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COMPLEMENTARITY OF SUBSPACES OF ℓ_∞ REVISITED

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Abstract. We present a simple criterion for complementarity of subspaces of ℓ_∞ induced by certain bounded linear operators. As applications, it is shown that some typical and well-known subspaces such as mean or almost convergent sequence spaces are uncomplemented in ℓ_∞. We also note that there exists a weak* closed uncomplemented subspace of ℓ_∞.

1. Introduction

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent or null sequences, respectively. The first example of an uncomplemented subspace of ℓ_∞ is c (and c_0). This folklore result was given by Phillips in his 1940 paper [14]. The original proof is based on a detailed study of representation of linear operators. Nearly a quarter century later, Whitley [16] drastically simplified the proof of Phillips' result by using an idea due to Nakamura and Kakutani [13]. Precisely, he showed that $(\ell_\infty/c_0)^*$ has no countable total subset; and it suffices to conclude that c_0 is not complemented in ℓ_∞ since the property that the dual space has a countable total subset is preserved under taking subspaces or by linear isomorphisms.

Complementarity of subspaces of ℓ_∞ had been deeply studied as a part of the main stream of the isomorphic theory. In 1967, Lindenstrauss [10] gave an important characterization of complemented subspaces of ℓ_∞ by showing that ℓ_∞ is a prime Banach space, that is, an infinite dimensional complemented subspace of ℓ_∞ must be isomorphic to ℓ_∞. (The converse implication follows from the fact that ℓ_∞ is an injective Banach space; see, for example, [1, Proposition 2.5.2].) It follows that, at least, there is no separable infinite dimensional complemented subspace of ℓ_∞. At this point, Phillips' result was significantly improved by Lindenstrauss.

The theoretical development for the study of Banach space structure of complemented subspaces of ℓ_∞ has been mostly reached the stage of satisfaction (since such spaces are isomorphically "the same" as ℓ_∞). However, this well-known characterization is not always effective in determining the complementarity of concrete non-separable subspaces of ℓ_∞. To do this, we still have to investigate for case by case; because we do not know whether checking an infinite dimensional subspace of ℓ_∞ is

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(not) isomorphic to ℓ_∞ is easier than examining the complementarity of the subspace directly.

In this paper, we present a simple criterion for complementarity of subspaces of ℓ_∞ induced by bounded linear operators admitting matrix representations. The proof employs the above mentioned argument of Whitley, that is, we check whether the dual spaces of the quotient of ℓ_∞ by such subspaces have countable total subsets. As an application, among other examples, we show that closed subspaces between c_0 and the mean convergent sequence space are all uncomplemented in ℓ_∞. From this and Lorentz's theorem [11], in particular, we conclude that the space of almost convergent sequences is also uncomplemented in ℓ_∞. On the other hand, we provide an example of a weak* closed uncomplemented subspace of ℓ_∞. It is known that if M is a weak* closed subspace of ℓ_∞ then $(\ell_\infty/M)^*$ always has a countable total subset. Consequently, we see that there is a limit to determining the complementarity of subspaces of ℓ_∞ by using Whitley's method.

2. Subspaces of ℓ_∞ induced by matrices

Let $B(\ell_\infty)$ denote the Banach space of bounded linear operators on ℓ_∞. Suppose that $T \in B(\ell_\infty)$. We consider the closed subspaces $c(T) := T^{-1}(c)$ and $c_0(T) := T^{-1}(c_0)$ of ℓ_∞, respectively. We note that $c(I) = c$ and $c_0(I) = c_0$ while $c(0) = c_0(0) = \ell_\infty$.

