Operators
and
Matrices
Volume 12, Number 4 (2018), 1099–1127
doi:10.7153/oam-2018-12-67

THE KUNZ–SOUILLARD APPROACH TO LOCALIZATION FOR JACOBI OPERATORS

VALMIR BUCAJ

(*Communicated by G. Teschl*)

Abstract. In this paper we study spectral properties of Jacobi operators. In particular, we prove two main results: (1) that perturbing the diagonal coefficients of Jacobi operator, in an appropriate sense, results in exponential localization, and purely pure point spectrum with exponentially decaying eigenfunctions; and (2) we present examples of decaying *potentials* b_n such that the corresponding Jacobi operator has purely pure point spectrum.

1. Introduction and setting

We will use the Kunz-Souillard approach to localization for random Schrödinger operators to prove that any Jacobi operator can be approximated by some *random* Jacobi operator, in operator norm, with purely pure point spectrum, and to also provide examples of Jacobi operators with decaying potentials having purely pure point spectrum.

Jacobi operators are important objects in mathematics. On one hand, the half line Jacobi operators with bounded coefficients correspond to compactly supported measures on the real line – such correspondence can be established via orthogonal polynomials or the Borel transform of the measure. So, as such, often many results from the spectral theoretic part can be translated back into the OPRL setting which otherwise would have been harder to achieve with direct tools. For a more elaborate discussion see [19].

On the other hand, the study of random Jacobi operators is of particular importance because of their usefulness in modeling disordered media (e.g. amorphous solids). In some instances, as it is the case for crystals, the structure of the solid is completely regular; that is, the atoms are distributed periodically on some lattice. Then, mathematically, in such regular crystals, the total potential that a single particle (e.g. electron) at some position in $\mathbb{R}^d$ feels is periodic with respect to the lattice at hand. Schrödinger operators with periodic potentials are well understood, see for example [16], [10], and [21].

However, as it is often the case in nature, if the positions of the atoms in the solid deviate from a lattice in some highly non-regular way, then it is natural to view the

Mathematics subject classification (2010): 47B37.
Keywords and phrases: Jacobi operators, decaying potentials, spectrum.
The author was supported in part by NSF grant DMS–1361625.

potential that a single particle feels at some position as some random quantity. Mathematically, this can be studied via Jacobi operators with random potentials. So, understanding spectral properties of such operators is of particular interest since, via the RAGE theorem, we can obtain good insights into the quantum transport phenomena of the quantum particles they model, something of great interest to physics and engineering since it also relates to conducting materials and insulators. Said differently, in some appropriate sense, one can answer the question of whether the wave packets are localized in space-time or disperse to infinity. Researchers have long studied this phenomenon, which in literature is known as the *Anderson localization*.

There are typically two separate statements referring to localization: a spectral statement and a dynamical one. *Spectral localization* asserts that the operators almost surely have pure point spectrum, with exponentially decaying eigenfunctions. On the other hand, different notions of *dynamical localization* have been used in literature. However, in essence, dynamical localization refers to an absence of transport in a random medium. This is typically quantified via (almost-sure) bounds on the moments of wave packets such as

$$\sup_t \sum_{n\in\mathbb{Z}} |n|^p \left|\langle \delta_n, e^{-itJ_\omega} \delta_0 \rangle\right|^2 < \infty, \tag{1}$$

for all $p > 0$. In some instances, one can prove stronger statements, also referred to as *strong dynamical localization*, such as replacing the *almost sure* condition by an expectation $\mathbb{E}(\cdot)$, as is the case via the Kunz-Souillard approach to localization in dimension one, which is the main focus of this paper.

The first mathematically rigorous proof of strong dynamical localization at all energies for one dimensional discrete Schrödinger operators, was originally given by H. Kunz and B. Souillard in 1980, see [13]. For a while it was not clear whether it was possible for a Schrödinger operator with *slow enough* decaying potential to exhibit purely pure point spectrum. Two years later, in [18], Simon answered this question in affirmative by providing examples of Schrödinger operators with slow enough decaying potentials who have purely pure point spectrum. In this paper we extend this result to Jacobi operators. In loose terms, which will be made precise later, we prove that *Jacobi operators with slow-enough decaying potentials display strong dynamical localization, and hence have purely pure point spectrum.*

We wish to mention some of the main new challenges we faced in this paper that resulted from the presence of the non-constant off-diagonal entries a_n in the Jacobi matrices, J_ω. The first such challenge resulted from the need to define two different countable families of integral operators, in contrast to the original Schrödinger operator case where only two different operators are needed (excluding the energy dependence, which is true in both cases), and as a result many of the subsequent operator norm bounds needed to be uniform in the single-site position parameter, or at the very least hold for all indices, something that was not an issue before. Another challenge that resulted from the presence of a_n's, was the necessity to be able to choose certain constants uniformly in some crucial steps in the proofs of some essential lemmas. For example, one such significant instance is in the proof of Lemma 4.7, which is crucial in producing quantitative bounds for some of the families of operators in question. The rest of the challenges were more superficial and were related mostly to finding the appropriate ways of translating the old techniques in this new setting.

The Kunz-Souillard approach to localization has since attracted more interest because, while there are other methods that can be used to prove localization in one-dimension (e.g. fractional moments method, spectral averaging, MSA), it is the only one which establishes localization at all energies and any disorder without requiring ergodicity. For example, it can handle models with decaying potentials and models with a fixed background potential which in turn allows one to obtain these pertubative statements, such as our main result below, Theorem 2.1 . It also tackles dynamical localization directly and completely avoids appealing to positivity of Lyapunov exponents, something that other methods typically need as an input.

On the other hand, the shortcomings of the method are mainly because it applies only in one-dimension, and that it is known to work only for single-site distributions that are purely absolutely continuous, nevertheless, the conclusions are very strong. Whether this method can be extended to single-site distributions with a non-trivial singular part, still remains open.

Originally, the Kunz-Souillard work for Schrödinger operators was done in the discrete setting, see [13]. There have been a few extensions of this method, including the current work, in different directions, for example see [3], [5], and [7].

In this paper we consider the model where the diagonal entries of the Jacobi operators are generated by i.i.d random variables, and the off diagonal entries are uniformly bounded away from zero. Specifically, suppose $r:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$ is bounded, measurable, and compactly supported with $\|r\|_1 = 1$. Let $c \in \ell^\infty(\mathbb{Z})$. Define a measure μ_n on $\mathbb{R}$ via $d\mu_n(E) = r_n(E)dE$, where $r_n(x) = d_n^{-1} r\left(d_n^{-1}x\right)$, and d_n is some fixed sequence. Let

$$\begin{aligned} M &= \sup\{|E| : E \in \operatorname{supp}(r)\} \\ M_n &= \sup\{|E| : E \in \operatorname{supp}(r_n)\} \\ I_n &= [c(n) - M_n, c(n) + M_n] \\ \Omega &= \prod_{n\in\mathbb{Z}} I_n \\ d\mu(x) &= \prod_{n\in\mathbb{Z}} r_n(x_n - c(n))dx_n. \end{aligned}$$

We wish to point out that r quantifies the deviation of our random potential from the background potential c. In the second situation we will consider, the sequence d_n will serve as a damping parameter that we will use to force decay of the random potential.

Next, we define $b_\omega(n) = \omega(n)$ for each $\omega \in \Omega$. Notice, that each $b_\omega(n)$ is the sum of a random *i.i.d* with distribution μ_n and some fixed background *potential* $c(n)$. With this notation, we define a one parameter family of Jacobi operators, J_ω, on $\ell^2(\mathbb{Z})$ as follows

$$(J_\omega\phi)(n) = a(n)\phi(n+1) + a(n-1)\phi(n-1) + b_\omega(n)\phi(n), \tag{2}$$

where $a \in \ell^\infty(\mathbb{Z})$ with $a(n) \geqslant \delta > 0$ for all $n \in \mathbb{Z}$.

In general, if one assumes that supp r contains more than one element-by construction, this is the case for us-the resulting family $\{J_\omega\}_{\omega\in\Omega}$ of operators, with $a(n)\equiv 1$ and $d_n\equiv 1$, is referred to as the *Anderson model*. There is also a generalization of this model, which in literature is known as *discrete generalized Anderson model*, for which the Kunz-Souillard method has been shown to work as well, see [3]. The, simplest non-trivial case, where supp r contains precisely two elements is known as the *Bernoulli-Anderson model*.

As mentioned above, one interesting property to study for this model is the phenomenon of *Anderson localization*. In an appropriate formulation, it is known that dynamical localization implies spectral localization, while the converse is not true in general. For example, the so called random dimer model serves as a counterexample to this implication (see [14] and [15] for a more elaborate description). One typically needs "spectral localization $+\varepsilon$" to imply dynamical localization in some suitable formulation. This relationship was studied by del Rio, Jitomirskaya, Last, and Simon in [17].

There are different approaches to localization: *Spectral averaging* can be used to study spectral localization; one can also study both spectral and dynamical localization via methods such as, *multi-scale analysis*, developed around 1983 by Fröhlich and Spencer in [11]; *fractional moments method*, initially introduced by Aizenman and Molchanov in [1] (for a nice expository treatment of this method one may also consult [20]); and also, which is what we do in this paper, the *Kunz-Souillard* method. For another approach to localization based on positivity of the Lyapunov exponent and Large Deviation Estimates you may consult [2].

The basic idea behind the Kunz-Souillard method is fairly simple. one begins by restricting the operator J_ω to some finite box, decomposing it in terms of its eigenspaces, and then via a change of variables rewriting the latter in terms of some integral operators. So, the main challenge is figuring out the appropriate change of variables and estimating the norms of the integral operators, which for the decaying case one needs to have quantitative bounds. Another positive factor of this paper is that not only it demonstrates the power of Kunz-Souillard even in the case of Jacobi operators but it also gives further hope that one might be able to extend this technique to show strong dynamical localization even for the CMV matrices, which is the next natural step.

2. Main results

Our main goal is to prove that given any Jacobi operator with bounded coefficients, it is possible to completely destroy the absolutely continuous spectrum by perturbing the diagonal entries with appropriate slow enough decaying potentials. In fact, if we do not require the pertubation to be done by decaying potentials, then one can actually obtain a stronger result, as stated below.

THEOREM 2.1. *For all $a_n, b_n\in\mathbb{R}$ bounded, with $a_n\geqslant\delta>0$, and for every $\varepsilon>0$, there exist $\tilde{a}_n, \tilde{b}_n$, with $\|\tilde{a}-a\|_\infty<\varepsilon$ and $\|\tilde{b}-b\|_\infty<\varepsilon$, such that the Jacobi operator, $\widetilde{J}\stackrel{def}{=}\widetilde{J}(\tilde{a},\tilde{b})$, has purely pure point spectrum with exponentially decaying eigenfunctions.*

REMARK 2.2. We wish to point out that if in Theorem 2.1 one wishes to approximate by decaying diagonal entries, $\tilde{b}_n$, then the first part of the conclusion still holds but not necessarily the latter; that is, $\tilde{J}(\tilde{a},\tilde{b})$, above, still has purely pure point spectrum, but is no longer guaranteed to have exponentially decaying eigenfunctions.

This theorem is actually a rather straightforward consequence of Theorems 2.3 and 2.4 below, for the decaying and non-decaying case, respectively.

THEOREM 2.3. *With the same notation as above, if d_n is a fixed sequence with $0 \leqslant d_n \leqslant 1$ and $d_n \geqslant C|n|^{-\zeta}$ for $\zeta < \frac{1}{2}$, then for μ-almost every ω, the Jacobi operator J_ω has purely pure point spectrum.*

Because it is well known how to go from *strong dynamical localization* to *spectral localization*, we would essentially be done if we could prove *strong dynamical localization* for the family of Jacobi operators, J_ω, defined above. Indeed, in Theorems 2.4 and 2.6 below, we prove precisely this, with exponential and sub-exponential bounds, respectively.

THEOREM 2.4. *With Ω, μ, and J_ω as above, and $d_n = 1$ for all n, there exist constants $C,\gamma \in (0,\infty)$ such that*

$$\int_\Omega \left(\sup_{t\in\mathbb{R}} \left|\langle \delta_m, e^{-itJ_\omega}\delta_0\rangle\right| \right) d\mu(\omega) \leqslant Ce^{-\gamma|m|},$$

for all $m \in \mathbb{Z}$.

