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A NOTE ON REPRESENTATIONS OF COMMUTATIVE C^*–ALGEBRAS IN SEMIFINITE VON NEUMANN ALGEBRAS

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Abstract. In the current paper, we generalize the "compact operator" part of D. Voiculescu's non-commmutative Weyl-von Neumann theorem on approximately unitary equivalence of unital $*$-homomorphisms of a separable commutative C^* algebra $\mathscr{A}$ into a semifinite von Neumann algebra. A result of D. Hadwin for approximate summands of representations into a finite von Neumann factor $\mathscr{R}$ is also extended.

1. Introduction

In 1976, as a non-commutative version of the Weyl-von Neumann theorem [2, 11, 14], Voiculescu [13] characterized approximately unitary equivalence of two unital representations $\phi, \psi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, where $\mathscr{A}$ is a separable unital C^*-algebra and $\mathscr{H}$ is a complex separable Hilbert space. A different beautiful proof was given by Arveson [1] in 1977. Two representations ϕ and ψ of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ are said to be *approximately* (*unitarily*) *equivalent*, denoted by $\phi \sim_a \psi$, if there exists a net $\{U_\lambda\}_{\lambda\in\Lambda}$ of unitary operators in $\mathscr{B}(\mathscr{H})$ such that

$$\lim_{\lambda\in\Lambda} \left\| U_\lambda^* \phi(A) U_\lambda - \psi(A) \right\| = 0, \ \forall A \in \mathscr{A}. \tag{1.1}$$

When $\mathscr{A}$ is separable, $\{U_\lambda\}_{\lambda\in\Lambda}$ can be chosen to be a sequence. Let $\mathscr{K}(\mathscr{H})$ denote the set of the compact operators on $\mathscr{H}$. We say that two representations ϕ and ψ of a separable C^*-algebra $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ are approximately unitarily equivalent relative to $\mathscr{K}(\mathscr{H})$, denoted by $\phi \sim_{\mathscr{A}} \psi$, $\mod \mathscr{K}(\mathscr{H})$, if there exists a sequence $\{U_n\}_{n=1}^\infty$ of unitary operators in $\mathscr{B}(\mathscr{H})$ satisfying (1.1) and

$$U_n^* \phi(A) U_n - \psi(A) \in \mathscr{K}(\mathscr{H})$$

for all $n \geqslant 1$ and every $A \in \mathscr{A}$. If $\mathscr{A}$ is a non-unital C^*-algebra and $\sigma : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is a $*$-homomorphism, then let $\mathscr{H}_1 = \cap\{\ker \sigma(A) : A \in \mathscr{A}\}$. It follows the equality

$$\sigma = \mathbf{0} \oplus \sigma_1$$

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relative to the direct sum $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_1^{\perp}$. Thus σ_1 is said to be the *nonzero part* of σ.

The following is the theorem that Voiculescu proved in [13].

THEOREM 1.1. *Suppose $\mathscr{A}$ is a separable unital C*-algebra, $\mathscr{H}$ is a separable Hilbert space and $\phi, \psi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ are unital $*$-homomorphisms. The following are equivalent:*

1. $\phi \sim_a \psi$.
2. $\phi \sim_{\mathscr{A}} \psi \mod \mathscr{K}(\mathscr{H})$.
3. $\ker\phi = \ker\psi$, $\phi^{-1}(\mathscr{K}(\mathscr{H})) = \psi^{-1}(\mathscr{K}(\mathscr{H}))$, *and the nonzero parts of the restrictions* $\phi|_{\phi^{-1}(\mathscr{K}(\mathscr{H}))}$ *and* $\psi|_{\psi^{-1}(\mathscr{K}(\mathscr{H}))}$ *are unitarily equivalent.*

In [7], the first author gave a different characterization of approximate equivalence. For $T \in \mathscr{B}(\mathscr{H})$, we let $\operatorname{rank}(T)$ denote the Hilbert-space dimension of the closure of the range $\operatorname{ran}(T)$ of T.

In the same paper, the first author (Lemma 2.3 of [7]) proved an analogue for *approximate summands* as follows.

THEOREM 1.2. *Suppose $\mathscr{A}$ is a separable unital C*-algebra, $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces, and $\phi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, $\psi : \mathscr{A} \to \mathscr{B}(\mathscr{K})$ are unital representations. The following are equivalent:*

1. *There is a representation $\gamma : \mathscr{A} \to \mathscr{B}(\mathscr{K}_1)$ for some Hilbert space $\mathscr{K}_1$ such that*
$$\psi \oplus \gamma \sim_a \phi.$$
2. *For every $A \in \mathscr{A}$,*
$$\operatorname{rank}(\psi(A)) \leqslant \operatorname{rank}(\phi(A)).$$

In her 1994 doctoral dissertation (see also [6]), Huiru Ding extended some of these results to the case in which $\mathscr{B}(\mathscr{H})$ is replaced by a von Neumann algebra. The following are some terms adopted in this paper.

Suppose $\mathscr{R}$ is a von Neumann algebra and $T \in \mathscr{R}$. We define the $\mathscr{R}$-rank of T (denoted by $\mathscr{R}\text{-rank}(T)$) to be the *Murray-von Neumann equivalence class* of the projection onto the closure of $\operatorname{ran}(T)$. Suppose that $\mathscr{A}$ is a unital C*-algebra. Let ϕ and ψ be unital $*$-homomorphisms of $\mathscr{A}$ into $\mathscr{R}$. Then, the homomorphisms ϕ and ψ are said to be *approximately equivalent in $\mathscr{R}$*, denoted by $\phi \sim_a \psi$ in $\mathscr{R}$, if there is a net $\{U_\lambda\}_{\lambda\in\Lambda}$ of unitary operators in $\mathscr{R}$ such that, for every $A \in \mathscr{A}$,
$$\lim_{\lambda\in\Lambda} \left\| U_\lambda^* \phi(A) U_\lambda - \psi(A) \right\| = 0.$$

THEOREM 1.3. (Corollary 3 of [6]) *Suppose that $\mathscr{A}$ is a unital C*-algebra which is a direct limit of finite direct sums of commutative C*-algebras tensored with matrix algebras. Let ϕ and ψ be unital $*$-homomorphisms of $\mathscr{A}$ into $\mathscr{R}$, a von Neumann algebra acting on a separable Hilbert space, then the following are equivalent:*

1. *$\phi \sim_a \psi$ in $\mathscr{R}$.*
2. *For every $A \in \mathscr{A}$,*

$$\mathscr{R}\text{-rank}\,(\phi\,(A)) = \mathscr{R}\text{-rank}\,(\psi\,(A))\,.$$

In the setting of von Neumann algebras, the compact ideal $\mathscr{K}(\mathscr{H})$ of $\mathscr{B}(\mathscr{H})$ can be extended in the following way.

