Operators and Matrices
Volume 12, Number 4 (2018), 1145–1157 doi:10.7153/oam-2018-12-69

WEYL'S THEOREM AND ITS PERTURBATIONS FOR THE FUNCTIONS OF OPERATORS

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(*Communicated by R. Curto*)

Abstract. In this paper, we study the stability of Weyl's theorem under compact perturbations, and characterize those operators satisfying that the stability of Weyl's theorem does not hold for any integer powers of the operator.

1. Introduction

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. H will always denote an infinite dimensional separable complex Hilbert space and let $B(H)$ ($\mathscr{K}(\mathscr{H})$) denote the algebra of all bounded linear operators (compact operators) on H.

For an operator $T \in B(H)$, we denote by $\sigma(T)$, $N(T)$ and $R(T)$ the spectrum, the kernel and the range of T, respectively. Also, we write $n(T) = dimN(T)$ and $d(T) = codimR(T)$. The ascent $asc(T)$ and the descent $des(T)$ of T are defined by

$$asc(T) = inf\{n \geqslant 0 : N(T^n) = N(T^{n+1})\} \text{ and } des(T) = inf\{n \geqslant 0 : R(T^n) = R(T^{n+1})\},$$

respectively. Let

$$\mathscr{F}_+(H) := \{T \in B(H) : n(T) < \infty \text{ and } R(T) \text{ is closed}\}$$

be the class of all upper semi-Fredholm operators, and let

$$\mathscr{F}_-(H) := \{T \in B(H) : d(T) < \infty \text{ and } R(T) \text{ is closed}\}$$

be the class of all lower semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $\mathscr{F}_\pm(H) := \mathscr{F}_+(H) \cup \mathscr{F}_-(H)$ and the class of all Fredholm operators is defined by

$$\mathscr{F}(H) := \mathscr{F}_+(H) \cap \mathscr{F}_-(H).$$

Also, the semi-Fredholm spectrum $\sigma_{SF}(T)$ is defined by

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathscr{F}_\pm(H)\}$$

Mathematics subject classification (2010): 47B53, 47A55, 47A10.
Keywords and phrases: Weyl's theorem, Browder's theorem, spectrum.
This work is supported by the NSF of China (No. 11471200) and the Fundamental Research Funds for the Central Universities (No. GK201601004).

and let $\rho_{SF}(T) = \mathbb{C}\backslash\sigma_{SF}(T)$. If T is an upper(or lower) semi-Fredholm operator, the index of T is written as $ind(T) = n(T) - d(T)$. As we all know, $\rho_{SF}(T) = \rho_{SF}(T+K)$ and $ind(T) = ind(T+K)$ for any $K \in K(H)$. In addition, if $T \in \mathscr{F}_+(H)$, then there exists an $\varepsilon > 0$ such that $N(T-\lambda I) \subseteq \bigcap\limits_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda| < \varepsilon$.

Recall that an operator T is said to be a Weyl operator if it is a Fredholm operator of index zero and T is said to be a Browder operator if it is a Fredholm operator of finite ascent and descent, equivalently, T is a Browder operator if and only if T is a Fredholm operator and 0 is the boundary point of $\sigma(T)$. The classes of operators defined above generate the following spectra: The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\}.$$

Let $\rho_w(T) = \mathbb{C}\backslash\sigma_w(T)$ and $\rho_b(T) = \mathbb{C}\backslash\sigma_b(T)$. The set of all normal eigenvalues of T consists with $\sigma_0(T)$, that is, $\sigma_0(T) = \sigma(T)\backslash\sigma_b(T)$.

Let $T \in B(H)$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T)\backslash\sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma;T)$ denote the Riesz idempotent of T corresponding to σ, i.e.,

$$E(\sigma;T) = \frac{1}{2\pi i}\int_{\mathscr{T}} (\lambda I - T)^{-1} d\lambda,$$

where $\mathscr{T} = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote by $H(\sigma;T) = R(E(\sigma;T))$. Clearly, if $\lambda \in iso\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$. We write $H(\lambda;T)$ instead of $H(\{\lambda\};T)$; if, in addition, $dimH(\lambda;T) < \infty$, then $\lambda \in \sigma_0(T)$.

It follows from [4] that Weyl's theorem holds for $T \in B(H)$ if there is an equality $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$, where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < n(T-\lambda I) < \infty\}$. Recall that Browder's theorem ([6, Definition 1]) holds for T if $\sigma_w(T) = \sigma_b(T)$. Evidently "Weyl's theorem" implies "Browder's theorem". The study of Weyl's theorem for bounded linear operators has a long history. In 1909, Weyl ([16]) proved that Weyl's theorem held for self-adjoint operators. Later, this Weyl's theorem has been studied by many mathematicians. Variants and Perturbations of Weyl's theorem have been considered by P. Aiena and M. T. Biondi([1]), R.Harte and W. Y. Lee ([6]) and others. Nowadays, Weyl's theorem has been extended to more and more different classes of Banach and Hilbert space operators ([2], [3], [4], [9], [12]). Also Weyl's theorem for functions of operators and the stability of Weyl's theorem under some compact operators are investigated by many mathematicians ([5]).

The organization of this paper is as follows. Using the property of generalized Weyl spectrum which is defined in section 2, we investigate the stability of Weyl's theorem for $T \in B(H)$ under compact perturbations. In addition, we characterize those operators satisfying that the stability of Weyl's theorem does not hold for any integer powers of the operator.

2. Main results

DEFINITION 2.1. Let $T \in B(H)$. We say that T is a generalized Weyl operator if there exist two closed T invariant subspaces M and N such that $H = M \oplus N$, where T acts as a Weyl operator on M and acts as a quasinilpotent operator on N.