Actually, we can loosen the condition on the sequence d_n; that is, the statement holds true as long as $d_n \in \ell^\infty(\mathbb{R})$ is positive and uniformly bounded away from zero.

For more pleasant exposition let

$$a(m,n) = \int_\Omega \left(\sup_{t\in\mathbb{R}} \left|\langle \delta_m, e^{-itJ_\omega}\delta_n\rangle\right| \right) d\mu(\omega).$$

REMARK 2.5. We wish to point out that in a similar way one shows that

$$a(m,n) \leqslant Ce^{-\gamma|m-n|}. \tag{3}$$

For simplicity, we only work out the case $n = 0$.

PROPOSITION 1. *If there are constants $C,\gamma \in (0,\infty)$ such that*

$$\max_{n\in\{0,1\}} a(m,n) \leqslant Ce^{-\gamma|m|},$$

then for $\mu-$almost every $\omega \in \Omega$, J_ω, as defined in (2), has pure point spectrum with exponentially decaying eigenfunctions. More precisely, these eigenfunctions obey estimates of the form

$$|u(m)| \leqslant C_{\omega,\varepsilon,u}e^{-(\gamma-\varepsilon)|m|},$$

for small enough $\varepsilon \in (0,\gamma)$.

Proof. This is proved in almost identical way as in the case for random Schrödinger operators, so we direct the reader to [8] or [4]. □

Even if we do not insist on exponential bounds for $\max_{n\in\{0,1\}} a(m,n)$, we still obtain pure point spectrum, but we no longer get exponentially decaying eigenfunctions. We make this statement precise in the following two theorems.

THEOREM 2.6. *Let d_n be a fixed sequence with $0 \leqslant d_n \leqslant 1$ and $d_n \geqslant C|n|^{-\zeta}$ for $\zeta < \frac{1}{2}$ and some constant $C > 0$. With Ω, μ, and J_ω as above, there exist constants $C' > 0$ and $\gamma'' > 0$, such that*

$$\int_\Omega \left(\sup_{t\in\mathbb{R}} \left| \langle \delta_m, e^{-itJ_\omega} \delta_0 \rangle \right| \right) d\mu(\omega) \leqslant C' |m|^{\zeta/2} \exp\left(-\gamma'' |m|^{1-2\zeta} \right).$$

REMARK 2.7. As in Remark 2.5, we only work out the proof for $a(m,0)$, since the other cases are completely analogous.

Going from a strong form of dynamical localization to spectral localization, even with subexponential bounds such as in Theorem 2.6, is now considered a standard result, so we state the following proposition without a complete proof.

PROPOSITION 2. *If there exist constants $C'' > 0$ and $\tau > \frac{3}{2}$, such that*

$$\max_{n\in\{0,1\}} a(m,n) \leqslant \frac{C''}{m^\tau}, \tag{4}$$

then for μ-almost every $\omega \in \Omega$, the Jacobi operator J_ω, has purely pure point spectrum.

Proof. Since the bound in (4) is different from the one found in the standard formulation of this result, we only provide the details of the beginning of the proof and refer the reader to [4] for further reading. Let us define

$$a(m,n,\omega) = \sup_{t\in\mathbb{R}} \left| \langle \delta_m, e^{-itJ_\omega} \delta_n \rangle \right|,$$

so that we have

$$a(m,n) = \int_\Omega a(m,n,\omega) d\mu(\omega).$$

Let $\frac{1}{2} < \beta < \tau - 1$ be given, and consider the set

$$S_{\beta,m,n} = \left\{ \omega \in \Omega : a(m,n,\omega) > \frac{1}{m^\beta} \right\}.$$

Then

$$a(m,n) \geqslant \frac{1}{m^\beta} \mu\left(S_{\beta,m,n} \right),$$

for all $m,n \in \mathbb{Z}$. So, by the above observation and the hypothesis, for all m, and $n = 0,1$ we get

$$\mu\left(S_{\beta,m,n}\right) \leqslant m^{\beta} a(m,n) \leqslant \frac{C''}{m^{\tau-\beta}}. \tag{5}$$

Since the rest of the proof follows in an almost identical way as in [4] or [8], with the appropriate changes resulting from the slightly different condition in (4), we omit it here. $\square$

2.1. Proof of Theorem 2.1

Proof. Let $J \stackrel{\text{def}}{=} J(a_n, b_n)$ be a given Jacobi operator, where a_n, b_n are as in the statement of the theorem. We will actually prove a much stronger statement, indeed, we will construct an uncountable family of Jacobi operators with the desired property. Specifically, given $\varepsilon > 0$ we will construct $\widetilde{J}_\omega = \widetilde{J}(\tilde{a}, \tilde{b}_\omega)$ as follows. We pick $\tilde{a} \stackrel{\text{def}}{=} a$, and for $\omega \in \Omega$ we set $\tilde{b}_\omega(n) \stackrel{\text{def}}{=} b_\omega(n)$, where $b_\omega(n)$ is as above, with $c(n)$ replaced by $b(n)$, and $M < \varepsilon$. By construction, we clearly have $\|\tilde{a} - a\|_\infty < \varepsilon$ and $\left\|\tilde{b} - b\right\|_\infty < \varepsilon$. Then, by Theorem 2.4 and Proposition 1, for the non-decaying case, and by Theorem 2.3 for the decaying case, respectively, it follows that for μ almost every ω, the Jacobi operator $\widetilde{J}_\omega$ has purely pure point spectrum, which concludes the proof! $\square$

2.2. Proof of Theorem 2.3

Proof. This is an immediate consequence of Proposition 2 and Theorem 2.6. The main idea is that Theorem 2.6 implies the assumption of Proposition 2, given in (4), and hence the conclusion follows. More specifically, we claim that for large enough m and some $\tau > 3/2$, we have

$$|m|^{\zeta/2} \exp\left(-\gamma'' |m|^{1-2\zeta}\right) \leqslant \frac{1}{m^{\tau}}. \tag{6}$$

A quick calculation shows that

$$\lim_{m\to\infty} |m|^{\zeta/2+\tau} \exp\left(-\gamma'' |m|^{1-2\zeta}\right) = 0,$$

which, in turn, implies (6). Then, this observation and Theorem 2.6 imply that for $n = 0,1$, we have

$$a(m,n) \leqslant \frac{C''}{m^{\tau}}.$$

Thus, the result follows from Proposition 2. $\square$

3. Preparatory work

We turn to the task of proving Theorems 2.4 and 2.6. Given $L \in \mathbb{Z}_+$, denote by $J_\omega^{(L)}$ the restriction of J_ω to $\ell^2(-L,\ldots,L)$, and let

$$a_L(m,n) = \int_\Omega \left(\sup_{t\in\mathbb{R}} \left|\langle \delta_m, e^{-itJ_\omega^{(L)}} \delta_n \rangle\right|\right) d\mu(\omega).$$

That is,

$$J_\omega^{(L)} = \begin{pmatrix} b_\omega(-L) & a(-L) & & & 0 \\ a(-L) & b_\omega(-L+1) & & & 0 \\ 0 & a(-L+1) & & & \\ \vdots & & \ddots & a(L-2) & \vdots \\ & & & b_\omega(L-1) & a(L-1) \\ 0 & & \dots & a(L-1) & b_\omega(L) \end{pmatrix}.$$

Let $\{E_\omega^{L,k}\}_k$, and $\{\varphi_\omega^{L,k}\}_k$ be the eigenvalues and the corresponding normalized eigenfunctions of $J_\omega^{(L)}$, respectively. Define,

$$\rho_L(m,n) = \int_\Omega \left(\sum_k \left| \langle \delta_m, \varphi_\omega^{L,k} \rangle \right| \left| \langle \delta_n, \varphi_\omega^{L,k} \rangle \right| \right) d\mu(\omega),$$

and notice that this is a $(2L+1)$ fold integral, since $J_\omega^{(L)}$ depends only on the entries $\omega_{-L}, \dots \omega_L$.

The following two lemmas are standard results, for a discussion see [4, pp. 192–193], and are easy to prove as well, so we state them here without proof.

LEMMA 3.1. *For $m,n \in \mathbb{Z}$ we have*

$$a(m,n) \leqslant \liminf_{L\to\infty} a_L(m,n).$$

LEMMA 3.2. *For $L \in \mathbb{Z}_+$, and $m,n \in \mathbb{Z}$ we have*

$$a_L(m,n) \leqslant \rho_L(m,n).$$

Put

$$\Sigma_0 = \left[-2\|a\|_\infty - M - \|c\|_\infty, 2\|a\|_\infty + M + \|c\|_\infty\right].$$

Notice that Σ_0 contains the spectrum of both J_ω, and $J_\omega^{(L)}$. Now, in the spirit of [13], we define a family of operators appropriate for our setting.

DEFINITION 3.3. For $E \in \mathbb{R}$, define the operators $U, S_E^{(n)}, T_E^{(n)}$ on $L^p(\mathbb{R})$ by:

$$(Uf)(x) = |x|^{-1} f(x^{-1}).$$

$$\left(S_E^{(n)} f\right)(x) = \begin{cases} a_n \int r_n(E - a_n x - a_{n-1} y^{-1}) f(y) dy & , \ n < 0 \\ a_0 \int r_0(E - a_0 x - a_{-1} y^{-1}) f(y) dy & , \ n = 0 \\ a_{n-1} \int r_n(E - a_{n-1} x - a_n y^{-1}) f(y) dy & , \ n > 0 \end{cases}$$

and

$$\left(T_E^{(n)} f\right)(x) = \sqrt{a_{n-1} a_n} \int r_n(E - a_{n-1} x - a_n y^{-1}) |y|^{-1} f(y) dy, \quad n > 0.$$

$$r_{k;E}^{(n)}(x) = r_k(E - a_{n-1}x)$$

For convenience of notation we set

$$S_{E;m}^{(n)} \stackrel{\text{def}}{=} S_{E-c(m)}^{(n)}, \quad T_{E;m}^{(n)} \stackrel{\text{def}}{=} T_{E-c(m)}^{(n)} \text{ and } r_{k;E;m}^{(n)} \stackrel{\text{def}}{=} r_{k;E-c(m)}^{(n)}.$$

We wish to point out that U is a unitary operator on $L^2(\mathbb{R})$.

From now on, we will drop the subscript ω on the sequence b (i.e. $b_n = b_\omega(n) = \omega(n)$), this should cause no confusion and should be clear from the context. We want to compute the following:

$$\begin{aligned}\rho_L(m,0) &= \int_\Omega \left(\sum_k \left| \langle \delta_m, \varphi_{\bar{b}}^{L,k} \rangle \right| \left| \langle \delta_0, \varphi_{\bar{b}}^{L,k} \rangle \right| \right) d\mu(\omega) \\ &= \int \cdots \int \left(\sum_k \left| \langle \delta_m, \varphi_{\bar{b}}^{L,k} \rangle \right| \left| \langle \delta_0, \varphi_{\bar{b}}^{L,k} \rangle \right| \right) \prod_{n=-L}^{L} r_n(b_n - c_n) db_{-L} \ldots db_L, \quad (7)\end{aligned}$$

where $\bar{b} = (b_{-L}, \ldots, b_L)$. Let $\{E_{\bar{b}}^{L,k}\}_{-L \leqslant k \leqslant L}$ and $\{\varphi_{\bar{b}}^{L,k}\}$ be the eigenvalues and the corresponding normalized eigenvectors of

$$J_\omega^{(L)} = \begin{pmatrix} b_{-L} & a_{-L} & & & 0 \\ a_{-L} & b_{-L+1} & & & \\ 0 & a_{-L+1} & & & \\ \vdots & & \ddots & a_{L-2} & \vdots \\ & & & b_{L-1} & a_{L-1} \\ 0 & & \ldots & a_{L-1} & b_L \end{pmatrix}.$$

Let E be $E_{\bar{b}}^{L,k}$ and u be $\varphi_{\bar{b}}^{L,k}$, then we have

$$a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n = E u_n, \quad (8)$$

for $-L \leqslant n \leqslant L$, where $u_{-L-1} = u_{L+1} = 0$.