In the current paper, we let $\mathscr{R}$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight τ. Let

$$\begin{aligned} \mathscr{P}\mathscr{F}(\mathscr{R},\tau) &= \{P \;:\; P = P^* = P^2 \in \mathscr{R} \text{ and } \tau(P) < \infty\},\\ \mathscr{F}(\mathscr{R},\tau) &= \{XPY \;:\; P \in \mathscr{P}\mathscr{F}(\mathscr{R},\tau) \text{ and } X,Y \in \mathscr{R}\},\\ \mathscr{K}(\mathscr{R},\tau) &= \|\cdot\|\text{-norm closure of } \mathscr{F}(\mathscr{R},\tau) \text{ in } \mathscr{R}, \end{aligned} \tag{1.2}$$

be the sets of finite rank projections, finite rank operators, and compact operators in $(\mathscr{R},\tau)$, respectively.

For a von Neumann algebra $\mathscr{R}$, denoted by $\mathscr{K}(\mathscr{R})$ the $\|\cdot\|$-norm closed ideal generated by finite projections in $\mathscr{R}$. In general, $\mathscr{K}(\mathscr{R},\tau)$ is a subset of $\mathscr{K}(\mathscr{R})$. That is because a finite projection might not be a finite rank projection with respect to τ. However, if $\mathscr{R}$ is a countably decomposable semifinite factor, then Proposition 8.5.2 of [9] entails that

$$\mathscr{K}(\mathscr{R},\tau) = \mathscr{K}(\mathscr{R})$$

for a faithful, normal, semifinite tracial weight τ.

To extend the definition of approximate equivalence of two unital $*$-homomorphisms of a separable C*-algebra $\mathscr{A}$ into $\mathscr{R}$ (*relative to* $\mathscr{K}(\mathscr{R},\tau)$), we need to develop the following notation and definitions.

Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the set of bounded linear operators on $\mathscr{H}$. Suppose that $\{E_{i,j}\}_{i,j=1}^{\infty}$ is a system of matrix units of $\mathscr{B}(\mathscr{H})$.

For a countably decomposable, properly infinite von Neumann algebra $\mathscr{R}$ with a faithful normal semifinite tracial weight τ, there exists a sequence $\{V_i\}_{i=1}^{\infty}$ of partial isometries in $\mathscr{R}$ such that

$$V_iV_i^* = I_{\mathscr{R}}, \qquad \sum_{i=1}^{\infty} V_i^*V_i = I_{\mathscr{R}}, \qquad \text{and } V_jV_i^* = 0 \text{ when } i \neq j.$$

Let $\mathscr{R} \otimes \mathscr{B}(\mathscr{H})$ be a von Neumann algebra tensor product of $\mathscr{R}$ and $\mathscr{B}(\mathscr{H})$.

DEFINITION 1.4. For all $X \in \mathscr{R}$ and all $\sum_{i,j=1}^{\infty} X_{i,j} \otimes E_{i,j} \in \mathscr{R} \otimes \mathscr{B}(\mathscr{H})$, define

$$\phi : \mathscr{R} \to \mathscr{R} \otimes \mathscr{B}(\mathscr{H}) \quad \text{and} \quad \psi : \mathscr{R} \otimes \mathscr{B}(\mathscr{H}) \to \mathscr{R}$$

by

$$\phi(X) = \sum_{i,j=1}^{\infty} (V_iXV_j^*) \otimes E_{i,j} \quad \text{and} \quad \psi(\sum_{i,j=1}^{\infty} X_{i,j} \otimes E_{i,j}) = \sum_{i,j=1}^{\infty} V_i^*X_{i,j}V_j.$$

By Lemma 2.2.2 of [10], both ϕ and ψ are normal $*$-homomorphisms satisfying

$$\psi\circ\phi = id_{\mathscr{R}} \qquad \text{and} \qquad \phi\circ\psi = id_{\mathscr{R}\otimes\mathscr{B}(\mathscr{H})}.$$

DEFINITION 1.5. Define a mapping $\tilde{\tau}:(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}))^{+}\to[0,\infty]$ to be

$$\tilde{\tau}(y)=\tau(\psi(y)), \qquad \forall\, y\in(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}))^{+}.$$

By the above Definition, the following are proved in Lemma 2.2.4 of [10]:

(i) $\tilde{\tau}$ is a faithful, normal, semifinite tracial weight of $\mathscr{R}\otimes\mathscr{B}(\mathscr{H})$.

(ii) $\tilde{\tau}(\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j}) = \sum_{i=1}^{\infty}\tau(X_{i,i})$ for all $\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j}\in(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}))^{+}$.

(iii)

$$\begin{aligned}
\mathscr{P}\mathscr{F}(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}),\tilde{\tau}) &= \phi(\mathscr{P}\mathscr{F}(\mathscr{R},\tau)),\\
\mathscr{F}(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}),\tilde{\tau}) &= \phi(\mathscr{F}(\mathscr{R},\tau)),\\
\mathscr{K}(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}),\tilde{\tau}) &= \phi(\mathscr{K}(\mathscr{R},\tau)).
\end{aligned}$$

REMARK 1.6. It shows that $\tilde{\tau}$ is a natural extension of τ from $\mathscr{R}$ to $\mathscr{R}\otimes\mathscr{B}(\mathscr{H})$. If no confusion arises, $\tilde{\tau}$ will be also denoted by τ. By Proposition 2.2.9 of [10], the ideal $\mathscr{K}(\mathscr{R}\otimes\mathscr{B}(\mathscr{H}),\tilde{\tau})$ is independent of the choice of the system of matrix units $\{E_{i,j}\}_{i,j=1}^{\infty}$ of $\mathscr{B}(\mathscr{H})$ and the choice of the family $\{V_i\}_{i=1}^{\infty}$ of partial isometries in $\mathscr{R}$.

Now we are ready to introduce the definition of approximate equivalence of $*$-homomorphisms of a separable C*-algebra into $\mathscr{R}$ relative to $\mathscr{K}(\mathscr{R},\tau)$.

Let $\mathscr{A}$ be a separable C*-subalgebra of $\mathscr{R}$ with an identity $I_{\mathscr{A}}$. Suppose that ψ is a positive mapping from $\mathscr{A}$ into $\mathscr{R}$ such that $\psi(I_{\mathscr{A}})$ is a projection in $\mathscr{R}$. Then for all $0\leqslant X\in\mathscr{A}$, we have $0\leqslant\psi(X)\leqslant\|X\|\psi(I_{\mathscr{A}})$. Therefore, it follows that

$$\psi(X)\psi(I_{\mathscr{A}})=\psi(I_{\mathscr{A}})\psi(X)=\psi(X)$$

for all positive $X\in\mathscr{A}$. In other words, $\psi(I_{\mathscr{A}})$ can be viewed as an identity of $\psi(\mathscr{A})$. Or, $\psi(\mathscr{A})\subseteq\psi(I_{\mathscr{A}})\mathscr{R}\psi(I_{\mathscr{A}})$.

