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DIFFERENCE OF COMPOSITION OPERATORS ON THE BERGMAN SPACES OVER THE BALL

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Abstract. This paper characterizes the compactness of a linear combination of three composition operators on $A^p_\alpha(\mathbb{B}^N)$, the weighted Bergman space over the unit ball $\mathbb{B}^N$ in $\mathbb{C}^N$. In this setting, we show that there is no cancellation property for the compactness of double difference of composition operators, which extends Koo-Wang's results over the unit disk in [13]. In addition, we investigate the compactness and essential norm estimate of the differences of weighted composition operators between weighted Bergman spaces.

1. Introduction

Let $\mathbb{B}=\mathbb{B}^N$ be the open unit ball of the complex N-space $\mathbb{C}^N$, $\partial\mathbb{B}$ the boundary of $\mathbb{B}$, $\mathbb{D}=\mathbb{B}^1$ the open unit disk in the complex plane $\mathbb{C}$. For $z=(z_1,\dots,z_N)$ and $w=(w_1,\dots,w_N)$ in $\mathbb{C}^N$, the Hermitian inner product of z and w is denoted by

$$\langle z,w\rangle = z_1\overline{w_1}+\dots z_N\overline{w_N},$$

and we write

$$|z|=\sqrt{\langle z,z\rangle}=\sqrt{|z_1|^2+\dots|z_N|^2}.$$

Denote by $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. Let $\mathrm{S}=\mathrm{S}(\mathbb{B})$ be the class of all holomorphic self-maps of $\mathbb{B}$. Then, for each $\varphi\in\mathrm{S}$ and $u\in H(\mathbb{B})$, the weighted composition operator induced by u and φ is given by

$$uC_\varphi f := u\cdot f\circ\varphi,\ \ f\in H(\mathbb{B}).$$

We can regard this operator as a generalization of a multiplication operator M_u induced by u and a composition operator C_φ induced by φ, where $M_u f = u\cdot f$ and $C_\varphi f = f\circ\varphi$.

The investigations of composition operators have increasingly become a major driving force in the development of modern complex analysis. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. We refer to books by Cowen and MacCluer

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[6] and Shapiro [19] for various aspects on the theory of composition operators acting on some subclasses of $H(\mathbb{B})$.

We first recall our function spaces to work on.

For $\alpha > -1$, put

$$d\nu_\alpha(z) := c_\alpha (1-|z|^2)^\alpha d\nu(z)$$

where the constant $c_\alpha := \frac{\Gamma(N+\alpha+1)}{N!\Gamma(\alpha+1)}$ is chosen so that $\nu_\alpha(\mathbb{B}) = 1$ and $d\nu$ is the normalized volume measure on $\mathbb{B}$. Now, given $0 < p < \infty$, the α-weighted Bergman space $A^p_\alpha(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that the "norm"

$$\|f\|_{A^p_\alpha} := \left\{ \int_{\mathbb{B}} |f(z)|^p d\nu_\alpha(z) \right\}^{\frac{1}{p}}$$

is finite. As is well-known, for each $\alpha > -1$, the space $A^p_\alpha(\mathbb{B})$ equipped with the norm above is a Banach space for $1 \leqslant p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f,g) \mapsto \|f-g\|^p_{A^p_\alpha}$.

Recently, there has been an increasing interest in studying the compact differences for composition operators acting on different spaces of holomorphic functions. In 2005, Moorhouse [15] considered the compact differences of composition operators acting on the standard weighted Bergman spaces and necessary conditions were given on a large scale of weighted Dirichlet spaces. In 2012, Choe et al. [2] characterized the compactness of differences of composition operators over polydisks which was analogous to Moorhouse's results. Then, the compact differences of composition operators on the Bergman spaces over the ball were also investigated by Choe et al. in [3]. Moreover, in [11], the authors studied the compactness of the difference of two weighted composition operators acting from the weighted Bergman space to the weighted type space in the unit disk. In [20], the authors generalized the results of [11] to the unit ball. Motivated by the ideas of these investigations, in this paper, we give sufficient and necessary conditions for compactness and essential norm estimate of the differences of weighted composition operators between the weighted Bergman spaces. For further results about compact differences on various settings, we refer to [5, 7, 8, 9, 10, 12, 17, 18, 22, 21, 23] and references therein.