Rewriting (8) we get:

$$b_n = E - a_n \frac{u_{n+1}}{u_n} - a_{n-1} \frac{u_{n-1}}{u_n} \quad (9)$$

Let

$$x_n = \begin{cases} \dfrac{\varphi_{\bar{b}}^{L,k}(n+1)}{\varphi_{\bar{b}}^{L,k}(n)}, & n < 0 \\[2ex] \dfrac{\varphi_{\bar{b}}^{L,k}(n-1)}{\varphi_{\bar{b}}^{L,k}(n)}, & n > 0 \end{cases}$$

so that

$$b_n = \begin{cases} E - a_{n-1}x_{n-1}^{-1} - a_n x_n, & n < 0 \\ E - a_{-1}x_{-1}^{-1} - a_0 x_1^{-1}, & n = 0 \\ E - a_n x_{n+1}^{-1} - a_{n-1}x_n, & n > 0 \end{cases}$$

with the convention $x_{-L-1}^{-1} = x_{L+1}^{-1} = 0$.

This motivates the following change of variables

$$F_L : (x_{-L}, \ldots, x_{-1}, E, x_1, \ldots, x_L) \mapsto (b_{-L}, \ldots, b_0, \ldots, b_L).$$

The next step is to rewrite (7) using this change of variables. In order to do so, we need to compute the determinant of the Jacobian of this change of variables.

Observe that: $\frac{\partial b_n}{\partial E} = 1$, for all n; $\frac{\partial b_n}{\partial x_n} = -a_n$, for $n < 0$; $\frac{\partial b_n}{\partial x_n} = -a_{n-1}$, for $n > 0$; $\frac{\partial b_n}{\partial x_{n-1}} = a_{n-1}x_{n-1}^{-2}$, for $n \leqslant 0$; $\frac{\partial b_n}{\partial x_{n+1}} = a_n x_{n+1}^{-2}$, for $n \geqslant 0$; and $\frac{\partial b_n}{\partial x_m} = 0$, for all other m,n.

Thus, the corresponding matrix of F_L is:

$$\begin{pmatrix}
-a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & & & \\
& -a_{-L+1} & a_{-L+1}x_{-L+1}^{-2} & & & & & & & & \\
& & -a_{-L+2} & & & & & & & & \\
& & & \ddots & \ddots & & & & & & \\
& & & & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & & \\
1 & 1 & \cdots & & 1 & 1 & 1 & \cdots & & 1 & 1 \\
& & & & & a_0x_1^{-2} & -a_0 & & & & \\
& & & & & & a_1x_2^{-2} & -a_1 & & & \\
& & & & & & & \ddots & \ddots & & \\
& & & & & & & & a_{L-2}x_{L-1}^{-2} & -a_{L-2} & \\
& & & & & & & & & a_{L-1}x_L^{-2} & -a_{L-1}
\end{pmatrix}.$$

We claim that

$$\begin{aligned} \det F_L &= \left(\prod_{n=-L}^{L-1} a_n\right)\Big(1 + x_1^{-2}\{1 + x_2^{-2}\{1 + \ldots x_{L-1}^{-2}\{1 + x_L^{-2}\}\ldots\}\} \\ &\quad + x_{-1}^{-2}\{1 + x_{-2}^{-2}\{1 + \ldots x_{-L+1}^{-2}\{1 + x_{-L}^{-2}\}\ldots\}\}\Big) \\ &= \left(\prod_{n=-L}^{L-1} a_n\right)\left(\varphi_{\overline{b}}^{L,k}(0)\right)^{-2}. \end{aligned} \tag{10}$$

We prove this by induction on L. For $L = 1$ it is clear. Now, suppose that (10) holds for some L. Consider the determinant of matrix of F_{L+1}:

$$\begin{pmatrix}
-a_{-L-1} & a_{-L-1}x_{-L-1}^{-2} & & & & & & & & \\
& -a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & \\
& & -a_{-L+1} & & & & & & & \\
& & & \ddots & \ddots & & & & & \\
& & & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & & \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & & 1 & 1 \\
& & & & a_0x_1^{-2} & -a_0 & & & & \\
& & & & & a_1x_2^{-2} & -a_1 & & & \\
& & & & & & \ddots & \ddots & & \\
& & & & & & & a_{L-1}x_L^{-2} & -a_{L-1} & \\
& & & & & & & & a_Lx_{L+1}^{-2} & -a_L
\end{pmatrix}.$$

Expanding along the first column we get:

$$(-a_{-L-1})\det\begin{pmatrix}
-a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & \\
& -a_{-L+1} & & & & & & & \\
& & \ddots & \ddots & & & & & \\
& & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & & \\
1 & \cdots & 1 & 1 & 1 & \cdots & & 1 & 1 \\
& & & a_0x_1^{-2} & -a_0 & & & & \\
& & & & a_1x_2^{-2} & -a_1 & & & \\
& & & & & \ddots & \ddots & & \\
& & & & & & a_{L-1}x_L^{-2} & -a_{L-1} & \\
& & & & & & & a_Lx_{L+1}^{-2} & -a_L
\end{pmatrix}$$

$+$

$$(-1)^{L+1}\det\begin{pmatrix}
a_{-L-1}x_{-L-1}^{-2} & & & & & & & & \\
-a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & \\
& -a_{-L+1} & & & & & & & \\
& & \ddots & \ddots & & & & & \\
& & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & & \\
& & & a_0x_1^{-2} & -a_0 & & & & \\
& & & & a_1x_2^{-2} & -a_1 & & & \\
& & & & & \ddots & \ddots & & \\
& & & & & & a_{L-1}x_L^{-2} & -a_{L-1} & \\
& & & & & & & a_Lx_{L+1}^{-2} & -a_L
\end{pmatrix}.$$

Note that the second matrix is lower-triangular, so expanding along the first row, repeatedly, we eventually will get:

$$\left(\prod_{n=-L-1}^{L} a_n\right) x_{-L-1}^{-2}x_{-L}^{-2}\dots x_{-1}^{-2}.$$

Expanding the first determinant along the last column we get:

$$
(-1)^L \det \begin{pmatrix}
-a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & \\
& -a_{-L+1} & & & & & & & \\
& & \ddots & \ddots & & & & & \\
& & & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & \\
& & & & a_0x_1^{-2} & -a_0 & & & \\
& & & & & a_1x_2^{-2} & -a_1 & & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & a_{L-1}x_L^{-2} & -a_{L-1} \\
& & & & & & & & a_Lx_{L+1}^{-2}
\end{pmatrix}
$$

+

$$
(-a_L)\det \begin{pmatrix}
-a_{-L} & a_{-L}x_{-L}^{-2} & & & & & & & \\
& -a_{-L+1} & & & & & & & \\
& & \ddots & \ddots & & & & & \\
& & & -a_{-1} & a_{-1}x_{-1}^{-2} & & & & \\
1 & \dots & & 1 & 1 & 1 & \dots & & 1 \\
& & & & a_0x_1^{-2} & -a_0 & & & \\
& & & & & a_1x_2^{-2} & -a_1 & & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & a_{L-1}x_L^{-2} & -a_{L-1}
\end{pmatrix}.
$$

As before, computing the fist determinant by expanding along the first columns, repeatedly, we eventually get:

$$
\left(\prod_{n=-L}^{L} a_n\right) x_1^{-2}x_2^{-2}\dots x_L^{-2}x_{L+1}^{-2}.
$$

Combining all of these, and noting that the last determinant is simply $\det F_L$ we get:

$$
\begin{aligned}
\det F_{L+1} = (-a_{-L-1})&\left(\left(\prod_{n=-L}^{L} a_n\right) x_1^{-2}\dots x_{L+1}^{-2} + (-a_L)\det F_L\right) + \left(\prod_{n=-L-1}^{L} a_n\right) x_{-L-1}^{-2}\dots x_{-1}^{-2} \\
&= a_{-L-1}a_L \det F_L + \left(\prod_{n=-L-1}^{L} a_n\right) x_1^{-2}\dots x_{L+1}^{-2} + \left(\prod_{n=-L-1}^{L} a_n\right) x_{-L-1}^{-2}\dots x_{-1}^{-2} \\
&= a_{-L-1}a_L \prod_{n=-L}^{L-1} a_n \Big(1 + x_1^{-2}\{1 + x_2^{-2}\{1 + \dots x_{L-1}^{-2}\{1 + x_L^{-2}\}\dots\}\} \\
&\quad + x_{-1}^{-2}\{1 + x_{-2}^{-2}\{1 + \dots \{x_{-L+1}^{-2}\{1 + x_{-L}^{-2}\}\dots\}\}\Big) \\
&\quad + \prod_{n=-L-1}^{L} a_n x_1^{-2}x_2^{-2}\dots x_L^{-2}x_{L+1}^{-2} + \prod_{n=-L-1}^{L} a_n x_{-L-1}^{-2}x_{-L}^{-2}\dots x_{-1}^{-2} \\
&= \prod_{n=-L-1}^{L} a_n \Big(1 + x_1^{-2}\{1 + x_2^{-2}\{1 + \dots x_{L-1}^{-2}\{1 + x_L^{-2}\}\dots\}\} \\
&\quad + x_{-1}^{-2}\{1 + x_{-2}^{-2}\{1 + \dots \{x_{-L+1}^{-2}\{1 + x_{-L}^{-2}\}\dots\}\} \\
&\quad + x_1^{-2}x_2^{-2}\dots x_L^{-2}x_{L+1}^{-2} + x_{-L-1}^{-2}x_{-L}^{-2}\dots x_{-1}^{-2}\Big)
\end{aligned}
$$

$$= \prod_{n=-L-1}^{L} a_n \Big(1 + x_1^{-2}\{1 + x_2^{-2}\{1 + \ldots x_L^{-2}\{1 + x_{L+1}^{-2}\}\ldots\}\}$$
$$+ x_{-1}^{-2}\{1 + x_{-2}^{-2}\{1 + \ldots x_{-L}^{-2}\{1 + x_{-L-1}^{-2}\}\ldots\}\}\Big)$$

as desired. The following two relations are straightforward computations:

$$x_1^{-2}\{1 + x_2^{-2}\{1 + \ldots x_{L-1}^{-2}\{1 + x_L^{-2}\}\ldots\}\} = \sum_{n=1}^{L} \frac{\varphi_{\bar{b}}^{L,k}(n)^2}{\varphi_{\bar{b}}^{L,k}(0)^2}$$

$$x_{-1}^{-2}\{1 + x_{-2}^{-2}\{1 + \ldots x_{-L+1}^{-2}\{1 + x_{-L}^{-2}\}\ldots\}\} = \sum_{n=-1}^{-L} \frac{\varphi_{\bar{b}}^{L,k}(n)^2}{\varphi_{\bar{b}}^{L,k}(0)^2}$$

Thus, using the fact that the eigenfunctions are normalized, we get the second expression for the determinant in (10).

We also note that

$$\left|x_1^{-1} \ldots x_m^{-1}\right| = \left|\varphi_{\bar{b}}^{L,k}(0)\right|^{-1} \left|\varphi_{\bar{b}}^{L,k}(m)\right|.$$

Now, we are in a position to carry out the substitution:

$$\begin{aligned}
\rho_L(m,0) &= \int \ldots \int \left(\sum_k \left|\langle \delta_m, \varphi_{\bar{b}}^{L,k}\rangle\right| \left|\langle \delta_0, \varphi_{\bar{b}}^{L,k}\rangle\right|\right) \prod_{n=-L}^{L} r_n(b_n - c_n) db_{-L} \ldots db_L \\
&= \sum_k \int \ldots \int \left|\varphi_{\bar{b}}^{L,k}(m)\right| \left|\varphi_{\bar{b}}^{L,k}(0)\right| \prod_{n=-L}^{L} r_n(b_n - c_n) db_{-L} \ldots db_L \\
&= \left(\prod_{n=-L}^{L-1} a_n\right) \sum_k \int \ldots \int \left|\varphi_{\bar{b}}^{L,k}(m)\right| \left|\varphi_{\bar{b}}^{L,k}(0)\right|^{-1} \prod_{n=-L}^{L} r_n(b_n - c_n) \\
&\quad \times \left(\prod_{n=-L}^{L-1} a_n\right)^{-1} \left|\varphi_{\bar{b}}^{L,k}(0)\right|^2 db_{-L} \ldots db_L \\
&\leqslant \left(\prod_{n=-L}^{L-1} a_n\right) \int_{\Sigma_0} \int_{\mathbb{R}^{2L}} \left|x_1^{-1} \ldots x_m^{-1}\right| \left(\prod_{n=-1}^{-L} r_n(E - a_{n-1}x_{n-1}^{-1} - a_n x_n - c_n)\right) \\
&\quad \times r_n(E - a_{-1}x_{-1}^{-1} - a_0 x_1^{-1} - c_0) \\
&\quad \times \left(\prod_{n=1}^{L} r_n(E - a_n x_{n+1}^{-1} - a_{n-1}x_n - c_n)\right) dx_{-L} \ldots dx_{-1} dx_1 \ldots dx_L dE
\end{aligned}$$