DEFINITION 1.7. (Definition 2.3.1 of [10]) Suppose $\{E_{i,j}\}_{i,j\geqslant 1}$ is a system of matrix units of $\mathscr{B}(\mathscr{H})$. Let $M,N\in\mathbb{N}\cup\{\infty\}$. Suppose that $\psi_1,\dots,\psi_M$ and $\phi_1,\dots,\phi_N$ are positive mappings from $\mathscr{A}$ into $\mathscr{R}$ such that $\psi_1(I_{\mathscr{A}}),\dots,\psi_M(I_{\mathscr{A}})$, $\phi_1(I_{\mathscr{A}}),\dots,\phi_N(I_{\mathscr{A}})$ are projections in $\mathscr{R}$.

(a) Let $\mathscr{F}\subseteq\mathscr{A}$ be a finite subset and $\varepsilon>0$. Say $\psi_1\oplus\cdots\oplus\psi_M$ is $(\mathscr{F},\varepsilon)$-*strongly-approximately-unitarily-equivalent* to $\phi_1\oplus\cdots\oplus\phi_N$ over $\mathscr{A}$, denoted by

$$\psi_1\oplus\psi_2\oplus\cdots\oplus\psi_M\sim_{\mathscr{A}}^{(\mathscr{F},\varepsilon)}\phi_1\oplus\phi_2\oplus\cdots\oplus\phi_N, \qquad \bmod \mathscr{K}(\mathscr{R},\tau)$$

if there exists a partial isometry V in $\mathscr{R}\otimes\mathscr{B}(\mathscr{H})$ such that

(i) $V^*V = \sum_{i=1}^{M} \psi_i(I_{\mathscr{A}}) \otimes E_{i,i}$ and $VV^* = \sum_{i=1}^{N} \phi_i(I_{\mathscr{A}}) \otimes E_{i,i}$;

(ii) $\sum_{i=1}^{M} \psi_i(X) \otimes E_{i,i} - V^* \left(\sum_{i=1}^{N} \phi_i(X) \otimes E_{i,i} \right) V \in \mathscr{K}(\mathscr{R} \otimes \mathscr{B}(\mathscr{H}), \tau)$ for all $X \in \mathscr{A}$;

(iii) $\| \sum_{i=1}^{M} \psi_i(X) \otimes E_{i,i} - V^* \left(\sum_{i=1}^{N} \phi_i(X) \otimes E_{i,i} \right) V \| < \varepsilon$ for all $X \in \mathscr{F}$.

(b) Say $\psi_1 \oplus \cdots \oplus \psi_M$ is *strongly-approximately-unitarily-equivalent* to $\phi_1 \oplus \cdots \oplus \phi_N$ over $\mathscr{A}$, denoted by

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathscr{A}} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N, \qquad \mod \mathscr{K}(\mathscr{R}, \tau)$$

if, for any finite subset $\mathscr{F} \subseteq \mathscr{A}$ and $\varepsilon > 0$,

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathscr{A}}^{(\mathscr{F},\varepsilon)} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N, \qquad \mod \mathscr{K}(\mathscr{R}, \tau).$$

In this paper we address the question of approximate summands and "compact" operators for semifinite von Neumann algebras $\mathscr{R}$ and commutative separable C*-algebras $\mathscr{A}$. In Section 2, relative to finite von Neumann algebras, we characterize the approximate summands of $*$-homomorphisms by virtue of a natural condition. Precisely, we prove the following theorem.

THEOREM 2.2. *Suppose $\mathscr{A}$ is a separable unital commutative C*-algebra and $\mathscr{R}$ is a finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. Suppose P is a projection in $\mathscr{R}$, $\pi : \mathscr{A} \to \mathscr{R}$ is a unital $*$-homomorphism and $\rho : \mathscr{A} \to P\mathscr{R}P$ is a unital $*$-homomorphism such that, for every $X \in \mathscr{A}$, we have*

$$\mathscr{R}\text{-rank}\,(\rho\,(X)) \leqslant \mathscr{R}\text{-rank}\,(\pi\,(X)).$$

Then there is a unital $*$-homomorphism $\gamma : \mathscr{A} \to P^{\perp}\mathscr{R}P^{\perp}$ such that

$$\gamma \oplus \rho \sim_a \pi \ \text{ in } \mathscr{R}.$$

In Section 3, for two $*$-homomorphisms ϕ and ψ of a commutative C^*-algebra into a semifinite von Neumann factor $\mathscr{R}$ with a faithful normal semifinite tracial weight τ, the main theorem states that the approximately unitary equivalence of ϕ and ψ implies that these two $*$-homomorphisms are strongly-approximately-unitarily-equivalent over $\mathscr{A}$ (defined as in Definition 1.7). Precisely, we obtian the following theorem.

THEOREM 3.3. *Let X be a compact metric space. Suppose that ϕ and ψ are two unital $*$-homomorphisms of $C(X)$ into a countably decomposable, properly infinite, semifinite factor $\mathscr{R}$ with a faithful normal semifinite tracial weight τ acting on a separable Hilbert space $\mathscr{H}$. Then the following are equivalent:*

1. *$\phi \sim_a \psi$ in $\mathscr{R}$,*

2. *$\phi \sim_{C(X)} \psi$,* $\mod \mathscr{K}(\mathscr{R}, \tau)$.

2. Representations relative to finite von Neumann algebras

THEOREM 2.1. *Suppose $\mathscr{A}$ is a separable unital commutative C*-algebra and $\mathscr{R}$ is a type II_1 factor with a faithful normal normalized trace τ, acting on a separable Hilbert space $\mathscr{H}$. Suppose P is a projection in $\mathscr{R}$, $\pi : \mathscr{A} \to \mathscr{R}$ is a unital $*$-homomorphism and $\rho : \mathscr{A} \to P\mathscr{R}P$ is a unital $*$-homomorphism such that, for every $X \in \mathscr{A}$, we have*

$$\mathscr{R}\text{-rank}\left(\rho\left(X\right)\right) \leqslant \mathscr{R}\text{-rank}\left(\pi\left(X\right)\right).$$

Then there is a unital $*$-homomorphism $\gamma : \mathscr{A} \to P^{\perp}\mathscr{R}P^{\perp}$ such that

$$\gamma \oplus \rho \sim_a \pi \ \text{ in } \mathscr{R}.$$

Proof. It follows from Lemma 2.2 of [12] that π and ρ can be extended to normal unital $*$-homomorphisms with domain, the second dual $\mathscr{A}^{\#\#}$ of $\mathscr{A}$, so that

$$\mathscr{R}\text{-rank}(\rho(X)) \leqslant \mathscr{R}\text{-rank}(\pi(X))$$

holds for all $X \in \mathscr{A}^{\#\#}$. Since $\mathscr{A}$ is separable, we can choose a countable family $\{Q_1, Q_2, \ldots\}$ of projections in $\mathscr{A}^{\#\#}$ such that

$$\mathscr{A} \subseteq C^*\left(Q_1, Q_2, \ldots\right).$$

However, if we let $A = \sum_{k=1}^{\infty} 3^{-k} Q_k$, then $C^*\left(A\right) = C^*\left(Q_1, Q_2, \ldots\right)$. It is also true that, for every $X \in C^*\left(A\right)$,

$$\mathscr{R}\text{-rank}\left(\rho\left(X\right)\right) \leqslant \mathscr{R}\text{-rank}\left(\pi\left(X\right)\right).$$

It is easily seen that if we prove the theorem for the restrictions of π and ρ to $C^*\left(A\right)$, we will have proved the theorem for π and ρ on $\mathscr{A}$. Hence, we can assume that $\mathscr{A} = C^*\left(A\right)$ and $0 \leqslant A \leqslant 1$.