More generally, study on linear combinations of composition operators has been a topic of growing interest. Krite and Moorhouse [14] first considered some general results on compact linear combinations on $A^2_\alpha(\mathbb{D})$. Recently, the compact linear combinations of three composition operators on $A^p_\alpha(\mathbb{D})$ were completely characterized by Koo-Wang in [13]. These results show a quite rigid behavior of compact linear combinations of three composition operators in the sense that the double difference cancellation is impossible. More precisely, for distinct C_{φ_1}, C_{φ_2} and C_{φ_3}, the form $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ is compact on $A^p_\alpha(\mathbb{D})$ if and only if both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_3} - C_{\varphi_1}$ are individually compact. In this paper, we extend such result to the unit ball setting under a suitable restriction on inducing maps using different techniques and the restriction is automatically satisfied in the case of the disk. For further results on linear combinations of composition operators on various settings, see [1, 3, 4, 10, 13, 14] and references therein.

Constants. In the rest of the paper, we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of C will be often specified in a parenthesis. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \leqslant CY$ for some constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

In this section, we will give some notation and recall some well-known results on the weighted Bergman space $A^p_\alpha(\mathbb{B})$ in the sequel.

2.1. Pseudo-hyperbolic distance

Recall that for $a \in \mathbb{B}$, the involutive automorphism of the unit ball $\mathbb{B}$ which interchanges 0 and a is given by

$$\sigma_a(z) = \frac{a - P_a(z)}{1 - \langle z, a\rangle} + \sqrt{1 - |a|^2}\frac{P_a(z) - z}{1 - \langle z, a\rangle},$$

where P_a, the orthogonal projection from C^N onto the one dimensional complex line generated by a, is defined as

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a \quad \text{if } a \neq 0$$

and $P_0(z) = 0$. More details about the automorphism of the unit ball can be found in [24, Sections 1.2].

The pseudo-hyperbolic distance between $z, w \in \mathbb{B}$ is given by

$$\rho(z, w) := |\sigma_z(w)|.$$

It is well known that $\rho(z, w)$ is a metric on $\mathbb{B}$ (see [24, Corollary 1.22]). By a straight calculation, we obtain by [24, Lemma 1.2] that

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w\rangle|^2} \tag{1}$$

and

$$\frac{1 - |z|^2}{|1 - \langle z, w\rangle|} = \sqrt{1 - \rho^2(z, w)}\sqrt{\frac{1 - |z|^2}{1 - |w|^2}} \tag{2}$$

for $z, w \in \mathbb{B}$. In particular, we have

$$\rho^2(z, w) \leqslant 1 - \left(\frac{1 - |z|^2}{|1 - \langle z, w\rangle|}\right)^2 \quad \text{for } |w| \leqslant |z| \tag{3}$$

which is quite useful for our purpose.

Throughout the paper, we assume the maps $\varphi_j : \mathbb{B} \to \mathbb{B}$ are holomorphic ($j \in \mathbb{N}$) and $\varphi_i \neq \varphi_j$ if $i \neq j$. Let $\rho_{ij}(z) = \rho(\varphi_i(z), \varphi_j(z))$ for $i \neq j$. From (1), it is easy to observe that

$$1 - \rho_{ij}^2(z) = \frac{(1 - |\varphi_i(z)|^2)(1 - |\varphi_j(z)|^2)}{|1 - \langle \varphi_i(z), \varphi_j(z) \rangle|^2}. \tag{4}$$

The pseudo-hyperbolic ball centered at $z \in \mathbb{B}$ with radius $r \in (0,1)$ is defined by

$$D_r(z) := \{w \in \mathbb{B} : \rho(z,w) < r\}.$$

Then, for given $0 < r < 1$ and $\alpha > -1$, by [24, Lemma 1.23] we have

$$v_\alpha[D_r(z)] \approx (1 - |z|^2)^{N+1+\alpha},$$

where the constants suppressed in these estimates depend only on r, α and N. This yields the sub-mean value type inequality:

$$|f(z)|^p \leqslant \frac{C}{(1-|z|^2)^{N+\alpha+1}} \int_{D_r(z)} |f(w)|^p dv_\alpha(w), \ z \in \mathbb{B}$$

for all $f \in H(\mathbb{B})$, $0 < p < \infty$ and some constant $C = C(r, \alpha)$; for more details, see [24, Lemma 2.24]. In particular, we have

$$|f(z)|^p \leqslant \frac{C}{(1-|z|^2)^{N+\alpha+1}} \|f\|_{A_\alpha^p}^p, \ z \in \mathbb{B} \tag{5}$$

for all $f \in A_\alpha^p(\mathbb{B})$.

The following lemma is a vital tool in the proof of our main results, see [20, Lemma 4].

LEMMA 1. *There exists a constant $C > 0$ such that*

$$\left|(1-|z|^2)^{\frac{N+\alpha+1}{p}} f(z) - (1-|w|^2)^{\frac{N+\alpha+1}{p}} f(w)\right| \leqslant C \|f\|_{A_\alpha^p} \rho(z,w)$$

for all $f \in A_\alpha^p(\mathbb{B})$ and for all $z, w \in \mathbb{B}$.