Let $\phi_{k;E;m}^{(n)}(x) = r_k(E - c_m - a_n x)$. Then, a quick computation shows:

$$\begin{aligned}
\left(S_{E;0}^{(0)} \ldots S_{E;-L+1}^{(-L+1)} \phi_{-L;E;-L}^{(-L)}\right)(x_1) &= \left(\prod_{n=0}^{-L+1} a_n\right) \int_{\mathbb{R}^L} r_0(E - a_{-1}x_{-1}^{-1} - a_0 x_1 - c_0) \\
&\quad \times \prod_{n=-1}^{-L} r_n(E - a_{n-1}x_{n-1}^{-1} - a_n x_n - c_n) dx_{-1} \ldots dx_{-L}.
\end{aligned}$$

Thus,

$$\left(US_{E;0}^{(0)}\dots S_{E;-L+1}^{(-L+1)}\phi_{-L;E;-L}^{(-L)}\right)(x_1)=\left(\prod_{n=0}^{-L+1}a_n\right)\int_{\mathbb{R}^L}|x_1|^{-1}r_0(E-a_{-1}x_{-1}^{-1}-a_0x_1^{-1}-c_0)$$
$$\times\prod_{n=-1}^{-L}r_n(E-a_{n-1}x_{n-1}^{-1}-a_nx_n-c_n)dx_{-1}\dots dx_{-L}.$$

Similarly,

$$\left(T_{E;1}^{(1)}\dots T_{E;m-1}^{(m-1)}S_{E;m}^{(m)}\dots S_{E;L-1}^{(L-1)}\phi_{L;E;L}^{(L-1)}\right)(x_1)$$
$$=\frac{\sqrt{a_0a_{m-1}}}{a_{L-1}}\left(\prod_{n=1}^{L-1}a_n\right)\int_{\mathbb{R}^{L-1}}\left|x_2^{-1}\dots x_m^{-1}\right|\prod_{n=1}^{L}r_n(E-a_nx_{n+1}^{-1}-a_{n-1}x_n-c_n)dx_L\dots dx_2.$$

Combining these results, we have thus proved the following lemma:

LEMMA 3.4. *With notation as above we have*

$$\rho_L(m,0)$$
$$\leqslant\frac{\sqrt{a_0a_{m-1}}}{a_{-L}a_{L-1}}\int_{\Sigma_0}\left\langle T_{E;1}^{(1)}\dots T_{E;m-1}^{(m-1)}S_{E;m}^{(m)}\dots S_{E;L-1}^{(L-1)}\phi_{L;E;L}^{(L-1)},US_{E;0}^{(0)}\dots S_{E;-L+1}^{(-L+1)}\phi_{-L;E;-L}^{(-L)}\right\rangle_{L^2(\mathbb{R},dx_1)}dE.$$

4. Norm estimates

DEFINITION 4.1. The norm of an operator $A:L^p(\mathbb{R})\to L^q(\mathbb{R})$ will be denoted by $\|A\|_{p,q}$.

REMARK 4.2. We want to point out that the following results hold for any $\alpha\in\mathbb{R}$, but since we will eventually care only for $\alpha\in\Sigma_0$ we state them in this form.

LEMMA 4.3. *For all $\alpha\in\Sigma_0$, we have*

$$\left\|S_\alpha^{(n)}\right\|_{1,1}\leqslant 1,$$

for all n.

Proof. We prove the statement for $n>0$, the cases $n=0$ and $n<0$ are proved similarly. For $f\in L^1(\mathbb{R})$ we have:

$$\begin{aligned}\left\|S_\alpha^{(n)}f\right\|_1&=\int\left|\left(S_\alpha^{(n)}f\right)(x)\right|dx\\&\leqslant a_{n-1}\int\int\left|d_n^{-1}r\left(d_n^{-1}\left(\alpha-a_{n-1}x-a_ny^{-1}\right)\right)\right||f(y)|dydx\\&=\frac{a_{n-1}}{d_n}\int\left(\frac{d_n}{a_{n-1}}\int r(\bar{x})d\bar{x}\right)|f(y)|dy=\|f\|_1.\end{aligned}$$

We have used the change of variables $\bar{x}=d_n^{-1}\left(\alpha-a_{n-1}x-a_ny^{-1}\right)$, the fact that r is nonnegative, and $\|r\|_1=1$. $\square$

LEMMA 4.4. *For all $\alpha \in \Sigma_0$ and all n we have*

$$\left\| S_\alpha^{(n)} \right\|_{1,2} \leqslant \sqrt{d_n^{-1} a_{n-1} \|r\|_\infty} < \infty$$

Proof. We prove for the case $n > 0$, the cases $n = 0$ and $n < 0$ are proved similarly. For $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}\left\| S_\alpha^{(n)} f \right\|_2^2 &= \int \left| \left(a_{n-1} \int r_n(\alpha - a_{n-1}x - a_n y^{-1}) f(y) dy \right) \left(a_{n-1} \int r_n(\alpha - a_{n-1}x - a_n z^{-1}) f(z) dz \right) \right| dx \\ &\leqslant \frac{a_{n-1}^2}{d_n} \|r\|_\infty \int \left(\int |f(y)| dy \right) \left(\int r_n(\alpha - a_{n-1}x - a_n z^{-1}) |f(z)| dz \right) dx \\ &= \frac{a_{n-1}^2}{d_n} \|r\|_\infty \|f\|_1 \int \frac{1}{a_{n-1}} \left(\int r_n(\bar{x}) d\bar{x} \right) |f(z)| dz = \frac{a_{n-1}}{d_n} \|r\|_\infty \|f\|_1^2 .\end{aligned}$$

So,

$$\left\| S_\alpha^{(n)} f \right\|_2 \leqslant \sqrt{d_n^{-1} a_{n-1} \|r\|_\infty}\, \|f\|_1 . \quad \square$$

LEMMA 4.5. *For all $\alpha \in \Sigma_0$ we have*

$$\left\| T_\alpha^{(n)} \right\|_{2,2} \leqslant 1.$$

Proof. Define an operator $\bar{U}^{(n)}$ by

$$\left(\bar{U}^{(n)} f \right)(x) = \sqrt{\frac{a_n}{a_{n-1}}} |x|^{-1} f\left(-\frac{a_n}{a_{n-1}} x^{-1} \right).$$

We first note that $\bar{U}^{(n)}$ is an isometry on $L^2(\mathbb{R})$. Indeed, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\left\| \bar{U}^{(n)} f \right\|_2^2 &= \int \left| \left(\bar{U}^{(n)} f \right)(x) \right|^2 dx = \int \left| \sqrt{\frac{a_n}{a_{n-1}}} |x|^{-1} f\left(-\frac{a_n}{a_{n-1}} x^{-1} \right) \right|^2 dx \\ &= \frac{a_n}{a_{n-1}} \int \left| \frac{a_{n-1}}{a_n} |u| f(u) \right|^2 \frac{a_n}{a_{n-1}} u^{-2} du = \int |f(u)|^2 du = \|f\|_2^2 .\end{aligned}$$

In the second line we have used the substitution $u = -\frac{a_n}{a_{n-1}} x^{-1}$. Since $\bar{U}^{(n)}$ is linear, it follows that, in fact, it is a unitary operator as surjectivity is easily established. Next, let us define an operator $K_{k;\alpha}^{(n)}$ by $K_{k;\alpha}^{(n)} f = r_{k;\alpha}^{(n)} * f$; that is

$$\left(K_{k;\alpha}^{(n)} f \right)(x) = \left(r_{k;\alpha}^{(n)} * f \right)(x) = \int r_{k;\alpha}^{(n)}(x-y) f(y) dy = \int r_k(\alpha - a_{n-1}x + a_{n-1}y) f(y) dy.$$

Then,

$$\begin{aligned}\left(K_{n;\alpha}^{(n)} \bar{U}^{(n)} f \right)(x) &= \left(r_{n;\alpha}^{(n)} * \bar{U}^{(n)} f \right)(x) = \int r_{n;\alpha}^{(n)}(x-y) \left(\bar{U}^{(n)} f \right)(y) dy \\ &= \int r_n(\alpha - a_{n-1}x + a_{n-1}y) \sqrt{\frac{a_n}{a_{n-1}}} |y|^{-1} f\left(-\frac{a_n}{a_{n-1}} y^{-1} \right) dy\end{aligned}$$

$$= \sqrt{\frac{a_n}{a_{n-1}}} \int r_n(\alpha - a_{n-1}x - a_n u^{-1}) \frac{a_{n-1}}{a_n} |u| f(u) \frac{a_n}{a_{n-1}} u^{-2} du$$
$$= \sqrt{\frac{a_n}{a_{n-1}}} \int r_n(\alpha - a_{n-1}x - a_n u^{-1}) |u|^{-1} f(u)\, du = \frac{1}{a_{n-1}} \left(T_\alpha^{(n)} f\right)(x).$$

We have used the substitution $u = -\frac{a_n}{a_{n-1}} y^{-1}$. So, we have $T_\alpha^{(n)} = a_{n-1} K_{n;\alpha}^{(n)} \bar{U}^{(n)}$. Then, it follows

$$\begin{aligned}
\left\|T_\alpha^{(n)} f\right\|_2 &= \left\|a_{n-1} K_{n;\alpha}^{(n)} \bar{U}^{(n)} f\right\|_2 = a_{n-1} \left\| r_{n;\alpha}^{(n)} * \bar{U}^{(n)} f \right\|_2 \\
&= a_{n-1} \left\| \widehat{r_{n;\alpha}^{(n)} * \bar{U}^{(n)} f} \right\|_2 = a_{n-1} \left\| \widehat{r_{n;\alpha}^{(n)}} \cdot \widehat{\bar{U}^{(n)} f} \right\|_2 \\
&\leqslant a_{n-1} \left\| \widehat{r_{n;\alpha}^{(n)}} \right\|_\infty \left\| \widehat{\bar{U}^{(n)} f} \right\|_2 \leqslant a_{n-1} \left\| r_{n;\alpha}^{(n)} \right\|_1 \|f\|_2 = a_{n-1} \|f\|_2 \int \left| r_{n;\alpha}^{(n)}(x) \right| dx \\
&= a_{n-1} \|f\|_2 \int r_n(\alpha - a_{n-1}x) dx = \|f\|_2 \int r_n(\bar{x}) d\bar{x} = \|r_n\|_1 \|f\|_2 = \|f\|_2 .
\end{aligned}$$

Hence, the result. $\square$

LEMMA 4.6. *For all $\alpha, \beta \in \Sigma_0$ the operator $T_\alpha^{(n)} T_\beta^{(n+1)}$ is compact.*

Proof. Let $K_{k;\alpha}^{(n)}$ and $\bar{U}^{(n)}$ be as before, and let F be the Fourier transform, $F : f \mapsto \widehat{f}$; that is

$$F[f](s) = \widehat{f}(s) = \int_{\mathbb{R}} e^{-2\pi i s x} f(x) dx.$$

Consider the operators $\bar{K}_{k;\alpha}^{(n)} = F K_{k;\alpha}^{(n)} F^{-1}$ and $\mathscr{U}^{(n)} = F \bar{U}^{(n)} F^{-1}$. Then

$$\begin{aligned}
T_\alpha^{(n)} T_\beta^{(n+1)} &= \left(a_{n-1} K_{n;\alpha}^{(n)} \bar{U}^{(n)}\right) \left(a_n K_{n+1;\beta}^{(n+1)} \bar{U}^{(n+1)}\right) \\
&= a_{n-1} a_n F^{-1} \bar{K}_{n;\alpha}^{(n)} \mathscr{U}^{(n)} \bar{K}_{n+1;\beta}^{(n+1)} \mathscr{U}^{(n+1)} F.
\end{aligned}$$

Since F and $\mathscr{U}^{(m)}$ are unitary operators, it suffices to show that $\bar{K}_{n;\alpha}^{(n)} \mathscr{U}^{(n)} \bar{K}_{n+1;\beta}^{(n+1)}$ is compact. We will actually show that it is a Hilbert-Schmidt operator, by showing that it is an integral operator with an L^2 kernel, and thus compact. Observe that

$$\bar{K}_{n;\alpha}^{(n)} f = F K_{n;\alpha}^{(n)} F^{-1} f = \widehat{r_{n;\alpha}^{(n)} * \check{f}} = \widehat{r_{n;\alpha}^{(n)}} \cdot f.$$