Let $S = \rho\left(A\right) \in P\mathscr{R}P$ and $T = \pi\left(A\right) \in \mathscr{R}$. Thus the following inequality

$$\mathscr{R}\text{-rank}\left(f\left(S\right)P\right) \leqslant \mathscr{R}\text{-rank}\left(f\left(T\right)\right)$$

holds for every $f \in C\left(\sigma\left(A\right)\right)$. This leads to the inequality

$$\tau\left(f\left(S\right)P\right) \leqslant \tau\left(f\left(T\right)\right)$$

for every $f \in C\left(\sigma\left(A\right)\right)_+$. The Riesz representation theorem implies that there exist two regular Borel measures μ_ρ and μ_π on $\sigma(A)$ such that the inequality

$$\tau\left(f\left(S\right)P\right) = \int_{\sigma(A)} f d\mu_\rho \leqslant \int_{\sigma(A)} f d\mu_\pi = \tau\left(f\left(T\right)\right)$$

holds for every $f \in C\left(\sigma\left(A\right)\right)$. It follows from Lusin's theorem that the preceding line holds for every bounded Borel measurable function $f : \sigma\left(A\right) \to \mathbb{C}$. Hence $\mu_\rho \leqslant \mu_\pi$ and, for every $z \in \sigma\left(A\right)$, we have $\tau\left(\chi_{\{z\}}\left(S\right)\right) \leqslant \tau\left(\chi_{\{z\}}\left(T\right)\right)$.

Since τ is faithful, the set L_S of $z \in \sigma(S)$ satisfying $\chi_{\{z\}}(S) \neq 0$ is countable. Hence $\sum_{z \in L_S} z\chi_{\{z\}}(S)$ is a direct summand of S and $\sum_{z \in L_S} z\chi_{\{z\}}(T)$ is a summand of T.

Since, for each $z \in L_S$, the projection $\chi_{\{z\}}(S)$ is unitarily equivalent to a subprojection of $\chi_{\{z\}}(T)$, without loss of generality, $\sum_{z \in L_S} z\chi_{\{z\}}(S)$ can be assumed to be a direct summand of T. Thus this summand can be removed from both S and T. Therefore, it can be assumed that S has no eigenvalues.

By the same way, the set $L_T = \left\{z \in \sigma(T) : \chi_{\{z\}}(T) \neq 0\right\}$ is countable. Hence $S\chi_{L_T}(S) = 0$. Therefore, for every bounded nonnegative measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\tau(f(S)P) = \tau\left(\left(\chi_{\mathbb{C}\setminus L_T} f\right)(S)P\right) \leqslant \tau\left(\chi_{\mathbb{C}\setminus L_T}(T) f(T)\right) d\mu_\pi.$$

This yields that T can be replaced with $T\left(1 - \chi_{\mathbb{C}\setminus L_T}(T)\right)$ and $\mathscr{R}$ can be replaced with

$$\left(1 - \chi_{\mathbb{C}\setminus L_T}(T)\right)\mathscr{R}\left(1 - \chi_{\mathbb{C}\setminus L_T}(T)\right).$$

Hence we can assume that $\chi_{L_T}(T) = 0$.

Similarly, since the equality

$$f(S) = \left(\chi_{\sigma(S)} f\right)(S)$$

holds for every bounded measurable function f, the operator T can be replaced with $\chi_{\sigma(S)}(T)T$. Hence we can assume that $\sigma(S) = \sigma(T) = \sigma(A)$. Thus $\mu_\rho \leqslant \mu_\pi$ are both non-atomic measures with supports satisfying $\sigma(S) = \sigma(T) = \sigma(A)$. Moreover, we have the equalities

$$\mu_\rho(\sigma(A)) = \tau(P) \qquad \text{and} \qquad \mu_\pi(\sigma(A)) = 1.$$

It follows that $\nu = \mu_\pi - \mu_\rho$ is a nonatomic measure and $\nu(\sigma(A)) = 1 - \tau(P)$. Thus there is a unital weak*-continuous $*$-isomorphism $\Delta_S : L^\infty[0, \tau(P)] \to L^\infty(\mu_\rho)$ such that for every $f \in L^\infty[0, \tau(P)]$,

$$\int_{\sigma(A)} \Delta_S(f)\, d\mu_\rho = \int_0^{\tau(P)} f(x)\, dx.$$

Similarly, there is an isomorphism $\Delta_\nu : L^\infty[\tau(P), 1] \to L^\infty(\nu)$ such that the equality

$$\int_{\sigma(A)} \Delta_\nu(f)\, d\nu = \int_{\tau(P)}^1 f(x)\, dx.$$

holds for every $f \in L^\infty[\tau(P), 1]$.

Moreover, we can choose a maximal chain $\mathscr{C} = \{Q_t : 0 \leqslant t \leqslant 1 - \tau(P)\}$ of projections in $P^\perp \mathscr{R} P^\perp$ with $\tau(Q_t) = t$ for $0 \leqslant t \leqslant 1 - \tau(P)$. Thus there exists a weak*-continuous unital $*$-homomorphism $\Delta_1 : L^\infty[\tau(P), 1] \to W^*(\mathscr{C})$ such that, for every $t \in [0, 1 - \tau(P)]$, we have $\Delta_1(\chi_{[\tau(P), \tau(P)+t)}) = Q_t$, and such that, for every $f \in L^\infty[\tau(P), 1]$ we have

$$\tau(\Delta_1(f)) = \int_{\tau(P)}^1 f(x)\, dx.$$

Define $\Delta : C(\sigma(A)) \to P\mathscr{R}P + P^{\perp}\mathscr{R}P^{\perp} \subset \mathscr{R}$ by

$$\Delta(h) = h(S) \oplus \left(\Delta_1 \circ \Delta_\nu^{-1}\right)(h).$$

If $z(\lambda) = \lambda$ is the identity map on $\sigma(A)$, then $\Delta(z) = S \oplus B$ and

$$\begin{aligned}
\tau(\Delta(h)) &= \tau(h(S)) + \tau(\Delta_1(\Delta_\nu^{-1}(h))) \\
&= \int_{\sigma(A)} h d\mu_\rho + \int_{\tau(P)}^{1} \Delta_\nu^{-1}(h)(x)dx \\
&= \int_{\sigma(A)} h d\mu_\rho + \int_{\sigma(A)} h d\nu = \int_{\sigma(A)} h d\mu_\pi = \tau(h(T)).
\end{aligned}$$