It is easy to verify that if $T \in B(H)$ is a generalized Weyl operator, then there exists an $\varepsilon > 0$ such that $T - \lambda I$ is a Weyl operator and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda| < \varepsilon$. Now we study a variant of the Weyl spectrum. Let

$$\rho_{gw}(T) = \{\lambda \in \mathbb{C} : \text{there exists an } \varepsilon > 0 \text{ such that } T - \mu I \text{ is generalized Weyl if } 0 < |\mu - \lambda| < \varepsilon\}$$

and let $\sigma_{gw}(T) = \mathbb{C} \backslash \rho_{gw}(T)$. Clearly, $\rho_{gw}(T)$ is an open set and $\rho_w(T) \cup iso\sigma(T) \cup iso\sigma_w(T) \subseteq \rho_{gw}(T)$.

DEFINITION 2.2. Let $T \in B(H)$. We say that T has the stability of Weyl's theorem if $T + K$ satisfies Weyl's theorem for all $K \in \mathscr{K}(\mathscr{H})$.

LEMMA 2.1. ([11, Theorem 1.4]) *Let $T \in B(H)$. Then for any $\varepsilon > 0$, there exists a $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $T + K$ does not satisfy Weyl's theorem if and only if at least one of the following conditions hold:*
(1) *T does not satisfy Weyl's theorem;*
(2) *$iso[\sigma(T) \backslash \sigma_0(T)] \neq \emptyset$;*
(3) *$\rho_w(T)$ consists of infinitely many connected components.*

By the similar way as in the proof of [11, Theorem 1.4], we can obtain the following corollary whose proof is left to the reader.

COROLLARY 2.1. *Let $T \in B(H)$. Then $T + K$ satisfies Weyl's theorem for all $K \in \mathscr{K}(\mathscr{H})$ if and only if the following conditions hold:*
(1) *T satisfies Weyl's theorem;*
(2) *$iso[\sigma(T) \backslash \sigma_0(T)] = \emptyset$;*
(3) *$\rho_w(T)$ is connected.*

Here and elsewhere in this paper, for each $\lambda_0 \in \mathbb{C}$ and for a $\delta > 0$, we denote by $B(\lambda_0; \delta)$ the neighborhood of λ_0 as the center and δ as the radius, that is, $B(\lambda_0; \delta) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$. Moreover, we write $B^0(\lambda_0; \delta) = B(\lambda_0; \delta) \backslash \{\lambda_0\}$.

LEMMA 2.2. *Let $T \in B(H)$. Then T has the stability of Weyl's theorem if and only if $\rho_{gw}(T) = \rho_w(T)$ is connected.*

Proof. Assume T has the stability of Weyl's theorem. Since $\rho_{gw}(T) \supseteq \rho_w(T)$ is clear, for the opposite inclusion we only need to show $\rho_{gw}(T) \subseteq \rho_w(T)$. Suppose $\lambda_0 \in \rho_{gw}(T)$. From the definition of $\rho_{gw}(T)$, there exists an $\varepsilon > 0$ such that, for each $\lambda \in B^0(\lambda_0; \varepsilon)$, $T - \lambda I$ is a generalized Weyl operator. Set $\lambda \in B^0(\lambda_0; \varepsilon)$, from the definition of the generalized Weyl operator, there exists a $\delta > 0$ such that $B(\lambda; \delta) \subseteq B^0(\lambda_0; \varepsilon)$ and, for any $\mu \in B^0(\lambda; \delta)$, $T - \mu I$ is Weyl and $N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n]$. Since

T satisfies Weyl's theorem, it follows that $T-\mu I$ is a Browder operator, which implies $asc(T-\mu I)<\infty$. Then, from [14, Theorem 3.4], $T-\mu I$ is invertible. Now we claim that $\lambda \in iso\sigma(T)\cup\rho(T)$. By Corollary 2.1, we have $\lambda \in iso\sigma(T)\cup\rho(T)\subseteq \rho_w(T)$, which means $\lambda_0 \in iso\sigma_w(T)\cup\rho_w(T)$. Apply Corollary 2.1 again, it is easy to get that $iso\sigma_w(T)=\emptyset$ if T has the stability of Weyl's theorem, so $\lambda_0\in\rho_w(T)$. Hence $\rho_{gw}(T)\subseteq \rho_w(T)$. Also, since T satisfies the stability of Weyl's theorem, it follows from Corollary 2.1 that $\rho_w(T)$ is connected. Consequently, we can conclude that $\rho_{gw}(T)=\rho_w(T)$ is connected.

On the other hand, assume $\rho_{gw}(T)=\rho_w(T)$ is connected. Since $iso\sigma(T)\subseteq \rho_{gw}(T)$, it follows that $iso\sigma(T)=\sigma_0(T)=\pi_{00}(T)$. Also, from [7, Corollary 1.14], we can get that $\sigma(T)\backslash\sigma_w(T)=\sigma(T)\backslash\sigma_b(T)=\sigma_0(T)$. Thus, from Corollary 2.1, T has the stability of Weyl's theorem. $\square$

Next, we will use $\sigma_{gw}(T)$ to investigate the stability of Weyl's theorem for T^n for any $n\in\mathbb{N}$. In order to state our main theorem we first need to state the following fact.

REMARK 2.1. Let $T\in B(H)$. If $int\sigma_{gw}(T)=\emptyset$, we can not conclude that T^n has the stability of Weyl's theorem for any $n\in\mathbb{N}$. Here are two examples to explain this conclusion.

(1) Suppose $A=(a_{ij})\in B(\ell^2)$, where $a_{ij}=1$ for $|i-j|=1$ and $a_{ij}=0$ for $|i-j|\neq 1$. Then $\sigma(A)$ consists of the interval $[-2,\ 2]$ of the real axis ([15, P_{286}]). Let $T=\frac{A+2I}{4}$. Then $int\sigma_{gw}(T)=\emptyset$. Also, for any $n\in\mathbb{N}$, $\sigma_{gw}(T^n)=\sigma_w(T^n)=[0,1]$ is easily justified, which implies that $\rho_{gw}(T^n)=\rho_w(T^n)$ is connected, and hence, from Lemma 2.2, T^n has the stability of Weyl's theorem for any $n\in\mathbb{N}$.