Now, let $g_1 \in C_c^\infty(\mathbb{R})$ be such that it is identically 1 in some neighborhood of zero, and put $g_2 = 1 - g_1$. We define the following two operators

$$\left(U_1^{(n)} f\right)(x) = g_1\left(\frac{a_{n-1}}{a_n} x\right) \left(\bar{U}^{(n)} f\right)(x)$$

$$\left(U_2^{(n)} f\right)(x) = g_2\left(\frac{a_{n-1}}{a_n}x\right)\left(\bar{U}^{(n)} f\right)(x).$$

Note that $\bar{U}^{(n)} = U_1^{(n)} + U_2^{(n)}$. Then,

$$\begin{aligned}
\left(\widehat{U_1^{(n)} f}\right)(k) &= \int e^{-2\pi ikx}\left(U_1^{(n)} f\right)(x)dx = \int e^{-2\pi ikx} g_1\left(\frac{a_{n-1}}{a_n}x\right)\left(\bar{U}^{(n)} f\right)(x)dx \\
&= \int e^{-2\pi ikx} g_1\left(\frac{a_{n-1}}{a_n}x\right)\sqrt{\frac{a_n}{a_{n-1}}}|x|^{-1} f\left(-\frac{a_n}{a_{n-1}}x^{-1}\right)dx \\
&= \sqrt{\frac{a_n}{a_{n-1}}}\int e^{-2\pi i\frac{a_n}{a_{n-1}}k\bar{x}^{-1}} g_1\left(\bar{x}^{-1}\right)\frac{a_{n-1}}{a_n}|\bar{x}| f\left(-\bar{x}\right)\frac{a_n}{a_{n-1}}\bar{x}^{-2}d\bar{x} \\
&= \sqrt{\frac{a_n}{a_{n-1}}}\int e^{-2\pi i\frac{a_n}{a_{n-1}}k\bar{x}^{-1}} g_1\left(\bar{x}^{-1}\right)|\bar{x}|^{-1}\left(\int e^{-2\pi i\bar{x}p}\widehat{f}(p)dp\right)d\bar{x} \\
&= \sqrt{\frac{a_n}{a_{n-1}}}\int\left(\int e^{-2\pi i\frac{a_n}{a_{n-1}}k\bar{x}^{-1}-2\pi i\bar{x}p} g_1\left(\bar{x}^{-1}\right)|\bar{x}|^{-1}d\bar{x}\right)\widehat{f}(p)dp \\
&= \sqrt{\frac{a_n}{a_{n-1}}}\int\left(\int e^{-2\pi i\frac{a_n}{a_{n-1}}kx-2\pi ipx^{-1}} g_1(x)|x|^{-1}dx\right)\widehat{f}(p)dp \\
&= \sqrt{\frac{a_n}{a_{n-1}}}\int a_1^{(n)}(k,p)\widehat{f}(p)dp,
\end{aligned}$$

where

$$a_1^{(n)}(k,p) \stackrel{\text{def}}{=} \int e^{-2\pi i\frac{a_n}{a_{n-1}}kx-2\pi ipx^{-1}} g_1(x)|x|^{-1}dx$$

We have used the following two substitutions in this order $\bar{x} = \dfrac{a_n}{a_{n-1}}x^{-1}$ and $x = \bar{x}^{-1}$, in lines four and seven, respectively.

Similarly

$$\left(\widehat{U_2^{(n)} f}\right)(k) = \sqrt{\frac{a_n}{a_{n-1}}}\int a_2^{(n)}(k,p)\widehat{f}(p)dp,$$

where

$$a_2^{(n)}(k,p) \stackrel{\text{def}}{=} \int e^{-2\pi i\frac{a_n}{a_{n-1}}kx-2\pi ipx^{-1}} g_2(x)|x|^{-1}dx$$

We claim that

$$\left(\bar{K}_{n;\alpha}^{(n)}\mathscr{U}^{(n)}\bar{K}_{n+1;\beta}^{(n+1)} f\right)(k) = \sqrt{\frac{a_n}{a_{n-1}}}\int b^{(n)}(k,p)f(p)dp, \tag{11}$$

where

$$b^{(n)}(k,p) = \widehat{r_{n;\alpha}^{(n)}}(k)\left(a_1^{(n)}(k,p) + a_2^{(n)}(k,p)\right)\widehat{r_{n+1;\beta}^{(n+1)}}(p).$$

Observe that

$$\bar{K}_{n;\alpha}^{(n)}\mathscr{U}^{(n)}\bar{K}_{n+1;\beta}^{(n+1)} = \bar{K}_{n;\alpha}^{(n)} F U_1^{(n)} F^{-1}\bar{K}_{n+1;\beta}^{(n+1)} + \bar{K}_{n;\alpha}^{(n)} F U_2^{(n)} F^{-1}\bar{K}_{n+1;\beta}^{(n+1)},$$

where we have used the fact that $\bar{U}^{(n)} = U_1^{(n)} + U_2^{(n)}$. Next

$$\begin{aligned}\left(\bar{K}^{(n)}_{n;\alpha} F U_1^{(n)} F^{-1} \bar{K}^{(n+1)}_{n+1;\beta} f\right)(k) &= \widehat{r^{(n)}_{n;\alpha}}(k) \cdot \left(F U_1^{(n)} F^{-1} \bar{K}^{(n+1)}_{n+1;\beta} f\right)(k) \\ &= \sqrt{\frac{a_n}{a_{n-1}}} \int \widehat{r^{(n)}_{n;\alpha}}(k) a_1^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) f(p)\,dp.\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\bar{K}^{(n)}_{n;\alpha} F U_2^{(n)} F^{-1} \bar{K}^{(n+1)}_{n+1;\beta} f\right)(k) &= \widehat{r^{(n)}_{n;\alpha}}(k) \cdot \left(F U_2^{(n)} F^{-1} \bar{K}^{(n+1)}_{n+1;\beta} f\right)(k) \\ &= \sqrt{\frac{a_n}{a_{n-1}}} \int \widehat{r^{(n)}_{n;\alpha}}(k) a_2^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) f(p)\,dp.\end{aligned}$$

Combining these two expressions, we get (11). Next, we need to show that $b^{(n)}$ is in L^2. We have,

$$\begin{aligned}\left\| b^{(n)}(k,p) \right\|_{L^2(\mathbb{R},dk)\times L^2(\mathbb{R},dp)} \leqslant{}& \left\| \widehat{r^{(n)}_{n;\alpha}}(k) a_1^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) \right\|_{L^2(\mathbb{R},dk)\times L^2(\mathbb{R},dp)} \\ &+ \left\| \widehat{r^{(n)}_{n;\alpha}}(k) a_2^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) \right\|_{L^2(\mathbb{R},dk)\times L^2(\mathbb{R},dp)}. \qquad (12)\end{aligned}$$

Note that,

$$\begin{aligned}&\left\| \widehat{r^{(n)}_{n;\alpha}}(k) a_1^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) \right\|_{L^2(dk)\times L^2(dp)} \\ &= \left(\int_{\mathbb{R}\times\mathbb{R}} \left| \widehat{r^{(n)}_{n;\alpha}}(k) a_1^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) \right|^2 dk\,dp \right)^{1/2} \\ &\leqslant \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \widehat{r^{(n)}_{n;\alpha}}(k) \right|^2 \left| a_1^{(n)}(k,p) \right|^2 dk\,dp \right)^{1/2} \left\| \widehat{r^{(n+1)}_{n+1;\beta}} \right\|_{L^\infty(\mathbb{R},dp)} \\ &\leqslant \left(\int_{\mathbb{R}} \left| \widehat{r^{(n)}_{n;\alpha}}(k) \right|^2 \left(\int_{\mathbb{R}} \left| a_1^{(n)}(k,p) \right|^2 dp \right) dk \right)^{1/2} \left\| \widehat{r^{(n+1)}_{n+1;\beta}} \right\|_{L^\infty(\mathbb{R},dp)} \\ &\leqslant \left\| \widehat{r^{(n)}_{n;\alpha}} \right\|_{L^2(\mathbb{R},dk)} \sup_k \left(\int_{\mathbb{R}} \left| a_1^{(n)}(k,p) \right|^2 dp \right)^{1/2} \left\| \widehat{r^{(n+1)}_{n+1;\beta}} \right\|_{L^\infty(\mathbb{R},dp)} \\ &= \left\| \widehat{r^{(n)}_{n;\alpha}} \right\|_{L^2(\mathbb{R},dk)} \sup_k \left\| a_1^{(n)}(k,\cdot) \right\|_{L^2(\mathbb{R},dp)} \left\| \widehat{r^{(n+1)}_{n+1;\beta}} \right\|_{L^\infty(\mathbb{R},dp)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\left\| \widehat{r^{(n)}_{n;\alpha}}(k) a_2^{(n)}(k,p) \widehat{r^{(n+1)}_{n+1;\beta}}(p) \right\|_{L^2(\mathbb{R},dk)\times L^2(\mathbb{R},dp)} \\ &\leqslant \left\| \widehat{r^{(n)}_{n;\alpha}} \right\|_{L^\infty(dk)} \sup_p \left\| a_2^{(n)}(\cdot,p) \right\|_{L^2(dk)} \left\| \widehat{r^{(n+1)}_{n+1;\beta}} \right\|_{L^2(dp)}.\end{aligned}$$

Since, $r \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $r \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. So, by Plancherel's theorem, it follows that $\widehat{r} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Thus, it remains to show that

i. $\sup_k \left\| a_1^{(n)}(k,\cdot) \right\|_{L^2(\mathbb{R},dp)} < \infty$, and

ii. $\sup_p \left\| a_2^{(n)}(\cdot,p) \right\|_{L^2(\mathbb{R},dk)} < \infty$.

We begin by proving the first claim. To this end let

$$f_N^{(k)}(\bar{x}) = e^{-2\pi i \frac{a_n}{a_{n-1}} k\bar{x}^{-1}} \frac{g_1(\bar{x}^{-1})}{|\bar{x}|} \cdot \chi_{[-N,N]}(\bar{x})$$

$$f^{(k)}(\bar{x}) = e^{-2\pi i \frac{a_n}{a_{n-1}} k\bar{x}^{-1}} \frac{g_1(\bar{x}^{-1})}{|\bar{x}|}.$$

Since g_1 is compactly supported and is identically equal to 1 in a neighborhood of 0, it is not difficult to see that $f^{(k)}$ is an L^2 function, and that its L^2 norm is independent of k. From this, it is, also, not difficult to see that $f_N^{(k)}$ converges to $f^{(k)}$ in L^2 sense. As a result, it is straightforward to see that $\widehat{f_N^{(k)}}$ converges to $\widehat{f^{(k)}}$ in L^2 sense; where

$$\widehat{f^{(k)}}(p) = \int f^{(k)}(\bar{x}) e^{-2\pi i p\bar{x}} d\bar{x} = \int e^{-2\pi i \frac{a_n}{a_{n-1}} k\bar{x}^{-1} - 2\pi i p\bar{x}} \frac{g_1(\bar{x}^{-1})}{|\bar{x}|} d\bar{x}$$

$$\widehat{f_N^{(k)}} = \int_{|\bar{x}|<N} e^{-2\pi i \frac{a_n}{a_{n-1}} k\bar{x}^{-1} - 2\pi i p\bar{x}} \frac{g_1(\bar{x}^{-1})}{|\bar{x}|} d\bar{x}.$$

Note that,

$$\begin{aligned} a_1^{(n)}(k,p) \cdot \chi_{\{x:|x|>\frac{1}{N}\}}(x) &= \int_{|x|>\frac{1}{N}} e^{-2\pi i \frac{a_n}{a_{n-1}} kx - 2\pi i p x^{-1}} g_1(x)|x|^{-1} dx \\ &= \int_{|\bar{x}|<N} e^{-2\pi i \frac{a_n}{a_{n-1}} k\bar{x}^{-1} - 2\pi i p\bar{x}} \frac{g_1(\bar{x}^{-1})}{|\bar{x}|} d\bar{x} \\ &= \widehat{f_N^{(k)}}(p). \end{aligned}$$

