4 implies that $W_{\psi,\varphi}$ is compact.

(2) Let X be any reflexive Banach space, $\psi(z) \equiv V$ and $\varphi(z) = \frac{z+1}{2}$ for $z \in \mathbb{D}$, where $V \notin K(X)$ is a fixed operator. Then $W_{\psi,\varphi}$ is weakly compact, but not compact. Non-compactness of W_ψ is because of non-compactness of T_ψ (Theorem 8, $\psi(z) \equiv V \notin K(X)$). Since X is reflexive, V is weakly compact. So T_ψ is weakly compact by Theorem 8. Also the conditions (3.1) and (3.2) hold. Now $W_{\psi,\varphi}$ is weakly compact.

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