(2) Let $A\in B(\ell^2)$ and $B\in B(\ell^2)$ be defined by:

$$A(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,\cdots),\qquad B(x_1,x_2,x_3,\cdots)=(x_2,x_3,x_4,\cdots),$$

and suppose $T\in B(\ell^2\oplus\ell^2)$ and $T=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$. It is easy to verify $int\sigma_{gw}(T)=\emptyset$. In addition, since $\sigma_{gw}(T^n)=\sigma_w(T^n)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ for any $n\in\mathbb{N}$, we have $\rho_{gw}(T^n)=\rho_w(T^n)=\{\lambda\in\mathbb{C}:|\lambda|<1\}\cup\{\lambda\in\mathbb{C}:|\lambda|>1\}$, which and is not connected, then it follows from Lemma 2.2 that T^n does not have the stability of Weyl's theorem. $\square$

The following theorems are concerned with the stability of Weyl's theorem for T^n for any $n\in\mathbb{N}$.

THEOREM 2.1. *Suppose $T\in B(H)$ does not have the stability of Weyl's theorem and $int\sigma_{gw}(T)=\emptyset$. If $\{\mu\in\mathbb{C}:\mu^n=\lambda^n\}\subseteq\sigma_{gw}(T)\cup\rho_w(T)$ for $\lambda\in\sigma_{gw}(T)$ and $n\in\mathbb{N}$, then T^n does not have the stability of Weyl's theorem for any $n\geqslant 2$.*

Proof. Assume, to the contrary, T^n has the stability of Weyl's theorem for some $n\geqslant 2$. Then we will use the following four steps to get the contradiction.

Step 1. Browder's theorem holds for T.

Obviously, we only need to show that $\rho_w(T)\subseteq\rho_b(T)$. Let $\lambda_1\in\rho_w(T)$.

If $\lambda_1=0$, then T is a Weyl operator, which implies that T^n is Weyl. Since Weyl's theorem holds for T^n, it follows that T^n is Browder and hence T is Browder.

If $\lambda_1 \neq 0$, then there exists a $\delta > 0$ such that, for each $\lambda \in B^0(\lambda_1;\delta)$, $T-\lambda I$ is a Weyl operator and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$. We choose $\lambda_i \in \mathbb{C}$ satisfying $\lambda_i^n = \lambda_1^n$ for $i = 2,3,\cdots,n$. According to the condition that $int\sigma_{gw}(T) = \emptyset$, we can find a $\lambda_{21} \in B(\lambda_2;\delta)$ such that $\lambda_{21} \in \rho_{gw}(T)$. From the definition of $\rho_{gw}(T)$, there exists a $\lambda_{22} \in \rho_w(T) \cap B^0(\lambda_2;\delta)$, and thus there exists a $\delta_2 > 0$ such that $B(\lambda_{22};\delta_2) \subseteq \rho_w(T) \cap B^0(\lambda_2;\delta)$. Similarly, we let $\lambda_{31} \in B^0(\lambda_3;\delta)$ satisfy $\lambda_{31}^n = \lambda_{22}^n$. Applying the fact $int\sigma_{gw}(T) = \emptyset$ again, we can find a $\lambda_{32} \in B(\lambda_{31};\delta_2)$ such that $\lambda_{32} \in \rho_{gw}(T)$. Then there exists a $\lambda_{33} \in \rho_w(T) \cap B^0(\lambda_{31};\delta_2) \subseteq \rho_w(T) \cap B^0(\lambda_3;\delta)$, and thus there exists a $\delta_3 > 0$ such that $B(\lambda_{33};\delta_3) \subseteq \rho_w(T) \cap B^0(\lambda_3;\delta)$. Take the same step and keep going, for each $\lambda_i (i = 2,\cdots,n)$, there exists a $\lambda_{ii} \in \rho_w(T) \cap B^0(\lambda_i;\delta)$, and thus there exists a $\delta_i > 0$ such that $B(\lambda_{ii};\delta_i) \subseteq \rho_w(T) \cap B^0(\lambda_i;\delta)$. From the above mentioned, we can choose $\lambda_1' \in B^0(\lambda_1;\delta)$ and $\lambda_i' \in B(\lambda_{ii};\delta_i)(i = 2,\cdots,n)$ satisfying that $\lambda_1'^n = \lambda_2'^n = \cdots = \lambda_n'^n = \mu$ and $\lambda_i' \in \rho_w(T) (i = 1,2,\cdots,n)$. Then

$$T^n - \mu I = (T-\lambda_1' I)(T-\lambda_2' I)\cdots(T-\lambda_n' I)$$

is Weyl. Since Weyl's theorem holds for T^n, we have $T^n - \mu I$ is Browder, which follows from [14, Theorem 7.2] that $T-\lambda_1' I$ is a Browder operator. Moreover, since $N(T-\lambda_1' I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda_1' I)^n]$, we can conclude that $T-\lambda_1' I$ is invertible by [14, Theorem 3.4]. Since δ can be small enough, it follows that $\lambda_1 \in \partial\sigma(T)$, and thus $\lambda_1 \in \rho_b(T)$, hence Browder's theorem holds for T.

Step 2. $iso\sigma_w(T) = \emptyset$.

Assume $iso\sigma_w(T) \neq \emptyset$. If $\lambda_1 \in iso\sigma_w(T)$, then it is clear that $\lambda_1 \in \rho_{gw}(T) \cap \sigma_w(T)$.