Then, from our discussion above, it follows that $a_1^{(n)}(k,p) = \widehat{f^{(k)}}(p)$. Hence, by unitarity of the Fourier transform, and the fact that $f^{(k)}$ has L^2 norm independent of k, we get

$$\sup_k \left\| a_1^{(n)}(k,\cdot) \right\|_{L^2(\mathbb{R},dp)} = \sup_k \left\| \widehat{f^{(k)}} \right\|_{L^2(\mathbb{R},dp)} = \sup_k \left\| f^{(k)} \right\|_{L^2(\mathbb{R},dp)} = \left\| f^{(k)} \right\|_{L^2(\mathbb{R},dp)} < \infty. \tag{13}$$

Next, let

$$f^{(p)}(\bar{x}) = e^{-2\pi i \frac{a_n}{a_{n-1}} p\bar{x}^{-1}} g_2\left(\frac{a_{n-1}}{a_n}\bar{x}\right) |\bar{x}|^{-1}.$$

Since g_2 vanishes in a neighborhood of 0, it is easy to see that $f^{(p)}$ is an L^2 function, and that its norm is independent of p. Then,

$$\widehat{f^{(p)}}(k) = \int e^{-2\pi i k\bar{x}} f^{(p)}(\bar{x})\bar{x} = \int e^{-2\pi i k\bar{x} - 2\pi i \frac{a_n}{a_{n-1}} p\bar{x}^{-1}} g_2\left(\frac{a_{n-1}}{a_n}\bar{x}\right)|\bar{x}|^{-1} d\bar{x}$$
$$= \int e^{-2\pi i \frac{a_n}{a_{n-1}} kx - 2\pi i p x^{-1}} g_2(x)|x|^{-1} dx = a_2^{(n)}(k,p).$$

Hence,

$$\sup_p \left\| a_2^{(n)}(\cdot,p) \right\|_{L^2(\mathbb{R},dk)} = \sup_p \left\| \widehat{f^{(p)}} \right\|_{L^2(\mathbb{R},dk)} = \sup_p \left\| f^{(p)} \right\|_{L^2(\mathbb{R},dk)} = \left\| f^{(p)} \right\|_{L^2(\mathbb{R},dk)} < \infty. \tag{14}$$

This concludes that $b^{(n)}$ is an L^2 function. So, $T_\alpha^{(n)} T_\beta^{(n+1)}$ is Hilbert-Schmidt, and thus compact. □

Next, we adopt the technique developed in [18] to prove the following lemma.

LEMMA 4.7. *For some fixed constant C_0 we have*

$$\left\| T_\alpha^{(n)} T_\beta^{(n+1)} f \right\|_2 \leqslant A(n,n+1)\,\|f\|_2,$$

where

$$A(n,n+1) \stackrel{def}{=} \left(\frac{15}{16} + \frac{1}{16} \sup_{|k| \geqslant t_n \frac{C_0}{\|a\|_\infty}} |\widehat{r}(k)|^2 \right)^{\frac{1}{2}},$$

where $t_n = \min(d_n, d_{n+1})$.

Proof. Above we have shown that, in particular, $T_\alpha^{(n)} T_\beta^{(n+1)}$ is a Hilbert-Schmidt operator. Specifically,

$$T_\alpha^{(n)} T_\beta^{(n+1)} = F^{-1}\left(a_{n-1} a_n \bar{K}_{n;\alpha}^{(n)} \mathscr{U}^{(n)} \bar{K}_{n+1;\beta}^{(n+1)} \right) \mathscr{U}^{(n+1)} F.$$

So, it suffices to show that for $\|\varphi\|_2 = \|\psi\|_2 = 1$ we have

$$\left| \langle \varphi, a_{n-1} a_n \bar{K}_{n;\alpha}^{(n)} \mathscr{U}^{(n)} \bar{K}_{n+1;\beta}^{(n+1)} \psi \rangle \right| \leqslant A(n,n+1). \tag{15}$$

Pick C_0, such that

$$B\left(\int_{|k| \leqslant C_0} \int_{|p| \leqslant C_0} \left| a^{(n)}(k,p) \right|^2 dk\,dp \right)^{1/2} \leqslant \frac{7}{16}, \tag{16}$$

where $B = \sup \sqrt{\frac{a_n}{a_{n-1}}}$. We claim that, this is possible, since the left hand side of (16) goes to zero, as $C_0 \to 0$, and also that such a C_0 can be chosen independently of n.

Both of these facts are a byproduct of the proof of Lemma 4.6. More precisely, note that

$$
\begin{aligned}
&\left\|a^{(n)}(k,p)\right\|_{L^2([-C_0,C_0]^2,dkdp)} \\
&\leqslant \left\|a_1^{(n)}(k,p)\right\|_{L^2([-C_0,C_0]^2,dkdp)} + \left\|a_2^{(n)}(k,p)\right\|_{L^2([-C_0,C_0]^2,dkdp)} \qquad (17)\\
&\leqslant \sqrt{2C_0}\left(\sup_k \left\|a_1^{(n)}(k,\cdot)\right\|_{L^2([-C_0,C_0],dp)} + \sup_p \left\|a_2^{(n)}(\cdot,p)\right\|_{L^2([-C_0,C_0],dk)}\right)\\
&\leqslant \sqrt{2C_0}\left(\sup_k \left\|a_1^{(n)}(k,\cdot)\right\|_{L^2(\mathbb{R},dp)} + \sup_p \left\|a_2^{(n)}(\cdot,p)\right\|_{L^2(\mathbb{R},dk)}\right)\\
&= \sqrt{2C_0}\left(\left\|f^{(k)}\right\|_{L^2(\mathbb{R},dp)} + \left\|f^{(p)}\right\|_{L^2(\mathbb{R},dk)}\right)\\
&\leqslant \sqrt{2C_0}\times\sqrt{2\pi}\left(\left(\int_{\mathbb{R}} \frac{\left|g_1(p^{-1})\right|^2}{|p|^2}dp\right)^{1/2} + \sqrt{\frac{\|a\|_\infty}{\delta}}\left(\int_{\mathbb{R}} \frac{\left|g_2(y)\right|^2}{|y|^2}dy\right)^{1/2}\right), \qquad (18)
\end{aligned}
$$

where, going from line three to four, we have used expressions (13) and (14), and from line four to five we have performed a change of variables and used the fact that $0 < \delta \leqslant a_n \leqslant \|a\|_\infty$, for all n. So, using the fact that, as seen before, the integrals that appear above are finite, we can pick C_0 independently of n, such that the right hand side of (17) is less than $\frac{7}{16}$.

Let $\varphi_+ = \varphi\chi_{\{|k|\geqslant C_0\}}$ and $\psi_+ = \psi\chi_{\{|k|\geqslant C_0\}}$. We consider two cases

(i) $\|\varphi_+\|_2 \geqslant \frac{1}{4}$ or $\|\psi_+\|_2 \geqslant \frac{1}{4}$;

(ii) $\|\varphi_+\|_2 \leqslant \frac{1}{4}$ and $\|\psi_+\|_2 \leqslant \frac{1}{4}$.

First, using the fact that $\bar{K}^{(n)}_{n;\alpha} f = \widehat{r^{(n)}_{n;\alpha}}\cdot f$ and the fact that $\|\widehat{g}\|_\infty \leqslant \|g\|_1$, we get that

$$
\left\|\bar{K}^{(n)}_{n;\alpha} f\right\|_2 \leqslant \frac{1}{a_{n-1}}\|f\|_2, \tag{19}
$$

and also

$$
\begin{aligned}
\widehat{r^{(n)}_{n;\alpha}}(s) = \int e^{-2\pi i s x} r_n(\alpha - a_{n-1}x)dx &= \frac{1}{a_{n-1}} e^{-2\pi i s \frac{\alpha}{a_{n-1}}} \frac{1}{\sqrt{2\pi}} \int e^{-2\pi i\left(-\frac{d_n}{a_{n-1}}s\right)\bar{x}} r(\bar{x})d\bar{x}\\
&= \frac{1}{a_{n-1}} e^{-2\pi i s \frac{\alpha}{a_{n-1}}}\, \widehat{r}\left(-\frac{d_n}{a_{n-1}}s\right).
\end{aligned}
$$

So,

$$
\left|\widehat{r^{(n)}_{n;\alpha}}(s)\right| = \frac{1}{a_{n-1}}\left|\widehat{r}\left(-\frac{d_n}{a_{n-1}}s\right)\right|. \tag{20}
$$

Now, suppose that $\|\psi_+\|_2 \geqslant \frac{1}{4}$. Then,

$$
\begin{aligned}
&\left\|a_n \bar{K}^{(n+1)}_{n+1;\beta}\psi\right\|_2^2 \\
&= a_n^2 \int_{\mathbb{R}} \left|\left(\bar{K}^{(n+1)}_{n+1;\beta}\psi\right)(k)\right|^2 dk \\
&= a_n^2 \int_{\mathbb{R}} \left|\widehat{r}^{(n+1)}_{n+1;\beta}(k)\psi(k)\right|^2 dk = a_n^2 \int_{\mathbb{R}} \left|\widehat{r}^{(n+1)}_{n+1;\beta}(k)\right|^2 |\psi(k)|^2 dk \\
&= a_n^2 \int_{\mathbb{R}} \left|\frac{1}{a_n}\widehat{r}\left(-\frac{d_{n+1}}{a_n}k\right)\right|^2 |\psi(k)|^2 dk = \int_{\mathbb{R}} \left|\widehat{r}\left(-\frac{d_{n+1}}{a_n}k\right)\right|^2 |\psi(k)|^2 dk \\
&= \int_{\{|k|\geqslant K_0\}} \left|\widehat{r}\left(-\frac{d_{n+1}}{a_n}k\right)\right|^2 |\psi(k)|^2 dk + \int_{\{|k|<C_0\}} \left|\widehat{r}\left(-\frac{d_{n+1}}{a_n}k\right)\right|^2 |\psi(k)|^2 dk \\
&\leqslant \sup_{|k|\geqslant C_0} \left|\widehat{r}\left(-\frac{d_{n+1}}{a_n}k\right)\right|^2 \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk + \int_{\{|k|<C_0\}} |\psi(k)|^2 dk \\
&= \sup_{|k|\geqslant \frac{d_{n+1}}{a_n}C_0} |\widehat{r}(k)|^2 \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk + \int_{\{|k|<C_0\}} |\psi(k)|^2 dk \\
&\leqslant \sup_{|k|\geqslant \frac{C_0}{\|a\|_\infty}d_{n+1}} |\widehat{r}(k)|^2 \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk + \int_{\{|k|<C_0\}} |\psi(k)|^2 dk \\
&\quad + \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk - \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk \\
&= \left(\sup_{|k|\geqslant \frac{C_0}{\|a\|_\infty}d_{n+1}} |\widehat{r}(k)|^2 - 1\right) \int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk + 1 \\
&= 1 + \left(-\int_{\{|k|\geqslant C_0\}} |\psi(k)|^2 dk\right)\left(1 - \sup_{|k|\geqslant \frac{C_0}{\|a\|_\infty}d_{n+1}} |\widehat{r}(k)|^2\right) \\
&\leqslant 1 - \frac{1}{16}\left(1 - \sup_{|k|\geqslant \frac{C_0}{\|a\|_\infty}d_{n+1}} |\widehat{r}(k)|^2\right) \\
&= \frac{15}{16} + \frac{1}{16} \sup_{|k|\geqslant \frac{C_0}{\|a\|_\infty}d_{n+1}} |\widehat{r}(k)|^2 .
\end{aligned}
$$

Now, using Cauchy-Schwarz, (19), and the fact that $\mathscr{U}^{(n)}$ is unitary, we get

$$
\begin{aligned}
\left|\langle \varphi, a_{n-1}a_n \bar{K}^{(n)}_{n;\alpha}\mathscr{U}^{(n)}\bar{K}^{(n+1)}_{n+1;\beta}\psi\rangle\right| &\leqslant \|\varphi\|_2 \left\|a_{n-1}a_n \bar{K}^{(n)}_{n;\alpha}\mathscr{U}^{(n)}\bar{K}^{(n+1)}_{n+1;\beta}\psi\rangle\right\|_2 \\
&\leqslant \left\|a_n \bar{K}^{(n+1)}_{n+1;\beta}\psi\right\|_2 .
\end{aligned}
$$