A linear operator T on ℓ_∞ is said to *admits a matrix representation* if there exists an infinite matrix (t_{ij}) of complex numbers such that $(Ta)_n = \sum_{j=1}^\infty t_{nj}a_j$ for each $a = (a_n) \in \ell_\infty$. If $T \in B(\ell_\infty)$ admits a matrix representation, the spaces $c(T)$ and $c_0(T)$ are closely related to objects studied in the monograph [4]. In particular, $c(T)$ is called the *bounded summability field* of T; see also [5, 7]. For further information of the "summability domains" of matrices in normed spaces and the matrix transformations, the readers are referred to [2]

Let X be a Banach space. A subset F of X^* is said to be *total* if $f(x) = 0$ for each $f \in F$ implies that $x = 0$. Now suppose that M is a subspace of X, and that Y is a Banach space isomorphic to X. If X^* has a countable total subset then M^* and Y^* also have countable total subsets. Since ℓ_∞^* has a countable total subset consisting of coordinate functionals, it follows that each complemented subspace of ℓ_∞ must have such a set.

Now we present the main theorem. The proof is based on a combination of a *gliding hump argument* and Whitley's method [16].

THEOREM 2.1. *Let $T \in B(\ell_\infty)$ with a matrix representation (t_{ij}). Suppose that $c_0 \subset c_0(T) \subsetneq \ell_\infty$. If M is a closed subspace with $c_0 \subset M \subset c(T)$, then $(\ell_\infty/M)^*$ has no countable total subsets. Consequently, M is not complemented in ℓ_∞.*

Proof. Let $e_n = (0,\dots,0,1,0,\dots)$ for each $n \in \mathbb{N}$, where 1 is in the n-th position; and let $e_n^* a = a_n$ for each $n \in \mathbb{N}$ and each $a = (a_n) \in \ell_\infty$. We note that $t_{ij} = e_i^* T e_j$ for each $i, j \in \mathbb{N}$. Let γ_{ij} be a complex number such that $|\gamma_{ij}| = 1$ and $\gamma_{ij} t_{ij} = |t_{ij}|$ for each $i, j \in \mathbb{N}$. Since T is bounded, we have that $\sum_{j=1}^\infty |t_{ij}| \leqslant \|T\|$ for each i. Moreover, since $T(c_0) \subset c_0$, we have $t_{ij} = e_i^* T e_j \to 0$ as $i \to \infty$. This also shows $\sum_{j=1}^m |t_{ij}| \to 0$ for each $m \in \mathbb{N}$ as $i \to \infty$.

Take an arbitrary $a = (a_n) \in \ell_\infty \setminus c_0(T)$. Then there exists an increasing sequence (i_k) of natural numbers such that $e_{i_k}^* Ta \to \alpha \neq 0$. Removing finite number of elements from (i_k) if necessary, we have

$$|e_{i_k}^* Ta - \alpha| < |\alpha|/2$$

for each k. Since $e_{i_k}^* Ta = \sum_{j=1}^\infty t_{i_k j} a_j$, it follows that

$$\|a\|_\infty \sum_{j=1}^\infty |t_{i_k j}| \geqslant \left| \sum_{j=1}^\infty t_{i_k j} a_j \right| = |e_{i_k}^* Ta| > |\alpha|/2.$$

Hence, putting $M = |\alpha|/(2\|a\|_\infty) > 0$ yields $M < \sum_{j=1}^\infty |t_{i_k j}| \leqslant \|T\|$ for each k.

If we put $n_1 = i_1$ then there exists an m_1 such that $\sum_{j=m_1+1}^\infty |t_{n_1 j}| < M/4$. In this case, we have

$$\sum_{j=1}^{m_1} |t_{n_1 j}| = \sum_{j=1}^\infty |t_{n_1 j}| - \sum_{j=m_1+1}^\infty |t_{n_1 j}| > M/2.$$

Now we assume that there exist strictly increasing sequences $(n_p)_{p=1}^q, (m_p)_{p=1}^q$ satisfying

(i) $\sum_{j=1}^{m_{p-1}} |t_{n_p j}| < M/2^{p+1}$;