Case 1. $\lambda_1 = 0$. Since Browder's theorem holds for T, we can find a $\delta > 0$ such that $\sigma_1 = B(0;\delta) \cap \sigma(T)$ is a clopen subset of $\sigma(T)$ and σ_1 consists of 0 and at most countable normal eigenvalues of T. If there exist limit points in σ_1, then 0 is the unique limit point of σ_1. Without loss of generality, we suppose $dimH(\sigma_1;T) = \infty$. From [13, Theorem 2.10], T can be represented as

$$T = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \begin{matrix} H(\sigma_1;T) \\ H(\sigma_1;T)^{\perp} \end{matrix},$$

where $\sigma(A) = \sigma_1 = \{0\} \cup \sigma_0(A)$ and $\sigma(B) = \sigma(T) \backslash \sigma_1$. It is not difficult to verify that $\sigma(A) = \sigma_w(A) \cup \sigma_0(A)$. By [7, Theorem 3.48], there exists a compact operator $\overline{K}$ acting on $H(\sigma_1;T)$ such that $\sigma(A+\overline{K}) = \sigma_w(A+\overline{K}) = \{0\}$. Suppose $K = \begin{pmatrix} \overline{K} & 0 \\ 0 & 0 \end{pmatrix}$ is a compact operator. Then $(T+K)^n = T^n + K_1 = \begin{pmatrix} (A+\overline{K})^n & * \\ 0 & B^n \end{pmatrix}$ and $0 \in iso\sigma((T+K)^n) = iso\sigma(T^n + K_1)$, where $K_1 = T^{n-1}K + \cdots + TK^{n-1} + K^n$ is a compact operator. Since T^n has the stability of Weyl's theorem, it follows from Lemma 2.2 that $T^n + K_1$ is Weyl and hence T is a Weyl operator, which is in contradiction with the assumption that $0 \in iso\sigma_w(T)$. Therefore, in this case, $iso\sigma_w(T) = \emptyset$.

Case 2. $\lambda_1 \neq 0$. Let $\mu_0 = \lambda_1^n = \lambda_2^n = \cdots = \lambda_n^n$, where $\lambda_i \in \mathbb{C}(1 \leqslant i \leqslant n)$. We claim that $\lambda_i \in \rho_{gw}(T)$ for all $1 \leqslant i \leqslant n$. In fact, if $\lambda_i \in \sigma_{gw}(T)$ for some $2 \leqslant i \leqslant n$, it follows from $\{\mu \in \mathbb{C} : \mu^n = \lambda_i^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ that $\lambda_1 \in \sigma_{gw}(T) \cup \rho_w(T)$, which is in contradiction with $\lambda_1 \in \rho_{gw}(T) \cap \sigma_w(T)$. Consequently, there exists a $\delta > 0$ such that, for each λ_i, $T - \mu_i I$ is a generalized Weyl operator with any $\mu_i \in B^0(\lambda_i;\delta)$. Since Browder's theorem holds for T, it follows from the definition of the generalized Weyl operator that $B^0(\lambda_i;\delta) \subseteq iso\sigma(T) \cup \rho(T)$. Let $\lambda_1' \in B^0(\lambda_1;\delta)$, there exists a $\lambda_i' \in B^0(\lambda_i;\delta)$ $(i \geqslant 2)$ satisfying $(\lambda_1')^n = (\lambda_2')^n = \cdots = (\lambda_n')^n = \mu$. Without loss of generality, suppose $\lambda_i' \in iso\sigma(T)$ for all $i = 1,2,\cdots,n$. By [13, Theorem 2.10], T can be written as

$$T = \begin{pmatrix} T_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & T_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & T_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & T_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_{n+1} \end{pmatrix} \begin{matrix} H(\lambda_1';T) \\ H(\lambda_2';T) \\ H(\lambda_3';T) \\ \cdots \\ H(\lambda_n';T) \\ M \end{matrix},$$

where $\sigma(T_i) = \{\lambda_i'\}(i = 1,\cdots,n)$, $M = H(\sigma;T)$ and $\sigma = \sigma(T)\backslash\{\lambda_1',\lambda_2',\cdots,\lambda_n'\}$. Consequently,

$$T^n = \begin{pmatrix} T_1^n & 0 & 0 & \cdots & 0 & 0 \\ 0 & T_2^n & 0 & \cdots & 0 & 0 \\ 0 & 0 & T_3^n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & T_n^n & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_{n+1}^n \end{pmatrix},$$

and $\mu \in iso\sigma(T^n)$, which yields that $\mu \in \rho_{gw}(T^n) = \rho_w(T^n) = \rho_b(T^n)$ from Lemma 2.2, and then $T - \lambda_i' I$ is Weyl for each i. Hence $\lambda_i \in iso\sigma_w(T) \cup \rho_w(T)$ for any $1 \leqslant i \leqslant n$. Without loss of generality, we may suppose $\lambda_i \in \sigma(T)$ for all $i = 1,2,\cdots,n$ and let $\lambda_i \in \rho_w(T)$ for $2 \leqslant i \leqslant k$ and $\lambda_j \in iso\sigma_w(T)$ for $k+1 \leqslant j \leqslant n$. Note that Browder's theorem holds for T, if $2 \leqslant i \leqslant K$, it follows that $\lambda_i \in iso\sigma(T)$, if $j = 1$ or $k+1 \leqslant j \leqslant n$, then there exists a $\delta_j > 0$ such that $B(\lambda_j;\delta_j) \cap \sigma(T)$ is a clopen subset of $\sigma(T)$ and $B(\lambda_j;\delta_j) \cap \sigma(T)$ consists of λ_j and countable normal eigenvalues of T. Suppose $\sigma_i = \{\lambda_i\}$ for $2 \leqslant i \leqslant k$ and $\sigma_j = B(\lambda_j;\delta_j) \cap \sigma(T)$ for $k+1 \leqslant j \leqslant n$ and $j = 1$. Without loss of generality, we suppose $dimH(\sigma_j;T) = \infty$, and if there exist limit points in σ_j, then λ_j is the unique limit point of σ_j. From [7, Corollary 3.22] and [13, Theorem 2.10] together, T has the following operator matrix form

$$T = \begin{pmatrix} T_1 & * & * & \cdots & * & * \\ 0 & T_2 & * & \cdots & * & * \\ 0 & 0 & T_3 & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & 0 & 0 & \cdots & T_n & * \\ 0 & 0 & 0 & \cdots & 0 & T_{n+1} \end{pmatrix} \begin{matrix} H(\sigma_1;T) \\ H(\sigma_2;T) \\ H(\sigma_3;T) \\ \cdots \\ H(\sigma_n;T) \\ N \end{matrix} \sim \begin{pmatrix} T_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & T_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & T_3 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & T_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_{n+1} \end{pmatrix} \begin{matrix} H(\sigma_1;T) \\ H(\sigma_2;T) \\ H(\sigma_3;T) \\ \cdots \\ H(\sigma_n;T) \\ N \end{matrix},$$

where $\sigma(T_1) = \sigma_1$, $\sigma(T_i) = \{\lambda_i\}$ for $2 \leqslant i \leqslant k$, $\sigma(T_j) = \sigma_j$ for all $k+1 \leqslant j \leqslant n$, $N = H(\sigma;T)$ and $\sigma = \sigma(T)\backslash\{\sigma_1,\sigma_2,\cdots,\sigma_n\}$.