Hence for every $h \in C(\sigma(A))$, we have $\tau(h(S \oplus B)) = \tau(h(T))$. Define a unital $*$-homomorphism $\gamma : C(\sigma(A)) \to P^{\perp}\mathscr{R}P^{\perp}$ by

$$\gamma(h) = P^{\perp}h(B).$$

By Theorem 1.3, the above equality yields that $\rho \oplus \gamma \sim_a \pi$ in $\mathscr{R}$. This completes the proof. $\square$

THEOREM 2.2. *Suppose $\mathscr{A}$ is a separable unital commutative C*-algebra and $\mathscr{R}$ is a finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. Suppose P is a projection in $\mathscr{R}$, $\pi : \mathscr{A} \to \mathscr{R}$ is a unital $*$-homomorphism and $\rho : \mathscr{A} \to P\mathscr{R}P$ is a unital $*$-homomorphism such that, for every $X \in \mathscr{A}$, we have*

$$\mathscr{R}\text{-rank}(\rho(X)) \leqslant \mathscr{R}\text{-rank}(\pi(X)).$$

Then there is a unital $*$-homomorphism $\gamma : \mathscr{A} \to P^{\perp}\mathscr{R}P^{\perp}$ such that

$$\gamma \oplus \rho \sim_a \pi \text{ in } \mathscr{R}.$$

Proof. First, we suppose $\mathscr{R}$ is a II_1 von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. By applying the central decomposition technique of von Neumann algebras, we can then write

$$\mathscr{H} = L^2\left(\mu, \ell^2\right) = \int_{\Omega}^{\oplus} \ell^2 d\mu(\omega) \text{ and } \mathscr{R} = \int_{\Omega}^{\oplus} \mathscr{R}_\omega d\mu(\omega),$$

where (Ω, μ) is a probability space and each $\mathscr{R}_\omega$ is a II_1 factor with a unique trace τ_ω. Furthermore, a faithful normal tracial state τ on $\mathscr{R}$ can be defined in the following form

$$\tau\left(\int_{\Omega}^{\oplus} A(\omega) d\mu(\omega)\right) = \int_{\Omega} \tau_\omega(A(\omega)) d\mu(\omega).$$

Similarly, the projection $P \in \mathscr{R}$ can be written in the form

$$P = \int_{\Omega}^{\oplus} P(\omega) d\mu(\omega),$$

where $P(\omega)$ is a projection in $\mathscr{R}_\omega$ a.e. (μ). Thus $P\mathscr{R}P$ can be written in the form

$$P\mathscr{R}P = \int_\Omega^\oplus P(\omega)\mathscr{R}_\omega P(\omega)\,d\mu(\omega).$$

By Theorem 2.1, we can assume that $\mathscr{A} = C^*(A)$ and $0 \leqslant A \leqslant 1$. Thus, for the identity map $z(\lambda) = \lambda$ on $\sigma(A)$, suppose that $\pi(z) = T$ and $\rho(z) = S \in P\mathscr{R}P$. Then we can write

$$T = \int_\Omega^\oplus T(\omega)\,d\mu(\omega)$$

and

$$S = PSP = \int_\Omega^\oplus S(\omega)\,d\mu(\omega) = \int_\Omega^\oplus P(\omega)S(\omega)P(\omega)\,d\mu(\omega).$$

It follows that, for every $f \in C(\sigma(A))$,

$$\pi(f) = f(T) = \int_\Omega^\oplus f(T(\omega))\,d\mu(\omega) = \int_\Omega^\oplus \pi_\omega(f)\,d\mu(\omega).$$

If f is in $C(\sigma(A))$ and $Q_{f(T)}$ is the projection onto the closure of the range of $f(T)$, then

$$Q_{f(T)} = \int_\Omega^\oplus Q_{f(T(\omega))}d\mu(\omega).$$

Similarly, if $Q_{f(S)P}$ is the range projection of $f(S)P$, then

$$Q_{f(S)P} = \int_\Omega^\oplus Q_{f(S(\omega))P(\omega)}d\mu(\omega).$$

If $\mathscr{R}$-rank$(f(S)P) \leqslant \mathscr{R}$-rank$(f(T))$, then $Q_{f(S)P}$ is Murray-von Neumann equivalent to a subprojection of $Q_{f(T)}$. Hence, for every central projection D, we have $DQ_{f(S)P}$ is Murray-von Neumann equivalent to a subprojection of $DQ_{f(T)}$. Thus for every measurable subset $E \subset \Omega$,

$$\tau\left(\chi_E Q_{f(S)P}\right) \leqslant \tau\left(\chi_E Q_{f(T)}\right),$$

which means that

$$\int_E \tau_\omega\left(Q_{f(S(\omega))P(\omega)}\right)d\mu(\omega) \leqslant \int_E \tau_\omega\left(Q_{f(T(\omega))}\right)d\mu(\omega).$$

This yields that

$$\tau_\omega\left(Q_{f(S(\omega))P(\omega)}\right) \leqslant \tau_\omega\left(Q_{f(T(\omega))}\right) \text{ a.e. } (\mu).$$

Since $C(\sigma(A))$ is separable, we conclude that, except for a subset of Ω of measure 0, for all $f \in C(\sigma(A))$,

$$\tau_\omega\left(Q_{f(S(\omega))P(\omega)}\right) \leqslant \tau_\omega\left(Q_{f(T(\omega))}\right).$$

We can now use Theorem 2.1 and measurably choose $B(\omega) = B(\omega)^* \in P(\omega)^\perp \mathscr{R}_\omega P(\omega)^\perp$ and define

$$\gamma_\omega : C(A) \to P(\omega)^\perp \mathscr{R}_\omega P(\omega)^\perp \quad \text{by} \quad \gamma_\omega(f) = f(B(\omega))P(\omega)^\perp$$

so that

$$\pi_\omega \sim_a \rho_\omega \oplus \gamma_\omega \text{ in } \mathscr{R}_\omega.$$

It easily follows that if we define $\gamma(f) = \int_\Omega^\oplus \gamma_\omega(f)\, d\mu(\omega)$, then $\pi \sim_a \rho \oplus \gamma$ in $\mathscr{R}$. This completes the proof. $\square$

3. Representations relative to semifinite infinite von Neumann algebras

As shown in the proof of Theorem 2.1, it is sufficient to replace a separable commutative C*-algebra with some certain $C(X)$ on a compact metric space X.

In the rest of this section, we assume that $\mathscr{R}$ is a countably decomposable, properly infinite, semifinite von Neumann factor with a faithful, normal, semifinite tracial weight τ. For an operator $T \in \mathscr{R}$, denote by $R(T)$ the range projection onto the closure of the range of T. The following two lemmas are useful in the sequel.