(ii) $\sum_{j=m_p+1}^\infty |t_{n_p j}| < M/2^{p+1}$; and

(iii) $\sum_{j=m_{p-1}+1}^{m_p} |t_{n_p j}| > (1 - 1/2^p)M$

for each $p = 1, 2, \ldots, q$, where $m_0 = 0$. Since $\sum_{j=1}^{m_q} |t_{ij}| \to 0$ as $i \to \infty$, there exists an $n_{q+1} \in (i_k)$ such that $\sum_{j=1}^{m_q} |t_{n_{q+1} j}| < M/2^{q+2}$. For this n_{q+1}, there exists an $m_{q+1} \in \mathbb{N}$ with $m_{q+1} > m_q$ such that $\sum_{j=m_{q+1}+1}^\infty |t_{n_{q+1} j}| < M/2^{q+2}$. It follows that

$$\sum_{j=m_q+1}^{m_{q+1}} |t_{n_{q+1} j}| = \sum_{j=1}^\infty |t_{n_{q+1} j}| - \sum_{j=1}^{m_q} |t_{n_{q+1} j}| - \sum_{j=m_{q+1}+1}^\infty |t_{n_{q+1} j}|$$
$$> (1 - 1/2^{q+1})M.$$

Thus, by an induction, we have infinite sequences $(n_p)_{p=1}^\infty, (m_p)_{p=1}^\infty$ satisfying

(i) $\sum_{j=1}^{m_{p-1}} |t_{n_p j}| < M/2^{p+1}$;

(ii) $\sum_{j=m_p+1}^\infty |t_{n_p j}| < M/2^{p+1}$; and

(iii) $\sum_{j=m_{p-1}+1}^{m_p} |t_{n_p j}| > (1 - 1/2^p)M$

for each $p \in \mathbb{N}$. Let $N_p = \{m_{p-1}+1, m_{p-1}+2, \ldots, m_p\}$ for each $p \in \mathbb{N}$, where $m_0 = 0$.

It is known that there exists a family $(A_\lambda)_{\lambda \in I}$ of subsets of $\mathbb{N}$ with the following properties:

(i) The index set I is uncountable.

(ii) A_λ is an infinite set for each $\lambda \in I$.

(iii) $A_\lambda \cap A_\mu$ is finite whenever $\lambda \neq \mu$.

See, for example, [12, Lemma 3.2.19]. For each $\lambda \in I$, define the bounded sequence $a^{(\lambda)} = (a_n^{(\lambda)})$ by

$$a_n^{(\lambda)} = \begin{cases} \gamma_{n_p,n} & (n \in N_p,\ p \in A_\lambda) \\ 0 & (n \in N_p,\ p \notin A_\lambda) \end{cases}.$$

We show that $a^{(\lambda)} \notin c(T)$. Indeed, we have

$$e_{n_p}^* T a^{(\lambda)} = \sum_{j=1}^{m_{p-1}} t_{n_p j} a_j^{(\lambda)} + \sum_{j=m_{p-1}+1}^{m_p} t_{n_p j} a_j^{(\lambda)} + \sum_{j=m_p+1}^{\infty} t_{n_p j} a_j^{(\lambda)},$$

and hence, if $p \in A_\lambda$ then

$$|e_{n_p}^* T a^{(\lambda)}| \geqslant \sum_{j=m_{p-1}+1}^{m_p} |t_{n_p j}| - \sum_{j=1}^{m_{p-1}} |t_{n_p j}| - \sum_{j=m_p+1}^{\infty} |t_{n_p j}| > (1 - 1/2^{p-1})M.$$

However, in the case of $p \notin A_\lambda$, one obtains

$$|e_{n_p}^* T a^{(\lambda)}| \leqslant \sum_{j=1}^{m_{p-1}} |t_{n_p j}| + \sum_{j=m_p+1}^{\infty} |t_{n_p j}| < M/2^p.$$

Since A_λ and $\mathbb{N} \setminus A_\lambda$ are both infinite set, the sequence $Ta^{(\lambda)}$ cannot converge.