For any $k+1 \leqslant j \leqslant n$ and $j=1$, it follows from [7, Theorem 3.48] that there exist compact operators $\overline{K_j}$ acting on $H(\sigma_j;T)$ with $\sigma(T_j+\overline{K_j})=\{\lambda_j\}$. Let

$$K=\begin{pmatrix} \overline{K_1} & 0 & \cdots\cdots & \cdots & \cdots\cdots & 0 \\ 0 & 0 & \cdots\cdots & \cdots & \cdots\cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 & \cdots & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 & \overline{K_{k+1}} & 0 \ \cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots\cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & 0 \ \overline{K_n} & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots\cdots & 0 \end{pmatrix} \begin{array}{c} H(\sigma_1;T) \\ H(\sigma_2;T) \\ \cdots \\ H(\sigma_k;T) \\ H(\sigma_{k+1};T) \\ \cdots \\ H(\sigma_n;T) \\ N \end{array}.$$

Then it is clear that $K \in B(H)$ is a compact operator and
$(T+K)^n = T^n + K_1 =$

$$\begin{pmatrix} (T_1+\overline{K_1})^n & 0 & \cdots\cdots & \cdots & \cdots & \cdots & 0 \\ 0 & T_2^{\,n} & 0 \ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 \ \ T_k^{\,n} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots\cdots & 0 & (T_{k+1}+\overline{K_{k+1}})^n & 0 & \cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & 0 & (T_n+\overline{K_n})^n & 0 \\ 0 & \cdots & \cdots\cdots & \cdots & \cdots & 0 & T_{n+1}^n \end{pmatrix},$$

where $K_1 = T^{n-1}K+\cdots+TK^{n-1}+K^n$ is a compact operator. It is easy to verify that $\sigma(T_i^n)=\{\lambda_i^n\}=\{\mu_0\}$ $(2 \leqslant i \leqslant k)$ and $\sigma((T_j+\overline{K_j})^n)=\{\mu_0\}$ ($j=1$ and $k+1 \leqslant j \leqslant n$), but $\mu_0 \notin \sigma(T_{n+1}^n)$. Hence, $\mu_0 \in iso\sigma((T+K)^n) = iso\sigma(T^n+K_1) \subseteq iso\sigma_w(T^n+K_1) \cup \rho_w(T^n+K_1) = iso\sigma_w(T^n) \cup \rho_w(T^n)$. Note that T^n has the stability of Weyl's theorem, it then follows from Lemma 2.2 that $\mu_0 \in \rho_{gw}(T^n)=\rho_w(T^n)=\rho_b(T^n)$. Since

$$T^n - \mu_0 I = (T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I),$$

it follows that $\lambda_i \in \rho_w(T)$ for all $1 \leqslant i \leqslant n$, which is in contradiction with $\lambda_j \in iso\sigma_w(T)$ for $j=1$ and $k+1 \leqslant j \leqslant n$. Hence $iso\sigma_w(T)=\emptyset$.

Step 3. $\rho_{gw}(T)=\rho_w(T)$.

We only need to show that $\rho_{gw}(T) \subseteq \rho_w(T)$. Take $\lambda_0 \in \rho_{gw}(T)$, since Browder's theorem holds for T, then there exists a $\delta>0$ such that $B^0(\lambda_0;\delta) \subseteq iso\sigma(T) \cup \rho(T)$. Furthermore, it follows from $iso\sigma(T) \subseteq iso\sigma_w(T) \cup \rho_w(T)$ and $iso\sigma_w(T)=\emptyset$ together that $B^0(\lambda_0;\delta) \subseteq \rho_w(T)$, and thus $\lambda_0 \in iso\sigma_w(T) \cup \rho_w(T)$. Applying the fact that $iso\sigma_w(T)=\emptyset$ again, we get $\lambda_0 \in \rho_w(T)$, which yields $\rho_{gw}(T) \subseteq \rho_w(T)$.

Step 4. $\rho_w(T)$ is connected.

Assume $\rho_w(T)$ is not connected. Then there is a bounded connected component Ω of $\rho_w(T)$, it is clear that $\partial\Omega \subseteq \sigma_{SF}(T)$, where $\partial\Omega$ is the set of the boundary points of Ω. From [10, Lemma 2.10], we can find a compact operator K_1 such that $T+K_1 = \begin{pmatrix} N & * \\ 0 & A \end{pmatrix}$, where N is normal and $\sigma(N)=\sigma_{SF}(T)=\partial\Omega$. By [8, Theorem 3.1], we may choose a compact operator K_2 such that $T+K_1+K_2=\begin{pmatrix}\overline{N} & * \\ 0 & A\end{pmatrix}$ and $\sigma(\overline{N})=\overline{\Omega}$, where $\overline{N}$ is a compact perturbation of N and $\overline{\Omega}$ is the closure of Ω, and then $\Omega\subseteq\sigma(\overline{N})\backslash\sigma_w(\overline{N})$. If $\lambda_1\in\Omega\subseteq\rho_w(T)$, then $T-\lambda_1 I$ is Weyl. As in the proof of "Step 1" in the beginning of this theorem, there exists a small enough $\delta_1>0$ such that $\delta_1\subseteq\Omega$, and there exists a sequence $\{\lambda_i'\}_{i=1}^n\in\rho_w(T)\cap B^0(\lambda_i;\delta_1)$ such that $(\lambda_1')^n=(\lambda_2')^n=\cdots=(\lambda_n')^n=\mu$. Suppose $K=K_1+K_2$. Then