LEMMA 3.1. *For an operator A in $\mathscr{R}$, the following are equivalent:*

1. *A is in $\mathscr{K}(\mathscr{R},\tau)$;*
2. *$|A|$ is in $\mathscr{K}(\mathscr{R},\tau)$;*
3. *for every $\varepsilon > 0$, $\tau(\chi_{[0,\varepsilon)}(|A|)) = \infty$ and $\tau(\chi_{[\varepsilon,\infty)}(|A|)) < \infty$;*
4. *for every $\varepsilon > 0$, $\tau(\chi_{[0,\varepsilon]}(|A|)) = \infty$ and $\tau(\chi_{(\varepsilon,\infty)}(|A|)) < \infty$.*

Proof. For an operator A in $\mathscr{R}$, Let $A = V|A|$ be the polar decomposition of A. If A is in $\mathscr{K}(\mathscr{R},\tau)$, then so is $|A| = V^*A$. On the other hand, if $|A|$ is in $\mathscr{K}(\mathscr{R},\tau)$, then so is $A = V|A|$. That $(2) \Leftrightarrow (3)$ is equivalent to $(2) \Leftrightarrow (4)$. Thus, we only need to prove $(2) \Leftrightarrow (3)$. Suppose that $|A|$ belongs to $\mathscr{K}(\mathscr{R},\tau)$ and π is the canonical $*$-homomorphism of $\mathscr{R}$ onto $\mathscr{R}/\mathscr{K}(\mathscr{R},\tau)$. If $\tau(\chi_{[0,\varepsilon)}(|A|)) < \infty$, then $\pi(\chi_{[0,\varepsilon)}(|A|)) = \pi(\chi_{[0,\varepsilon)}(|A|)|A|) = 0$. It follows that

$$\pi(|A|) = \pi(\chi_{[0,\varepsilon)}(|A|) + \chi_{[\varepsilon,\infty)}(|A|)|A|).$$

Note that $\chi_{[0,\varepsilon)}(|A|) + \chi_{[\varepsilon,\infty)}(|A|)|A|$ is invertible in $\mathscr{R}$, so $\pi(|A|)$ is invertible in $\mathscr{R}/\mathscr{K}(\mathscr{R},\tau)$. This is a contradiction. By a similar method, if $|A|$ belongs to $\mathscr{K}(\mathscr{R},\tau)$, then

$$\chi_{[\varepsilon,\infty)}(|A|) = \chi_{[\varepsilon,\infty)}(|A|)|A||A|^{-1}\chi_{[\varepsilon,\infty)}(|A|) \in \mathscr{K}(\mathscr{R},\tau).$$

If $\chi_{[\varepsilon,\infty)}(|A|) \in \mathscr{K}(\mathscr{R},\tau)$, then, for every $n \in \mathbb{N}$, there exists a positive operator A_n such that

(i) $\tau(R(A_n)) < \infty$;

(ii) $0 \leqslant A_n \leqslant \chi_{[\varepsilon,\infty)}(|A|)$;

(iii) $\|\chi_{[\varepsilon,\infty)}(|A|) - A_n\| < 1/n$.

It is easy to obtain that $0 \leqslant A_n \leqslant R(A_n) \leqslant \chi_{[\varepsilon,\infty)}(|A|)$. Thus

$$\|\chi_{[\varepsilon,\infty)}(|A|) - R(A_n)\| < 1/n.$$

A routine calculation shows that $\chi_{[\varepsilon,\infty)}(|A|)$ is unitarily equivalent to $R(A_n)$. Therefore, we obtain that $\tau(\chi_{[\varepsilon,\infty)}(|A|)) < \infty$ holds for every $\varepsilon > 0$. By the definition of $\mathscr{K}(\mathscr{R},\tau)$, we can prove $(3) \Rightarrow (2)$. This completes the proof. $\square$

LEMMA 3.2. *Let X be a compact metric space. Suppose that ϕ and ψ are two unital $*$-homomorphisms of $C(X)$ into a countably decomposable, properly infinite, semifinite factor $\mathscr{R}$ with a faithful normal semifinite tracial weight τ acting on a separable Hilbert space $\mathscr{H}$. If $\phi \sim_a \psi$ in $\mathscr{R}$, then, for f in $C(X)$,*

$$\phi(f) \in \mathscr{K}(\mathscr{R},\tau) \quad \Leftrightarrow \quad \psi(f) \in \mathscr{K}(\mathscr{R},\tau).$$

Proof. First, we need to extend ϕ and ψ to $\hat{\phi}$ and $\hat{\psi}$ as two normal unital $*$-homomorphisms of $C(X)^{\#\#}$ into $\mathscr{R}$, respectively. Given any open subset Δ of X, there exists a continuous function f such that

$$\begin{cases} 0 < f(x) \leqslant 1, & \text{if } x \in \Delta, \\ f(x) = 0, & \text{if } x \notin \Delta. \end{cases}$$

Thus, the increasing sequence $\{f^{1/n}\}_{n\in\mathbb{N}}$ converges pointwise to χ_Δ. Furthermore, if $\{f^{1/n}\}_{n\in\mathbb{N}}$ are viewed as elements in $C(X)^{\#\#}$, then $f^{1/n}$ converges to χ_Δ in the weak* topology. Since $\phi(f^{1/n}) = \phi(f)^{1/n}$ and $\{\phi(f)^{1/n}\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of positive operators in $\mathscr{R}$ with the upper bound $I_{\mathscr{R}}$. By applying Lemma 5.1.5 of [8], $\phi(f)^{1/n}$ converges to the projection $R(\phi(f))$ in the strong operator topology. Therefore, ϕ can be extended to a unital normal $*$-homomorphism $\hat{\phi}$ of $\mathscr{B}(X)$, the $*$-subalgebra of all the bounded Borel functions on X, into $\mathscr{R}$ unambiguously such that $\hat{\phi}(\chi_\Delta) = R(\phi(f))$. For details, the reader is referred to Theorem 5.2.6 and Theorem 5.2.8 of [8].