Thus, from above, in this case the result follows. Next, if $\|\varphi_+\|_2 \geqslant \frac{1}{4}$, then

$$
\begin{aligned}
&\left|\langle \varphi, a_{n-1}a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi\rangle\right| \\
&= \left| \int_{\mathbb{R}} \varphi(k) a_{n-1} a_n \left(\bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \right)(k) dk \right| \\
&\leqslant \int_{\mathbb{R}} \left| \varphi(k) a_{n-1} \widehat{r}^{(n)}_{n;\alpha}(k) \right| \left| a_n \left(\mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \right)(k) \right| dk \\
&\leqslant \left(\int_{\mathbb{R}} \left| \varphi(k) a_{n-1} \widehat{r}^{(n)}_{n;\alpha}(k) \right|^2 dk \right)^{1/2} \left(\int_{\mathbb{R}} \left| a_n \left(\mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \right)(k) \right|^2 dk \right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \left| \widehat{r}\left(-\frac{d_n}{a_{n-1}} k \right) \right|^2 |\varphi(k)|^2 dk \right)^{1/2} a_n \left\| \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \right\|_2 \\
&= \left(\int_{\mathbb{R}} \left| \widehat{r}\left(-\frac{d_n}{a_{n-1}} k \right) \right|^2 |\varphi(k)|^2 dk \right)^{1/2} a_n \left\| \bar{K}^{(n+1)}_{n+1;\beta} \psi \right\|_2 \\
&\leqslant \left(\int_{\mathbb{R}} \left| \widehat{r}\left(-\frac{d_n}{a_{n-1}} k \right) \right|^2 |\varphi(k)|^2 dk \right)^{1/2} \\
&\leqslant \left(\frac{15}{16} + \frac{1}{16} \sup_{|k| \geqslant \frac{C_0}{\|a\|_\infty} d_n} |\widehat{r}(k)|^2 \right)^{1/2}.
\end{aligned}
$$

The last inequality follows via the same argument as before. Thus, again, the result follows.

Before we consider the second case, let $\varphi_- = \varphi\chi_{\{|k|<C_0\}}$, and $\psi_- = \psi\chi_{\{|k|<C_0\}}$. Then

$$
\begin{aligned}
&\left|\langle \varphi_-, a_{n-1}a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi_-\rangle\right| \\
&= \left| \int_{\mathbb{R}} \varphi_-(k) a_{n-1} a_n \left(\bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi_- \right)(k) dk \right| \\
&= \left| \int_{\mathbb{R}} \varphi_-(k) \sqrt{\frac{a_n}{a_{n-1}}} a_{n-1} a_n \int_{\mathbb{R}} \widehat{r}^{(n)}_{n;\alpha}(k) a^{(n)}(k,p) \widehat{r}^{(n+1)}_{n+1;\beta}(p) \psi_-(p) dp dk \right| \\
&= \left| \int_{\{|k|\leqslant C_0\}} \int_{\{|p|\leqslant C_0\}} \sqrt{\frac{a_n}{a_{n-1}}} a_{n-1} a_n \varphi(k) \widehat{r}^{(n)}_{n;\alpha}(k) a^{(n)}(k,p) \widehat{r}^{(n+1)}_{n+1;\beta}(p) \psi(p) dp dk \right| \\
&\leqslant \int_{\{|k|\leqslant C_0\}} \int_{\{|p|\leqslant C_0\}} \sqrt{\frac{a_n}{a_{n-1}}} a_{n-1} a_n |\varphi(k)\psi(p)| \left| \widehat{r}^{(n)}_{n;\alpha}(k) \right| \left| a^{(n)}(k,p) \right| \left| \widehat{r}^{(n+1)}_{n+1;\beta}(p) \right| dp dk \\
&\leqslant \sqrt{\frac{a_n}{a_{n-1}}} \int_{\{|k|\leqslant C_0\}} \int_{\{|p|\leqslant C_0\}} |\varphi(k)\psi(p)| \left| a^{(n)}(k,p) \right| dp dk \\
&\leqslant \sqrt{\frac{a_n}{a_{n-1}}} \left(\int_{\{|k|\leqslant C_0\}} \int_{\{|p|\leqslant C_0\}} |\varphi(k)|^2 |\psi(p)|^2 dp dk \right)^{1/2} \\
&\quad \times \left(\int_{\{|k|\leqslant C_0\}} \int_{\{|p|\leqslant C_0\}} \left| a^{(n)}(k,p) \right|^2 dp dk \right)^{1/2}
\end{aligned}
$$

$$= \sqrt{\frac{a_n}{a_{n-1}}} \left(\int_{\{|p| \leqslant C_0\}} |\psi(p)|^2 dp \right)^{1/2} \left(\int_{\{|k| \leqslant C_0\}} |\varphi(k)|^2 dk \right)^{1/2}$$
$$\times \left(\int_{\{|k| \leqslant C_0\}} \int_{\{|p| \leqslant C_0\}} \left| a^{(n)}(k,p) \right|^2 dp dk \right)^{1/2}$$
$$\leqslant B \left(\int_{\{|k| \leqslant C_0\}} \int_{\{|p| \leqslant C_0\}} \left| a^{(n)}(k,p) \right|^2 dp dk \right)^{1/2} \leqslant \frac{7}{16}.$$

Finally, if $\|\varphi_+\|_2 \leqslant \frac{1}{4}$, and $\|\psi_+\|_2 \leqslant \frac{1}{4}$, we have

$$\left| \langle \varphi, a_{n-1} a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \rangle \right|$$
$$= \left| \langle \varphi_+, a_{n-1} a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \rangle + \langle \varphi_-, a_{n-1} a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi \rangle \right|$$
$$\leqslant \|\varphi_+\|_2 + \left| \langle \varphi_-, a_{n-1} a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi_+ \rangle \right| + \left| \langle \varphi_-, a_{n-1} a_n \bar{K}^{(n)}_{\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{\beta} \psi_- \rangle \right|$$
$$\leqslant \|\varphi_+\|_2 + \|\psi_+\|_2 + \left| \langle \varphi_-, a_{n-1} a_n \bar{K}^{(n)}_{n;\alpha} \mathscr{U}^{(n)} \bar{K}^{(n+1)}_{n+1;\beta} \psi_- \rangle \right|$$
$$\leqslant \frac{1}{4} + \frac{1}{4} + \frac{7}{16} = \frac{15}{16} < A(n, n+1).$$

This concludes the proof of the lemma. □

We record the following as a corollary, so we can refer to it later.

COROLLARY 4.8. *Let d_n be such that $d_n = 1$ for all n. Then, there exists some constant $0 < q < 1$, such that*

$$\left\| T^{(n)}_{\alpha} T^{(n+1)}_{\beta} \right\|_{2,2} \leqslant q$$

for all n and all α, β.

Proof. This is an immediate consequence of Lemma 4.7 with

$$q \stackrel{\text{def}}{=} \left(\frac{15}{16} + \frac{1}{16} \sup_{|k| \geqslant \frac{C_0}{\|a\|_\infty}} |\widehat{r}(k)|^2 \right)^{\frac{1}{2}}. \quad \square$$

If the sequence $d_n \not\equiv 1$ we can no longer bound $\left\| T^{(n)}_{\alpha} T^{(n+1)}_{\beta} \right\|_{2,2}$ uniformly away from 1, however, we can still control the rate at which this norm converges to 1, as is established in the following Lemma.

LEMMA 4.9. *Let d_n be a fixed sequence with $0 \leqslant d_n \leqslant 1$ and $d_n \geqslant C|n|^{-\zeta}$ for $\zeta < \frac{1}{2}$, and some constant $C > 0$. Then,*

$$A(s) \stackrel{def}{=} \left(\frac{15}{16} + \frac{1}{16} \sup_{|k| \geqslant t_s \frac{C_0}{\|a\|_\infty}} |\widehat{r}(k)|^2 \right)^{\frac{1}{2}} \leqslant \exp\left(-\gamma' |s|^{-2\zeta} \right),$$

for some $\gamma' > 0$, *where* $t_s = \min(d_{2s-1}, d_{2s})$.

Proof. First let us show that $\frac{d^2}{dk^2}|\widehat{r}(k)|^2\Big|_{k=0} < 0$. We compute,

$$\begin{aligned}
\frac{d^2}{dk^2}|\widehat{r}(k)|^2\Big|_{k=0} &= \frac{d^2}{dk^2}\widehat{r}(k)\overline{\widehat{r}(k)}\Big|_{k=0} = \frac{d}{dk}\left(\widehat{r}(k)\frac{d}{dk}\overline{\widehat{r}(k)} + \overline{\widehat{r}(k)}\frac{d}{dk}\widehat{r}(k)\right)\Big|_{k=0} \\
&= 2\frac{d}{dk}\widehat{r}(k)\frac{d}{dk}\overline{\widehat{r}(k)}\Big|_{k=0} + \widehat{r}(k)\frac{d^2}{dk^2}\overline{\widehat{r}(k)}\Big|_{k=0} + \overline{\widehat{r}(k)}\frac{d^2}{dk^2}\widehat{r}(k)\Big|_{k=0} \\
&= 2\int xe^{-ikx}r(x)dx\int xe^{ikx}r(x)dx\Big|_{k=0} - 2\Re\left(\int e^{-ikx}r(x)dx\int x^2e^{ikx}r(x)dx\right)\Big|_{k=0} \\
&= 2\left(\int xr(x)dx\right)^2 - 2\left(\int x^2r(x)dx\right) < 0,
\end{aligned}$$

where the strict inequality, in the last line, follows by Cauchy-Schwarz. Before we proceed, we need the following result to be able to complete the proof of the lemma. Though this is a standard result, since we were unable to find a reference, for reader's convenience, we provide a proof as well.

CLAIM 4.10. For $k \neq 0$ we have

$$|\widehat{r}(k)| < 1.$$

Proof of Claim. First, we know that in general we have $|\widehat{r}(k)| \leqslant \|r\|_1 = 1$ with $|\widehat{r}(0)| = 1$. So, suppose that there is some $k \neq 0$ such that $|\widehat{r}(k)| = 1$. First, by taking the real part of $\widehat{r}(k)$ we get

$$\Re\widehat{r}(k) = \int\cos(kx)r(x)dx < \int r(x)dx = 1, \tag{21}$$

where the strict inequality follows from the fact that $r(k) \geqslant 0$ and for $k \neq 0$ we have $\cos(kx) < 1$, away from a set of measure zero. So, we can rotate $\widehat{r}(k)$ by an angle θ such that it is equal to one. That is, let $\theta \in (0, 2\pi)$ be such that $\widehat{r}(k)e^{-i\theta} = 1$. On the other hand, observe that for any such θ we also have

$$\Re\left(\widehat{r}(k)e^{-i\theta}\right) = \int\cos(kx+\theta)r(x)dx < \int r(x)dx = 1,$$

where, again, the strict inequality follows from the fact that $\cos(kx+\theta) < 1$ away from a countable set. This is clearly a contradiction to our choice of θ. $\square$

Thus, since $|\widehat{r}(k)| < 1$ for $k \neq 0$ and $|\widehat{r}(0)| = 1$, with $\frac{d^2}{dk^2}|\widehat{r}(k)|^2\Big|_{k=0} < 0$, by a Taylor series expansion around zero, for λ small enough we have

$$\sup_{|k|\geqslant\lambda}|\widehat{r}(k)|^2 \leqslant 1 - c\lambda^2 \leqslant e^{-\tilde{c}\lambda^2},$$

with $c \stackrel{\text{def}}{=} -\frac{d^2}{dk^2}|\widehat{r}(k)|^2\Big|_{k=0} > 0$, and some $\tilde{c} > 0$. Then,

$$\left(\frac{15}{16}+\frac{1}{16}\sup_{|k|\geqslant\lambda}|\widehat{r}(k)|^2\right)^{1/2} \leqslant \left(\frac{15}{16}+\frac{1}{16}-c_1\lambda^2\right)^{1/2} = \left(1-c_1\lambda^2\right)^{1/2} \leqslant e^{-c_2\lambda^2}.$$

As a result, up to possibly shrinking C_0, we have

$$A(s) \stackrel{\text{def}}{=} \left(\frac{15}{16}+\frac{1}{16}\sup_{|k|\geqslant t_s\frac{C_0}{\|a\|_\infty}}|\widehat{r}(k)|^2\right)^{\frac{1}{2}} \leqslant \exp\left(-c_2\left(t_s\frac{C_0}{\|a\|_\infty}\right)^2\right) \leqslant \exp\left(-\gamma'|s|^{-2\zeta}\right).$$

□

The following corollary is an immediate consequence of the arguments above.