$$(T+K)^n-\mu I=T^n+K_3-\mu I=(T+K-\lambda_1' I)(T+K-\lambda_2' I)\cdots(T+K-\lambda_n' I)$$

is Weyl, where K_3 is a compact operator. Since T^n has the stability of Weyl's theorem, then $(T+K)^n-\mu I$ is Browder, which implies that $T+K-\lambda_1' I$ is a Browder operator. Since $\lambda_1\in\rho_w(T)=\rho_w(T+K)$ and $\delta_1>0$ is small enough, so $N(T+K-\lambda_1' I)\subseteq\bigcap_{n=1}^{\infty}R[(T+K-\lambda_1' I)^n]$ for $\lambda_1'\in B^0(\lambda_1;\delta_1)\subseteq\Omega$. From [14, Theorem 3.4], we get that $T+K-\lambda_1' I$ is invertible. Now we can find a $\lambda_0\in\Omega$ such that $T+K-\lambda_0 I$ is invertible, which generates $\overline{N}-\lambda_0 I$ is injective. Moreover, since $\overline{N}-\lambda_0 I$ is Weyl, we now have $\overline{N}-\lambda_0 I$ is consequently invertible. It is in contradiction with the fact that $\sigma(\overline{N})=\overline{\Omega}$. So $\rho_w(T)$ is connected.

Combining step 3 and step 4 together, we can show that T has the stability of Weyl's theorem, which is a contradiction. Therefore T^n does not have the stability of Weyl's theorem for any $n\geqslant 2$. $\square$

EXAMPLE 2.1. Let $T\in B(\ell^2\oplus\ell^2)$ be defined as example 2 in Remark 2.1. Since $\sigma_{gw}(T)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$, it follows that $int\,\sigma_{gw}(T)=\emptyset$. In addition, T does not have the stability of Weyl's theorem and $\{\mu\in\mathbb{C}:\mu^n=\lambda^n\}\subseteq\sigma_{gw}(T)\cup\rho_w(T)$ for $\lambda\in\sigma_{gw}(T)$ and $n\in\mathbb{N}$. Hence, from Theorem 2.1, T^n does not have the stability of Weyl's theorem for any $n\geqslant 2$.

As an immediate application of Theorem 2.1, we have the following result.

COROLLARY 2.2. *Let $T\in B(H)$. Suppose $int\,\sigma_{gw}(T)=\emptyset$ and T does not have the stability of Weyl's theorem. If for some $k\in\mathbb{N}$, $\{\mu\in\mathbb{C}:\mu^k=\lambda^k\}\subseteq\sigma_{gw}(T)\cup\rho_w(T)$ with $\lambda\in\sigma_{gw}(T)$, then T^k does not have the stability of Weyl's theorem.*

REMARK 2.2. (1) If $int\,\sigma_{gw}(T)=\emptyset$ while $int\,\sigma(T)\neq\emptyset$, then T does not satisfy Browder's theorem.

In fact, if Browder's theorem holds for T, we let $\lambda_0\in int\,\sigma(T)$. Then there exists a $\delta>0$ such that $B(\lambda_0;\delta)\subseteq\sigma(T)$. Since $int\,\sigma_{gw}(T)=\emptyset$, we can find a $\lambda_1\in B(\lambda_0;\delta)$ satisfying $\lambda_1\in\rho_{gw}(T)$, and from the definition of $\rho_{gw}(T)$, there exists a $\lambda_2\in B(\lambda_0;\delta)$ such that $T-\lambda_2 I$ is a Weyl operator. Since Browder's theorem holds for T, if follows that $T-\lambda_2 I$ is a Browder operator, and then we can choose an element $\lambda\in B(\lambda_0;\delta)$ such that $\lambda\in\rho(T)$, which is in contradiction with $B(\lambda_0;\delta)\subseteq\sigma(T)$.

(2) If $\sigma_{gw}(T)=\emptyset$, then $\rho_{gw}(T)=\mathbb{C}$, but $\rho_w(T)\neq\mathbb{C}$ since $\sigma_w(T)$ is not an empty set, it follows from Theorem 2.1 that T does not have the stability of Weyl's theorem.

From the two cases and the proof of Theorem 2.1 together, we can immediately get the following corollaries.

COROLLARY 2.3. *Let $T \in B(H)$. Then the following statements hold:*

(1) *If $int\sigma_{gw}(T) = \emptyset$ while $int\sigma(T) \neq \emptyset$, then T^n does not have the stability of Weyl's theorem for any $n \in \mathbb{N}$;*

(2) *If $\sigma_{gw}(T) = \{\lambda \in \mathbb{C} : |\lambda| = R\}$ for some $R > 0$, then T^n does not have the stability of Weyl's theorem for any $n \in \mathbb{N}$;*

(3) *If $\sigma_{gw}(T) = \emptyset$, then T^n does not have the stability of Weyl's theorem for any $n \in \mathbb{N}$.*

COROLLARY 2.4. *Suppose T does not have the stability of Weyl's theorem. If $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$, then T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$.*

Proof. According to the condition that "$\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$", we can conclude that Browder's theorem holds for T. The remaining steps of the proof is the same to the proof of Theorem 2.1. $\square$

REMARK 2.3. The conditions in Theorem 2.1 are essential.

(1) The condition that "$int\sigma_{gw}(T) = \emptyset$" is essential, which can be verified by the following example.

Suppose $A \in B(\ell^2)$ is the forward shift and $B \in B(\ell^2)$ is the backward shift. Let $T \in B(\ell^2 \oplus \ell^2 \oplus \ell^2)$ be defined by: $T = \begin{pmatrix} A+I & 0 & 0 \\ 0 & B+I & 0 \\ 0 & 0 & A-I \end{pmatrix}$. Then

(a) $\sigma_{gw}(T) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda + 1| \leqslant 1\}$, thus $int\sigma_{gw}(T) \neq \emptyset$;

(b) T does not have the stability of Weyl's theorem since $\rho_{gw}(T)$ is not connected;

(c) $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$.