By applying Lemma 3.1, it is sufficient to suppose that $\phi(f)$ is a positive element in $\mathscr{K}(\mathscr{R},\tau)$. Thus, for every $\varepsilon > 0$, we have $\tau(\chi_{(\varepsilon,\infty)}(\phi(f))) < \infty$. Note that there exists a continuous function h defined as

$$h(x) = \begin{cases} \dfrac{x-\varepsilon}{x}, & \varepsilon < x, \\ 0, & 0 \leqslant x \leqslant \varepsilon \end{cases}$$

such that

$$\chi_{(\varepsilon,\infty)}(\phi(f)) = \chi_{(\varepsilon,\infty)}(\hat{\phi}(f)) = \hat{\phi}(\chi_{(\varepsilon,\infty)}(f)) = \hat{\phi}(\chi_{(0,\infty)}(h\circ f)) = R(\phi(h\circ f)).$$

The same equality holds for ψ and $\hat{\psi}$. By applying Theorem 1.3, the relation $\phi \sim_a \psi$ in $\mathscr{R}$ yields the following equality

$$\tau(\chi_{(\varepsilon,\infty)}(\phi(f))) = \tau(R(\phi(h\circ f))) = \tau(R(\psi(h\circ f))) = \tau(\chi_{(\varepsilon,\infty)}(\psi(f))).$$

A similar argument ensures that

$$\tau(\chi_{[0,\varepsilon)}(\phi(f))) = \tau(\chi_{[0,\varepsilon)}(\psi(f))).$$

Therefore, $\phi(f)$ in $\mathscr{K}(\mathscr{R},\tau)$ implies $\psi(f)$ in $\mathscr{K}(\mathscr{R},\tau)$, and vice versa. $\square$

Suppose that ϕ and ψ as assumed are two unital $*$-homomorphisms of $C(X)$ into $\mathscr{R}$. Then, by Definition 1.7, the relation $\phi \sim_{C(X)} \psi, \mod \mathscr{K}(\mathscr{R},\tau)$ implies that $\phi \sim_a \psi$ in $\mathscr{R}$. In the rest of this section, we aim to prove the converse of this.

THEOREM 3.3. *Let X be a compact metric space. Suppose that ϕ and ψ are two unital $*$-homomorphisms of $C(X)$ into a countably decomposable, properly infinite, semifinite factor $\mathscr{R}$ with a faithful normal semifinite tracial weight τ acting on a separable Hilbert space $\mathscr{H}$. Then the following are equivalent:*

1. *$\phi \sim_a \psi$ in $\mathscr{R}$,*
2. *$\phi \sim_{C(X)} \psi, \mod \mathscr{K}(\mathscr{R},\tau)$.*

Proof. Assume that ϕ and ψ are approximately unitarily equivalent relative to $\mathscr{R}$. By applying Theorem 1.3, for every f in $C(X)$, the equality

$$\mathscr{R}\text{-rank}(\phi(f)) = \mathscr{R}\text{-rank}(\psi(f))$$

holds and yields that $\tau(R(\phi(f))) = \tau(R(\psi(f)))$. Thus, the equality $\ker\phi = \ker\psi$ holds. This ensures that $\psi\circ\phi^{-1}$ is a well-defined unital $*$-isomorphism of $\phi(C(X))$ onto $\psi(C(X))$ and we denote this isomorphism by ρ. That is, for every A in $\phi(C(X))$ and every f in $C(X)$,

$$\rho(A) = \psi\circ\phi^{-1}(A), \quad \rho(\phi(f)) = \psi(f).$$

Therefore, the following two statements are equivalent

1. $\phi \sim_{C(X)} \psi, \mod \mathscr{K}(\mathscr{R},\tau)$;
2. $\mathrm{id} \sim_{\phi(\mathrm{C(X)})} \rho, \mod \mathscr{K}(\mathscr{R},\tau)$, where id stands for the identity mapping.

In the following, we need to partition X into two parts in order to reduce the proof into two special cases. Then we assemble them to complete the proof.

By a routine computation, it is easy to verify that the set

$$\mathscr{I} = \{f \in C(X) : \phi(f) \in \mathscr{K}(\mathscr{R},\tau)\}$$

is a closed ideal in $C(X)$. Note that, by Lemma 3.2, the equality

$$\phi(C(X)) \cap \mathscr{K}(\mathscr{R},\tau) = \psi(C(X)) \cap \mathscr{K}(\mathscr{R},\tau)$$

holds. This implies that the equality $\mathscr{I} = \{f \in C(X) : \psi(f) \in \mathscr{K}(\mathscr{R},\tau)\}$ also holds.

By applying Theorem 3.4.1 of [8], there exists a closed subset F of the compact metric space X such that

$$\mathscr{I} = \{f \in C(X) : f(x) = 0, \forall x \in F\}.$$

As shown in Lemma 3.2, we denote by $\hat{\phi}$ and $\hat{\psi}$ the normal extensions of $\mathscr{B}(X)$ into $\mathscr{R}$ induced by ϕ and ψ, respectively. Note that, for every f in $C(X)$, the projections $\hat{\phi}(\chi_F)$ and $\hat{\psi}(\chi_F)$ reduce $\phi(f)$ and $\psi(f)$, respectively.

To deal with one of the two special cases mentioned above, we adopt the classical method initiated by Voiculescu. That is, for every $A \in \phi(C(X))$ and $B \in \psi(C(X))$, we can define representations ρ_e and ρ_e' as follows

$$\rho_e(A) \triangleq \psi \circ \phi^{-1}(A)|_{\mathrm{ran}\ \hat{\psi}(\chi_F)}, \quad \rho_e'(B) \triangleq \phi \circ \psi^{-1}(B)|_{\mathrm{ran}\ \hat{\phi}(\chi_F)}.$$

Note that

$$\rho_e(\phi(C(X)) \cap \mathscr{K}(\mathscr{R}, \tau)) = \rho_e'(\psi(C(X)) \cap \mathscr{K}(\mathscr{R}, \tau)) = 0.$$

By applying Theorem 5.3.1 of [10], we have

$$\mathrm{id}_{\phi(C(X))} \sim_{\phi(C(X))} \mathrm{id}_{\phi(C(X))} \oplus \rho_e, \qquad \mod \mathscr{K}(\mathscr{R}, \tau),$$

$$\mathrm{id}_{\psi(C(X))} \sim_{\psi(C(X))} \mathrm{id}_{\psi(C(X))} \oplus \rho_e', \qquad \mod \mathscr{K}(\mathscr{R}, \tau).$$

Therefore, for every $f \in C(X)$, it follows that

$$\phi(f) \sim_{C(X)} \phi(f) \oplus (\psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_F)}), \quad \mod \mathscr{K}(\mathscr{R}, \tau) \tag{3.1}$$

and

$$\psi(f) \sim_{C(X)} \psi(f) \oplus (\phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_F)}), \quad \mod \mathscr{K}(\mathscr{R}, \tau). \tag{3.2}$$

Note that, for every $f \in C(X)$, the equalities

$$\phi(f) \oplus (\psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_F)}) = (\phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_{(X-F)})}) \oplus (\phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_F)}) \oplus (\psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_F)}) \tag{3.3}$$

and

$$\psi(f) \oplus (\phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_F)}) = (\psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_{(X-F)})}) \oplus (\psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_F)}) \oplus (\phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_F)}) \tag{3.4}$$

hold. Thus, the above relations from (3.1) to (3.4) imply that, to prove that

$$\phi \sim_{C(X)} \psi, \quad \mod \mathscr{K}(\mathscr{R}, \tau),$$

it is sufficient to prove that

$$\phi|_{\mathrm{ran}\ \hat{\phi}(\chi_{(X-F)})} \sim_{C(X)} \psi|_{\mathrm{ran}\ \hat{\psi}(\chi_{(X-F)})}, \quad \mod \mathscr{K}(\mathscr{R}, \tau).$$

And this is the other special case.