Next, we shall see that $a^{(\lambda)} - a^{(\mu)} \notin c(T)$ whenever $\lambda \neq \mu$. If $p \in A_\lambda \setminus A_\mu$, as in the preceding paragraph, we have

$$\mathrm{Re}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})] > (1 - 3/2^p)M,$$

while

$$\mathrm{Re}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})] < -(1 - 3/2^p)M,$$

for the case of $p \in A_\mu \setminus A_\lambda$. Remark that, in either case, one has

$$|\,\mathrm{Im}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})]| < M/2^{p-1}.$$

From these estimations, we deduce that the sequence $Ta^{(\lambda)} - Ta^{(\mu)}$ is not Cauchy, and thus it does not converge.

We now consider the value of $\|(\sum_{j=1}^n \alpha_j a^{(\lambda_j)}) + c_0\|$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are mutually distinct elements of I. Since $A_{\lambda_i} \cap A_{\lambda_j}$ is finite whenever $i \neq j$, after removing finitely many coordinates, we can easily show that $\|(\sum_{j=1}^n \alpha_j a^{(\lambda_j)}) + c_0\| \leqslant \max_{1 \leqslant j \leqslant n} |\alpha_j|$.

Finally, let M be a closed subspace of ℓ_∞ with $c_0 \subset M \subset c(T)$. Then one has $a^{(\lambda)}, a^{(\lambda)} - a^{(\mu)} \notin M$ whenever $\lambda \neq \mu$. Let $\varphi \in (\ell_\infty/M)^*$. For each $\lambda \in I$, there exists a $\delta_\lambda \in \mathbb{C}$ such that $|\delta_\lambda| = 1$ and $\delta_\lambda \varphi(a^{(\lambda)} + M) = |\varphi(a^{(\lambda)} + M)|$. Take an arbitrary finite subset J of I. Then we obtain

$$\begin{aligned} \|\varphi\| &\geqslant \|\varphi\| \left\| (\sum_{\lambda\in J} \delta_\lambda a^{(\lambda)}) + c_0 \right\| \geqslant \|\varphi\| \left\| (\sum_{\lambda\in J} \delta_\lambda a^{(\lambda)}) + M \right\| \\ &\geqslant \left| \varphi\left((\sum_{\lambda\in J} \delta_\lambda a^{(\lambda)}) + M \right) \right| = \sum_{\lambda\in J} |\varphi(a^{(\lambda)} + M)|, \end{aligned}$$

which implies that $I_{\varphi,n} = \{\lambda \in I : |\varphi(a^{(\lambda)} + M)| > 1/n\}$ is finite for each n. Hence $I_\varphi = \{\lambda \in I : \varphi(a^{(\lambda)} + M) \neq 0\} = \bigcup_n I_{\varphi,n}$ is countable. Now suppose that $\mathscr{C}$ is a countable subset of $(\ell_\infty/M)^*$. Then it follows that

$$\{\lambda \in I : \varphi(a^{(\lambda)} + M) \neq 0 \text{ for some } \varphi \in \mathscr{C}\} = \bigcup_{\varphi\in\mathscr{C}} I_\varphi$$

is countable, and therefore $\mathscr{C}$ cannot be total. This completes the proof. □

REMARK 2.2. We remark that the preceding theorem is not true in general without the assumption on matrix representability. Indeed, there exists an operator $T \in B(\ell_\infty)$ which satisfies $c_0 \subset c_0(T) \subsetneq \ell_\infty$, but the conclusion of Theorem 2.1 does not hold. Indeed, let φ be a Banach limit on ℓ_∞, and let $Ta = \varphi(a)\mathbf{1}$ for each $a \in \ell_\infty$. Then $T(c_0) = \{0\} \subset c_0$ and $T(\ell_\infty) = \mathbb{C}\mathbf{1} \not\subset c_0$. However the identity $c(T) = \ell_\infty$ holds. Hence the conclusion of Theorem 2.1 fails for this T.