In addition, it is easy to show that $\sigma(T^2) = \sigma_w(T^2) = \sigma_{gw}(T^2) = \{re^{i\theta} : r \leqslant 2(1+cos\theta)\}$ is connected. Hence, T^2 has the stability of Weyl's theorem by Lemma 2.2.

(2) The condition that "T does not have the stability of Weyl's theorem" is essential. We will give an example to explain the statement.

Let $A \in B(\ell^2)$ be defined as example (1) in Remark 2.1, and write $T = \frac{A+2I}{4}$. It is clear that

(a) $int\sigma_{gw}(T) = \emptyset$;

(b) $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$;

(c) T has the stability of Weyl's theorem since $\rho_w(T) = \rho_{gw}(T)$ is connected.

However, since $\sigma(T^n) = [0,1]$, it follows that $\rho_{gw}(T^n) = \rho_w(T^n)$ is connected. From Lemma 2.2, we can conclude that T^n has the stability of Weyl's theorem for any $n \geqslant 2$.

(3) The condition that "$\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$" is essential. Here is an example.

Assume $A \in B(\ell^2)$ is defined as example (1) in Remark 2.1 and $B \in B(\ell^2)$ is defined by $B(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{3}, \frac{x_3}{3}, \cdots)$. Let $T = \begin{pmatrix} \frac{A+2I}{4} & 0 \\ 0 & B-I \end{pmatrix}$. Then,

(a) $int\sigma_{gw}(T) = \emptyset$;

(b) T does not have the stability of Weyl's theorem since $0 \notin \rho_w(T)$ while $0 \in \rho_{gw}(T)$;

(c) $1 \in \sigma_{gw}(T)$ but $-1 \notin \sigma_{gw}(T) \cup \rho_w(T)$.

For each $n \in \mathbb{N}$, since $\sigma(T^{2n}) = [0,1]$, then $\rho_{gw}(T^{2n}) = \rho_w(T^{2n})$ is connected, which implies that T^{2n} has the stability of Weyl's theorem.

REMARK 2.4. (1) From the following example, we can find that the conditions in Theorem 2.1 are only sufficient, but not necessary.

Suppose $A \in B(\ell^2)$ is the forward shift and $B \in B(\ell^2)$ is the backward shift, and let $T \in B(\ell^2 \oplus \ell^2)$ be defined by $T = \begin{pmatrix} A+I & 0 \\ 0 & B-I \end{pmatrix}$. Obviously, T has the stability of Weyl's theorem, $int\,\sigma_{gw}(T) \neq \emptyset$ and $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$. We claim that T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$.

Indeed, for $T^{2n} \in B(H) (n \in \mathbb{N})$, let $0 < a < 2 - 2cos\frac{2\pi}{n}$ and

$$\lambda_k = aie^{-\frac{k\pi}{n}i} = ae^{i(\frac{\pi}{2} - \frac{k\pi}{n})}, \quad \mu_k = aie^{\frac{k\pi}{n}i} = ae^{i(\frac{\pi}{2} + \frac{k\pi}{n})}, \ k = 1,2,\cdots,n-1.$$

Since λ_k and μ_k are symmetrical with respect to y axis, it follows that $\lambda_k \in \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$, $\mu_k \in \{\lambda \in \mathbb{C} : |\lambda + 1| < 1\}$ and $ind(T - \lambda_k I) + ind(T - \mu_k I) = 0$ for any $1 \leqslant k \leqslant n-1$. Suppose that

$$\begin{aligned} &T^{2n} - (ai)^{2n} \\ &= (T - aiI)(T + aiI)(T - \lambda_1 I)(T - \mu_1 I)(T - \lambda_2 I)(T - \mu_2 I) \cdots (T - \lambda_{n-1} I)(T - \mu_{n-1} I). \end{aligned}$$

Then $T^{2n} - (ai)^{2n}$ is Weyl. But since $T - \lambda_i I$ is not Browder, it follows that T^{2n} does not have the stability of Weyl's theorem for any $n \in \mathbb{N}$.

For $T^{2n+1} \in B(H)$, take $\lambda_k = ie^{-\frac{2k\pi}{2n+1}}$ for $k = 1,2,\cdots,2n$. Note that λ_k and λ_{2n-k+1} are symmetrical with respect to y axis, we have $\lambda_i \in \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \cup \{\lambda \in \mathbb{C} : |\lambda + 1| < 1\} \cup \rho(T)$ and $ind(T - \lambda_k I) + ind(T - \lambda_{2n-k+1} I) = 0$. Let

$$T^{2n+1} - (i)^{2n+1} = (T - iI)(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_{2n} I).$$

Then $T^{2n+1} - (i)^{2n+1}$ is Weyl. But since $T - \lambda_i I$ is not Browder, it follows that T^{2n+1} does not have the stability of Weyl's theorem for any $n \in \mathbb{N}$.

(2) Suppose $int\,\sigma_{gw}(T) = \emptyset$ and T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$. If $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$, we can not conclude that T does not have the stability of Weyl's theorem. We use the following example explain the statement:

Let $A \in B(\ell^2)$ be defined as example (1) in Remark 2.1, and let $T = e^{\frac{A+2I}{4}\pi i}$. Then

(a) $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda = e^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$ and $int\,\sigma_{gw}(T) = \emptyset$;

(b) $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T) \cup \rho_w(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$;

(c) For any $n \geqslant 2$, $\sigma(T^n) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, which implies that $\sigma(T^n) = \sigma_w(T^n) = \sigma_{gw}(T^n)$.

By Theorem 2.1, we can find that T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$, but T has the stability of Weyl's theorem.