COROLLARY 4.11. *Let d_n be a fixed sequence with $0 \leqslant d_n \leqslant 1$ and $d_n \geqslant C|n|^{-\zeta}$ for $\zeta < \frac{1}{2}$. Then,*

$$\left\|T_\alpha^{(2s-1)}T_\beta^{(2s)}\right\|_{2,2} \leqslant \exp\left(-\gamma'|s|^{-2\zeta}\right).$$

Proof. From Lemma 4.7 we clearly have,

$$\left\|T_\alpha^{(2s-1)}T_\beta^{(2s)}\right\|_{2,2} \leqslant A(s),$$

hence the result follows from Lemma 4.9. □

LEMMA 4.12. *With notation as above we have*

$$A(1)\times\ldots\times A(s) \leqslant \exp\left(-\gamma'|s|^{1-2\zeta}\right),$$

for some constant $\gamma' > 0$.

Proof. We have,

$$\begin{aligned}
A(1)\times\ldots\times A(s) &= \exp\left(-\gamma'\left(1+2^{-2\zeta}+\ldots+(s-1)^{-2\zeta}+s^{-2\zeta}\right)\right)\\
&= \exp\left(-\gamma' s^{-2\zeta}\left(s^{2\zeta}+2^{-2\zeta}s^{2\zeta}+\ldots+(s-1)^{-2\zeta}s^{2\zeta}+1\right)\right)\\
&\leqslant \exp\left(-\gamma' s^{1-2\zeta}\right).
\end{aligned}$$

The last inequality follows from the fact that

$$s^{2\zeta}+2^{-2\zeta}s^{2\zeta}+\ldots+(s-1)^{-2\zeta}s^{2\zeta}+1 \geqslant 1+2^{-2\zeta}2^{2\zeta}+\ldots+(s-1)^{-2\zeta}(s-1)^{2\zeta}+1 = s.$$

□

4.1. Proof of Theorem 2.4

Proof. With the same notation as in the statement of the theorem, we have

$$\begin{aligned}
&\int_\Omega \left(\sup_{t\in\mathbb{R}} \left|\langle \delta_m, e^{-itJ_\omega}\delta_0\rangle\right|\right) d\mu(\omega) \\
&= a(m,0) \leqslant \liminf_{L\to\infty} a_L(m,0) \leqslant \liminf_{L\to\infty} \rho_L(m,0) \\
&\leqslant \liminf_{L\to\infty} \frac{\sqrt{a_0 a_{m-1}}}{a_{-L}a_{L-1}} \\
&\quad\times \int_{\Sigma_0} \left\langle T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}, U S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\rangle_{L^2(\mathbb{R},dx_1)} dE \\
&\leqslant \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}\right\|_2 \left\|U S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\|_2 dE \\
&= \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}\right\|_2 \left\|S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\|_2 dE \\
&\leqslant \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1}\right\|_{2,2} \left\|S^{(m)}_{E;m}\right\|_{1,2} \left\|S^{(m+1)}_{E;m+1}\right\|_{1,1}\cdots \left\|S^{(L-1)}_{E;L-1}\right\|_{1,1} \left\|\phi^{(L-1)}_{L;E;L}\right\|_1 \\
&\quad\times \left\|S^{(0)}_{E;0}\right\|_{1,2} \left\|S^{(1)}_{E;1}\right\|_{1,1}\cdots \left\|S^{(-L+1)}_{E;-L+1}\right\|_{1,1} \left\|\phi^{(-L)}_{-L;E;-L}\right\|_1 dE \\
&\leqslant \|a\|_\infty\cdot\delta^{-4}\liminf_{L\to\infty}\int_{\Sigma_0} q^{\frac{m-2}{2}}\sqrt{a_{m-1}\cdot a_{-1}}\,\|r\|_\infty dE \leqslant \|a\|^2_\infty\cdot\delta^{-4}\|r\|_\infty Leb(\Sigma_0) q^{\frac{m-2}{2}} = C\cdot e^{-\gamma|m|},
\end{aligned}$$

where $C = \|a\|^2_\infty\cdot\delta^{-4}\|r\|_\infty Leb(\Sigma_0)q^{-1}$, and $\gamma = \frac{1}{2}\log\left(q^{-1}\right)$. □

4.2. Proof of Theorem 2.6

Proof. With the same notation as in the statement of the theorem, we have

$$\begin{aligned}
&\int_\Omega \left(\sup_{t\in\mathbb{R}} \left|\langle \delta_m, e^{-itJ_\omega}\delta_0\rangle\right|\right) d\mu(\omega) \\
&= a(m,0) \leqslant \liminf_{L\to\infty} a_L(m,0) \leqslant \liminf_{L\to\infty} \rho_L(m,0) \\
&\leqslant \liminf_{L\to\infty} \frac{\sqrt{a_0 a_{m-1}}}{a_{-L}a_{L-1}} \\
&\quad\times \int_{\Sigma_0} \left\langle T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}, U S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\rangle_{L^2(\mathbb{R},dx_1)} dE \\
&\leqslant \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}\right\|_2 \left\|U S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\|_2 dE \\
&= \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1} S^{(m)}_{E;m}\cdots S^{(L-1)}_{E;L-1}\phi^{(L-1)}_{L;E;L}\right\|_2 \left\|S^{(0)}_{E;0}\cdots S^{(-L+1)}_{E;-L+1}\phi^{(-L)}_{-L;E;-L}\right\|_2 dE \\
&\leqslant \|a\|_\infty\cdot\delta^{-2}\liminf_{L\to\infty}\int_{\Sigma_0} \left\|T^{(1)}_{E;1}\cdots T^{(m-1)}_{E;m-1}\right\|_{2,2} \left\|S^{(m)}_{E;m}\right\|_{1,2} \left\|S^{(m+1)}_{E;m+1}\right\|_{1,1}\cdots \left\|S^{(L-1)}_{E;L-1}\right\|_{1,1} \left\|\phi^{(L-1)}_{L;E;L}\right\|_1 \\
&\quad\times \left\|S^{(0)}_{E;0}\right\|_{1,2} \left\|S^{(1)}_{E;1}\right\|_{1,1}\cdots \left\|S^{(-L+1)}_{E;-L+1}\right\|_{1,1} \left\|\phi^{(-L)}_{-L;E;-L}\right\|_1 dE \\
&\leqslant \|a\|_\infty\delta^{-4}\liminf_{L\to\infty}\int_{\Sigma_0} A(1)\times\ldots\times A\left(\left\lfloor\frac{m-1}{2}\right\rfloor\right)\sqrt{d_m^{-1}a_{m-1}}\sqrt{d_0^{-1}a_{-1}}\,\|r\|_\infty dE
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sqrt{d_0^{-1}}\,\|r\|_\infty \|a\|_\infty^2 \delta^{-4} d_m^{-1/2} A(1)\times\ldots\times A(k) = \tilde{C} d_m^{-1/2} A(1)\times\ldots\times A\left(\left\lfloor \frac{m-1}{2}\right\rfloor\right)\\
&\leqslant \tilde{C} d_m^{-1/2} \exp\left(-\gamma'|k|^{1-2\zeta}\right) \leqslant \tilde{C}\times C_1 |m|^{\zeta/2} \exp\left(-\gamma'\left|\left\lfloor \frac{m-1}{2}\right\rfloor\right|^{1-2\zeta}\right)\\
&\leqslant C'|m|^{\zeta/2}\exp\left(-\gamma''|m|^{1-2\zeta}\right)
\end{aligned}
$$

where $\tilde{C} = \sqrt{d_0^{-1}}\,\|r\|_\infty \|a\|_\infty^2 \delta^{-4} Leb(\Sigma_0)$, and $C' = \tilde{C}\times C_1$.

The first two inequalities follow from Lemmas 3.1 and 3.2. The third inequality follows from Lemma 3.4. The fourth inequality follows from the fact that $a_n \in \ell^\infty$ with $a_n \geqslant \delta > 0$, and the Cauchy-Schwartz inequality. The sixth inequality follows from Lemmas 4.3, 4.4, 4.5, and 4.7. Inequality eight follows from Lemma 4.12, and the rest are simple algebraic manipulations. □

Acknowledgements. The author would like to thank David Damanik for introducing him to this problem and also for his guidance throughout this project. The author is also grateful to Jake Fillman and the anonymous referee for their comments which have improved the exposition of the paper.

REFERENCES

[1] M. AIZENMAN, S. MOLCHANOV, *Localization at large disorder and at extreme energies: an elementary derivation*, Comm. Math. Phys. **157** (1993), 245–278.

[2] V. BUCAJ, D. DAMANIK, J. FILLMAN, V. GERBUZ, T. VANDENBOOM, F. WANG, Z. ZHANG, *Localization for the one dimensional Anderson model via positivity and large deviation for the Lyapunov exponent*, `arXiv:1706.06135` (2017).

[3] V. BUCAJ, *On the Kunz-Souillard approach to localization for the discrete one dimensional generalized Anderson model*, `arXiv:1608.01379`, 24 Oct. 2016.

[4] H. L. CYCON, R. G. FROESE, W. KIRSCH, B. SIMON, *Schrödinger operators with applications to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer Study Edition (1987).

[5] D. DAMANIK, G. STOLZ, *A continuum version of the Kunz-Souillard approach to localization in one dimension*, J. Reine Angew. Math. **660** (2011), 99–130.

[6] D. DAMANIK AND J. FILLMAN, *Spectral Theory of Discrete One-Dimensional Ergodic Schrödinger Operators*, preprint.

[7] D. DAMANIK, A. GORODETSKI, *An extension of the Kunz-Souillard approach to localization in one dimension and applications to almost-periodic Schrödinger operatos*, Advances in Mathematics, vol. 297 (2016), pp. 149–173.

[8] D. DAMANIK, *A Short Course on One-Dimensional Random Schrödinger Operators*, `arXiv:1107.1094v1`, 6 Jul. 2011, preprint.

[9] F. DELYON, H. KUNZ, B. SOUILLARD, *One-Dimensional Wave Equations in Disordered Media*, J. Phys. A: Math. Gen. **16** (1983) 25–42.

[10] M. S. P. EASTHAM, *The spectral theory of periodic differential equations*, Texts in Mathematics, Edinburgh-London: Scottish Academic Press, (1973).

[11] J. FRÖHLICH, T. SPENCER, *Absence of diffusion in teh Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. **88** (1983), 151–184.

[12] W. KIRSCH, *An Invitation to Random Schrödinger Operators*, Panor. Synthéses 25, Random Schrödinger operators, 1–119, Soc. Math. France, Paris, 2008.

[13] H. KUNZ, B. SOUILLARD, *Sur le spectre des opérateurs aux différences finies aléatoires*, Commun. Math. Phys. **78**, (1980), 201–246.

[14] S. JITOMIRSKAYA, H. SCHULZ-BLADES, *Upper bounds on wavepacket spreading for random Jacobi matrices*, Commun. Math. Phys. **273** (2007), 601–618.

[15] S. JITOMIRSKAYA, H. SCHULZ-BLADES, G. STOLZ, *Delocalization in random polymer models*, Commun. Math. Phys. **233** (2003), 27–48.

[16] M. REED, B. SIMON, *Methods of modern mathematical physics. IV: Analysis of operators*, New York-San Francisco-London: Academic Press (1978).

[17] R. DEL RIO, S. JITOMIRSKAYA, Y. LAST, B. SIMON, *Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one pertubations, and localization*, J. Anal. Math. **69** (1996), 153–200.

[18] B. SIMON, *Some Jacobi Matrices with Decaying Potential and Dense Point Spectrum*, Commun. Math. Phys. **87**, (1982) 253–258.

[19] B. SIMON, *Szegö's Theorem and Its Descendants: Spectral Theory for L^2 Pertubations of Orthogonal Polynomials*, Princeton University Press, (2010).

[20] G. STOLZ, *An Introduction to the Mathematics of Anderson Localization*, Contemp. Math. **552** (2011), 71–108.

[21] G. TECHL, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Grad. Stud. Math, Vol. 99, 2009.

(Received October 16, 2017)

Valmir Bucaj
Department of Mathematics
Rice University
Houston, TX 77005, USA
e-mail: valmir.bucaj@rice.edu