The rest of this section is devoted to presenting some applications of Theorem 2.1. Recall that a sequence $a = (a_n) \in \ell_\infty$ is said to be *convergent in the sense of Cesáro mean of order* 1 to α if the sequence $(n^{-1}\sum_{j=1}^n a_j)$ converges to α, and *almost convergent* to the *almost limit* α if $\varphi(a) = \alpha$ for each Banach limit φ on ℓ_∞. It is well-known as Lorentz's theorem [11] that $a = (a_n) \in \ell_\infty$ is almost convergent to α if and only if

$$\lim_m \sup_{n\in\mathbb{N}} \left| \frac{1}{m} \sum_{j=1}^m a_{n+j-1} - \alpha \right| = 0.$$

The spaces of all bounded sequences convergent in the sense of Cesáro mean of order 1 is denoted by $\widetilde{c}$. In [15], a similar sequence space (containing unbounded ones) was investigated by using the same symbol. The Banach spaces consisting of all almost convergent or almost null sequences are denoted by f and f_0, respectively. We note that $c_0 \subset f_0 \subset f \subset \widetilde{c}$ holds.

COROLLARY 2.3. *All the spaces $\widetilde{c}, f, f_0$ are closed and uncomplemented in ℓ_∞. Moreover, f_0 contains an isometric copy of ℓ_∞. Consequently, $\widetilde{c}, f, f_0$ are not prime.*

Proof. For each $a=(a_n)\in\ell_\infty$, let

$$Ta=\left(a_1,\frac{a_1+a_2}{2},\dots,\frac{a_1+a_2+\cdots+a_n}{n},\dots\right).$$

Then $T\in B(\ell_\infty)$ and admits a matrix representation (t_{ij}), where

$$t_{ij}=\begin{cases}1/i & (i\geqslant j)\\ 0 & (i<j)\end{cases}.$$

Moreover, we have $T(\mathbf{1})=\mathbf{1}\notin c_0$. Hence, by Theorem 2.1, all closed subspaces M of ℓ_∞ satisfying $c_0\subset M\subset c(T)=\widetilde{c}$ are not complemented in ℓ_∞.

For the fact that f_0 contains an isometric copy of ℓ_∞, we refer the readers to Lorentz [11] (see also [3, Theorem 3.2]). The proof is complete. □

COROLLARY 2.4. *Let d and d_0 be subspaces of ℓ_∞ given by*

$$c(\Delta)=\{a=(a_n)\in\ell_\infty:(a_n-a_{n+1})\ \textit{converges}\}$$
$$c_0(\Delta)=\{a=(a_n)\in\ell_\infty:(a_n-a_{n+1})\ \textit{converges to}\ 0\}$$

Then $c(\Delta),c_0(\Delta)$ are closed and uncomplemented in ℓ_∞. Moreover, $c_0(\Delta)$ contains an isomorphic copy of ℓ_∞. Consequently, $c(\Delta),c_0(\Delta)$ are not prime.

Proof. Let Δ be a bounded linear operator on ℓ_∞ given by $\Delta a=(a_n-a_{n+1})$ for each $a=(a_n)\in\ell_\infty$. Then Δ admits a matrix representation (t_{ij}), where $t_{ii}=1$ and $t_{i(i+1)}=-1$ for each i, and $t_{ij}=0$ for otherwise. We note that $\Delta(c)\subset c_0$ since each convergent sequence is Cauchy. Moreover, one has

$$\Delta(1,0,1,0,\dots)=(1,-1,1,-1,\dots)\notin c.$$

Hence Δ satisfies the assumption of Theorem 2.1. Now, it follows from $c_0\subset c_0(\Delta)\subset c(\Delta)$ that $c(\Delta)$ and $c_0(\Delta)$ are closed and not complemented in ℓ_∞.