For every f in $C(X)$, write

$$\phi_0(f) \triangleq \phi(f)|_{\mathrm{ran}\ \hat{\phi}(\chi_{(X-F)})} \quad \text{and} \quad \psi_0(f) \triangleq \psi(f)|_{\mathrm{ran}\ \hat{\psi}(\chi_{(X-F)})}.$$

Since X is a compact metric space and F is a closed subset of X, we can construct a continuous function h such that

$$h(x) = \mathrm{dist}(x,F), \quad \forall x \in X,$$

where $\mathrm{dist}(x,F)$ is the distance between x and F. This construction of h ensures that $\phi(h)$ is bounded and belongs to $\mathscr{K}(\mathscr{R},\tau)$. By applying Lemma 3.1, it follows that:

1. for every positive integer k, the projection $\hat{\phi}(\chi_{(\frac{1}{k},\infty)}(h))$ is finite, i.e.,
$$\tau(\hat{\phi}(\chi_{(\frac{1}{k},\infty)}(h))) < \infty;$$
2. for every positive integer k,
$$\hat{\phi}(\chi_{(\frac{1}{k},\infty)}(h)) \leqslant \hat{\phi}(\chi_{(\frac{1}{k+1},\infty)}(h));$$
3. as k goes to infinity, the projection $\hat{\phi}(\chi_{(\frac{1}{k},\infty)}(h))$ converges to $\hat{\phi}(\chi_{(X-F)})$ in the strong operator topology.

For a fixed $\delta > 0$, define a closed subset Δ of X by

$$\Delta \triangleq \{x \in X : \mathrm{dist}(x,F) = \delta\}.$$

Then $\hat{\phi}(\chi_\Delta)$ is a sub-projection of certain $\hat{\phi}(\chi_{(\frac{1}{k},\infty)}(h))$. Therefore, there exist at most countably many such $\hat{\phi}(\chi_\Delta)$ satisfying $\tau(\hat{\phi}(\chi_\Delta)) > 0$. This implies that there exists a decreasing sequence $\{\alpha_k\}_{k=1}^\infty$ in the unit interval converging to 0 such that

$$\hat{\phi}(\chi_{(\alpha_{k+1},\alpha_k)}(h)) = \hat{\phi}(\chi_{(\alpha_{k+1},\alpha_k]}(h)) = \hat{\phi}(\chi_{[\alpha_{k+1},\alpha_k]}(h)). \tag{3.5}$$

Write $\alpha_0 = +\infty$. For every k in $\mathbb{N}$,

$$\tau(\hat{\phi}(\chi_{(\alpha_{k+1},\alpha_k]}(h))) < \infty.$$

Note that, for every $k \geqslant 1$, $\Delta_k \triangleq \{x \in X : \alpha_k < \mathrm{dist}(x,F) < \alpha_{k-1}\}$ is open in X. Thus, there exists a positive continuous function h_k satisfying

1. $0 \leqslant h_k \leqslant 1, \quad \forall k \geqslant 1$;
2. $h_k(x) > 0, \forall x \in \Delta_k$;
3. $h_k(x) = 0, \forall x \in X \backslash \Delta_k$;
4. $R(\phi(h_k)) = \hat{\phi}(\chi_{(\alpha_k,\alpha_{k-1})}(h))$.

Since $\tau(R(\phi(h_k))) = \tau(R(\psi(h_k))) < \infty$, the reduced von Neumann algebras

$$\mathscr{N}_k = \hat{\phi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h))\mathscr{R}\hat{\phi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h))$$

and

$$\mathscr{M}_k = \hat{\psi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h))\mathscr{R}\hat{\psi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h))$$

are both type II_1 factors.

Furthermore, for every f in $C(X)$ and $k \geqslant 1$, define two $*$-homomorphisms ϕ_k and ψ_k of $C(X)$ into $\mathscr{R}$ by

$$\phi_k(f) = \hat{\phi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h)f), \quad \psi_k(f) = \hat{\psi}(\chi_{(\alpha_k,\alpha_{k-1}]}(h)f)$$

belonging to $\mathscr{N}_k$ and $\mathscr{M}_k$, respectively.

Note that the equality $\tau(R(\phi(h_k f))) = \tau(R(\psi(h_k f)))$ implies

$$\tau(R(\phi_k(f))) = \tau(R(\hat{\phi}(\chi_{(\alpha_{k+1},\alpha_k]}(h)f))) = \tau(R(\hat{\psi}(\chi_{(\alpha_{k+1},\alpha_k]}(h)f))) = \tau(R(\psi_k(f))). \tag{3.6}$$

Therefore, by applying (3.6) and Theorem 1.3, for all $k \geqslant 1$, it follows the relation

$$\phi_k \sim_{C(X)} \psi_k, \qquad \mathrm{mod}\, \mathscr{K}(\mathscr{R},\tau). \tag{3.7}$$

Since X is a compact matric space, there exists a sequence $\mathscr{B} = \{f_i\}_{i\in\mathbb{N}}$ dense in $C(X)$. By applying (3.7), there exists a sequence $\{V_{mk}\}_{m,k=1}^{\infty}$ of unitary operators from $\mathscr{M}_k$ to $\mathscr{N}_k$ such that

$$\|V_{mk}^*\phi_k(f_i)V_{mk} - \psi_k(f_i)\| < \frac{1}{2^m}\cdot\frac{1}{2^k}, \quad 1 \leqslant i \leqslant m+k.$$

Define a partial isometry V_m by $V_m \triangleq \oplus_{k=1}^{\infty} V_{mk}$. Then, it follows that

(a) for every $m \geqslant 1$,

$$V_m^*V_m = \psi_0(1) \quad \text{and} \quad V_mV_m^* = \phi_0(1);$$

(b) for every f in $C(X)$ and every $m \geqslant 1$ the limit

$$\sum_{k=1}^{\infty} \|V_{mk}^*\phi_k(f)V_{mk} - \psi_k(f)\| < \infty$$

shows that $V_m^*\phi_0(f)V_m - \psi_0(f)$ is in $\mathscr{K}(\mathscr{R},\tau)$;

(c) for every f in $C(X)$, there corresponds a sufficiently large m, such that

$$\|V_m^*\phi_0(f)V_m - \psi_0(f)\| < \frac{1}{2^m}.$$

By the definition, the above (a), (b), and (c) lead to that

$$\phi_0 \sim_{C(X)} \psi_0, \quad \mathrm{mod}\, \mathscr{K}(\mathscr{R},\tau).$$

Thus, combining the above reductions, we obtain that

$$\phi \sim_{C(X)} \psi, \quad \mathrm{mod}\, \mathscr{K}(\mathscr{R},\tau).$$

This completes the proof. $\square$

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