From the statement (2) in Remark 2.4, we want to get that T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$ if and only if T does not have the stability of Weyl's theorem. First, let us explore the equivalence of the stability of Weyl's theorem for T and T^2.

LEMMA 2.3. *Suppose $T \in B(H)$ with $int\sigma_{gw}(T) = \emptyset$. If $\sigma_{gw}(T)$ is symmetrical about the origin, then T has the stability of Weyl's theorem if and only if T^2 has the stability of Weyl's theorem.*

Proof. First, we claim that $\rho_{SF}(T) = \rho_w(T)$. From the inclusion we only need to show that $\rho_{SF}(T) \subseteq \rho_w(T)$. If $\lambda_0 \in \rho_{SF}(T)$, then we can find a $\delta > 0$ such that $T - \lambda I$ is a semi-Fredholm operator and $ind(T - \lambda I) = ind(T - \lambda_0 I)$ if $\lambda \in B(\lambda_0; \delta)$. Since $int\sigma_{gw}(T) = \emptyset$, it follows that there exists a $\lambda_1 \in B(\lambda_0; \delta)$ such that $\lambda_1 \in \rho_{gw}(T)$, then from the definition of $\rho_{gw}(T)$, we can choose a $\lambda_2 \in B(\lambda_0; \delta)$ such that $T - \lambda_2 I$ is a Weyl operator. Hence $ind(T - \lambda_0 I) = ind(T - \lambda_2 I) = 0$, which means that $\lambda_0 \in \rho_w(T)$. So $\rho_{SF}(T) = \rho_w(T)$.

Suppose T has the stability of Weyl's theorem.

(1) $\rho_w(T^2) = [\rho_w(T)]^2$ is connected.

Obviously, $\rho_w(T) = \rho_{gw}(T)$ is connected, which implies that $[\rho_w(T)]^2$ is connected. Note that $\rho_{SF}(T) = \rho_w(T)$, which follows that $\rho_w(T^2) \subseteq [\rho_w(T)]^2$. For the converse inclusion, take $\mu_0 = \lambda_0^2 \in [\rho_w(T)]^2$ with $\lambda_0 \in \rho_w(T)$. Since $\sigma_w(T)$ is symmetrical about the origin, it follows that $-\lambda_0 \in \rho_w(T)$. Hence $\mu_0 \in \rho_w(T^2)$, that is, $[\rho_w(T)]^2 \subseteq \rho_w(T^2)$.

(2) $\rho_{gw}(T^2) = \rho_w(T^2)$.

We only need to show $\rho_{gw}(T^2) \subseteq \rho_w(T^2)$. Suppose $\mu_0 \in \rho_{gw}(T^2)$ such that $\mu_0 = (\pm\lambda_0)^2$, then we can get an $\varepsilon > 0$ such that $T^2 - \mu I$ is a generalized Weyl operator if $\mu \in B^0(\mu_0; \varepsilon)$. For any $\mu \in B^0(\mu_0; \varepsilon)$, if $0 < |\mu' - \mu|$ is small enough, then $T^2 - \mu' I$ is a Weyl operator and $N(T^2 - \mu' I) \subseteq \bigcap_{n=1}^{\infty} R[(T^2 - \mu' I)^n]$. Now we suppose

$$T^2 - \mu' I = (T - \lambda_1' I)(T + \lambda_1' I),$$

where $\mu' = (\pm\lambda_1)^2$. Since $\rho_{SF}(T) = \rho_w(T)$, then both $T - \lambda_1' I$ and $T + \lambda_1' I$ are Weyl operators. By the condition that Weyl's theorem holds for T, it follows that $T - \lambda_1' I$ and $T + \lambda_1' I$ are Browder operators and hence $T^2 - \mu' I$ is a Browder operator. Moreover, it follows from [14, Theorem 3.4] that $T^2 - \mu' I$ is invertible, which means that $\mu \in iso\sigma(T^2) \cup \rho(T^2)$. Let

$$T^2 - \mu I = (T - \lambda_1 I)(T + \lambda_1 I).$$

Then $\pm\lambda_1 \in iso\sigma(T) \cup \rho(T) \subseteq \rho_{gw}(T) = \rho_w(T)$, and thus $T^2 - \mu I$ is a Weyl operator. Hence $\mu_0 \in iso\sigma_w(T^2) \cup \rho_w(T^2)$. Since

$$T^2 - \mu_0 I = (T - \lambda_0 I)(T + \lambda_0 I),$$

and $\rho_{SF}(T) = \rho_w(T)$, we can get $\pm\lambda_0 \in iso\sigma_w(T) \cup \rho_w(T) \subseteq \rho_{gw}(T) \subseteq \rho_w(T)$, and thus $\mu_0 \in \rho_w(T^2)$.

From the preceding proof and Lemma 2.2, we see that T^2 has the stability of Weyl's theorem.

Suppose T^2 has the stability of Weyl's theorem. It follows from Corollary 2.2 that T has the stability of Weyl's theorem. $\square$

If $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$, then $\sigma_{gw}(T)$ is symmetrical about the origin. Combining Lemma 2.3 and Theorem 2.1 together, we can immediately obtain the following theorem.

THEOREM 2.2. *Suppose $T \in B(H)$ with $int\,\sigma_{gw}(T) = \emptyset$. If $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$, then*

(1) *T^n does not have the stability of Weyl's theorem for any $n \geqslant 2$ if and only if T does not have the stability of Weyl's theorem;*

(2) *T^n has the stability of Weyl's theorem for any $n \geqslant 2$ if and only if T has the stability of Weyl's theorem;*

(3) *T^n has the stability of Weyl's theorem for any $n \in \mathbb{N}$ if and only if T^k has the stability of Weyl's theorem for some $k \in \mathbb{N}$.*

By (2) in Remark 2.4, the condition "If $\{\mu \in \mathbb{C} : \mu^n = \lambda^n\} \subseteq \sigma_{gw}(T)$ for $\lambda \in \sigma_{gw}(T)$ and $n \in \mathbb{N}$" in Theorem 2.2 is essential.

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(Received November 16, 2016)

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