We shall show that $c_0(\Delta)$ has an isomorphic copy of ℓ_∞ in it. Since $\sum_n 1/n=\infty$, we have an infinite sequence (m_k) such that $1/2\leqslant\sum_{j=m_{k-1}+1}^{m_k}1/j\leqslant 1$, where $m_0=0$. Put $M_k=\sum_{j=m_{k-1}+1}^{m_k}1/j$ for each k. Then there exists an $n_k\in\mathbb{N}$ such that $M_k/n_k\leqslant 1/m_k$. Put $q_0=0$. Define p_k and q_k inductively by $p_k=q_{k-1}+m_k-m_{k-1}$ and $q_k=p_k+n_k$ for each $k\in\mathbb{N}$. It follows that $q_0<p_1<q_1<p_2<\cdots$. Let $I_k=\{q_{k-1}+1,q_{k-1}+2,\dots,p_k\}$ and $J_k=\{p_k+1,p_k+2,\dots,q_k\}$ for each k. Then $|I_k|=m_k-m_{k-1}$, $|J_k|=n_k$ and

$$I_k\cup J_k=\{q_{k-1}+1,q_{k-1}+2,\dots,q_k\}.$$

Let $a=(a_n)$ be an element of ℓ_∞ given by

$$a_{q_{k-1}+l}=\sum_{j=m_{k-1}+1}^{m_{k-1}+l}1/j$$

for each $1 \leqslant l \leqslant m_k - m_{k-1}$, and

$$a_{p_k+l} = a_{p_k} - M_k l/n_k = (1 - l/n_k)M_k$$

for each $1 \leqslant l \leqslant n_k$. In particular, one has that $a_{p_k} = M_k$ and $a_{q_k} = 0$. Moreover, if $k \in \mathbb{N}$, then we note that $a_{q_{k-1}+l} - a_{q_{k-1}+l+1} = 1/(m_{k-1}+l)$ for each $1 \leqslant l \leqslant m_k - m_{k-1}$, and $a_{p_k+l} - a_{p_k+l+1} = M_k/n_k \leqslant 1/m_k$ for each $1 \leqslant l \leqslant n_k$. One has $a_{q_k} - a_{q_k+1} = -1/(m_k+1)$. These show that

$$\max_{n \in I_k \cup J_k} |a_n - a_{n+1}| = 1/(m_{k-1}+1)$$

for each $k \in \mathbb{N}$.

Now, for each $b = (b_n) \in \ell_\infty$, we define $(\Phi b)_n = a_n b_k$ for each $n \in I_k \cup J_k$. By the preceding paragraph and the fact that $a_{q_k+1} = 1/(m_k+1)$ for each k, we have $\Phi b \in c_0(\Delta)$. Moreover, since $1/2 \leqslant a_{p_k} = M_k \leqslant 1$ and $\|a\|_\infty \leqslant 1$, it follows that

$$\|b\|_\infty/2 \leqslant \|\Phi b\|_\infty \leqslant \|b\|_\infty.$$

This proves that $\Phi(\ell_\infty)$ is an isomorphic copy of ℓ_∞ in $c_0(\Delta)$. $\square$

We remark that the symbols $c(\Delta)$ and $c_0(\Delta)$ are used in [8] to denoting the spaces of all (possibly unbounded) difference convergent or difference null sequences.

3. A weak* closed subspace

In this section, we construct a weak* closed uncomplemented subspace of ℓ_∞. For this, we refer some results on projection constants; see König [9] and Foucart and Skrzypek [6]. Let M be a closed subspace of a Banach space X. Then the relative projection constant of M in X is given by

$$\lambda(M,X) := \inf\{\|P\| : P \text{ is a bounded projection from } X \text{ onto } M\}.$$

For each $m,N \in \mathbb{N}$, we consider the value

$$\lambda(m,N) := \max\{\lambda(M,\ell_\infty^N) : \dim M = m\}.$$

THEOREM 3.1. *There exists an uncomplemented weak* closed subspace W of ℓ_∞. Moreover, W contains an isometric copy of ℓ_∞.*

Proof. Let (p_m) be the increasing sequence of prime numbers with $p_1 = 5$. As in [9] (or [6]), for each m, there exists an p_m-dimensional subspace M_m of $\ell_\infty^{p_m^2}$ such that $\lim_m \lambda(M_m, \ell_\infty^{p_m^2})/\sqrt{m} = 1$. Fix an $m \in \mathbb{N}$. Let $\{e_1^{(m)}, e_2^{(m)}, \ldots, e_{p_m}^{(m)}\}$ be a basis for M_m. Then we have a basis $\{e_1^{(m)}, e_2^{(m)}, \ldots, e_{p_m^2}^{(m)}\}$ for the whole space $\ell_\infty^{p_m^2}$. Let $f_j^{(m)}(\sum_{i=1}^{N_m} a_i e_i^{(m)}) = a_{m+j}$ for each $j = 1,2,\ldots,p_m^2 - p_m$. Then one has $M_m = \bigcap_{j=1}^{p_m^2 - p_m} \ker f_j^{(m)}$.

Now let $A_1 = \{1,2,\dots,p_1^2\}$ and

$$A_m = \left\{ \left(\sum_{i=1}^{m-1} p_i^2\right)+1, \left(\sum_{i=1}^{m-1} p_i^2\right)+2,\dots,\sum_{i=1}^{m} p_i^2 \right\}.$$

Put $P_m(a) = a\cdot\chi_{A_m}$ for each m and for each $a\in\ell_\infty$. Then, for each m, there exists a natural identification $Q_m : \ell_\infty^{p_m^2} \to P_m(\ell_\infty)$. Define a subspace W of ℓ_∞ by the internal direct sum $\sum_{m=1}^{\infty}\oplus Q_m(M_m)$. In other words, $a\in W$ if and only if $P_m a \in Q_m(M_m)$ for each m. It follows that the space W can be written as

$$W = \bigcap\{\ker(Q_m^{-1}P_m)^* f_j^{(m)} : m\in\mathbb{N},\ 1\leqslant j\leqslant p_m^2 - p_m\}.$$

Since each projection P_m is weak*-to-norm continuous, all the functional of the form $(Q_m^{-1}P_m)^* f_j^{(m)}$ are weakly* continuous, which proves that W is weak* closed.

Suppose that P is a bounded projection from ℓ_∞ onto W. Then the operator $Q_m^{-1}P_mPQ_m$ is a bounded projection from $\ell_\infty^{p_m^2}$ onto M_m. Indeed, we have $P_mPa \in Q_m(M_m)\subset W$ for each $a\in\ell_\infty$, which implies that $(P_mP)^2 = P_mP$. Hence one has

$$1 = \lim_m \frac{\lambda(M_m,\ell_\infty^{p_m^2})}{\sqrt{m}} \leqslant \limsup_m \frac{\|Q_m^{-1}P_mPQ_m\|}{\sqrt{m}} \leqslant \lim_m \frac{\|P\|}{\sqrt{m}} = 0,$$

a contradiction. Thus there is no bounded projection from ℓ_∞ onto W, that is, W is uncomplemented in ℓ_∞.

Finally, take an arbitrary $x_m \in S_{Q_m(M_m)}$ for each m. Define $T : \ell_\infty \to W$ by $T(a_n) = w^*\text{-}\lim_m \sum_{i=1}^{m} a_i x_i$. It is routine to check that T is well-defined and isometric. The proof is complete. $\square$

On the other hand, it is known that if M is a weak* closed subspace of ℓ_∞ then $(\ell_\infty/M)^*$ always has a countable total subset. As a consequence, the property that $(\ell_\infty/M)^*$ has a countable total subset is necessary but not sufficient for assuring the complementarity of M in ℓ_∞. Hence there exists a limit to determining the complementarity of subspaces of ℓ_∞ by using Whitley's method while it still has interesting applications.

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