2.2. Test functions

It is well known that if $w \in \mathbb{B}$ and $c > 0$ then

$$\int_{\mathbb{B}} \frac{1}{|1 - \langle z, w \rangle|^{N+1+\alpha+c}} dv_\alpha(z) \approx (1 - |w|^2)^{-c} \tag{6}$$

as $|w| \to 1^-$ (see [6]). For $w \in \mathbb{B}$, let τ_w be the function on $\mathbb{B}$ defined by

$$\tau_w(z) := \frac{1}{1 - \langle z, w \rangle}.$$

Let $\alpha > -1$ and $s > 0$. By (6), it follows that if $ps > \alpha + N + 1$, $\tau_w^s \in A_\alpha^p$. Also, when $ps > \alpha + N + 1$, we have

$$\|\tau_w^s\|_{A_\alpha^p}^p \approx \frac{1}{(1-|w|^2)^{ps-\alpha-N-1}} \tag{7}$$

and thus

$$\frac{\tau_w^s}{\|\tau_w^s\|_{A_\alpha^p}} \to 0 \text{ uniformly on compact subsets of } \mathbb{B} \text{ as } |w| \to 1.$$

2.3. Angular derivatives

Given $\xi \in \partial\mathbb{B}$, a continuous function $\Lambda(t) : [0,1) \to \mathbb{B}$ with $\lim_{t\to 1} \Lambda(t) = \xi$ is said to be a restricted ξ-curve, if

$$\lim_{t\to 1} \frac{|\Lambda(t) - \langle \Lambda(t), \xi\rangle \xi|^2}{1 - |\langle \Lambda(t), \xi\rangle|^2} = 0 \text{ and } \sup_{0 \leqslant t < 1} \frac{|\xi - \langle \Lambda(t), \xi\rangle \xi|}{1 - |\langle \Lambda(t), \xi\rangle|} < \infty.$$

We say that $f : \mathbb{B} \to \mathbb{C}$ has restricted limit at ξ, naturally denoted by $f(\xi)$, if $\lim_{t\to 1^-} f(\Lambda(t)) = f(\xi)$ for every restricted ξ-curve Λ.

Let φ be a holomorphic self-map of $\mathbb{B}$. We recall that φ has finite angular derivative at $\xi \in \partial\mathbb{B}$, if there exists $\eta \in \partial\mathbb{B}$ so that

$$\frac{1 - \langle \varphi(z), \eta\rangle}{1 - \langle z, \xi\rangle}$$

has finite restricted limit $A_\varphi(\xi)$ at ξ.

In addition, we write $\varphi_\eta := \langle \varphi, \eta\rangle$ for the coordinate of φ in the direction of $\eta \in \partial\mathbb{B}$ and put $D_\xi = \frac{\partial}{\partial \xi}$ as the directional derivative in the direction of $\xi \in \partial\mathbb{B}$. The following is the Julia-Carathéodory Theorem of the ball (see [6, Theorem 2.81]).

THEOREM 1. *Let φ be a holomorphic self-map on $\mathbb{B}$ and $\xi \in \partial\mathbb{B}$, the following statements are equivalent:*

(1) φ has finite angular derivative at ξ.

(2)

$$d_\varphi(\xi) = \liminf_{z\to\xi} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty.$$

(3) φ has restricted limit $\eta \in \partial\mathbb{B}$ at ξ and $D_\xi \varphi_\eta(z) = \langle \varphi'(z)\xi, \eta\rangle$ has finite restricted limit at ξ.

Furthermore, when these conditions above hold, then

(4) $D_\xi \varphi_\eta(z)$ has restricted limit $d_\varphi(\xi)$ at ξ.

(5) $A_\varphi(\xi) = d_\varphi(\xi)$.

Here we use the following notation:

$$F_i = \{\xi \in \partial\mathbb{B} : \varphi_i \text{ has a finite angular derivative at } \xi\},$$

where each $\varphi_i : \mathbb{B} \to \mathbb{B}$ is a holomorphic map.

2.4. Compact operator

It seems better to clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when $0 < p < 1$. Suppose X and Y are topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $T : X \to Y$ is said to be compact if the image of every bounded sequence in X has a subsequence that converges in Y. If $T : X \to Y$ is a bounded linear operator, then the essential norm of the operator $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined as follows:

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ denotes the operator norm. Since the set of all compact operators is a closed subset of the space of bounded operators, it is obvious that the operator T is compact if and only if $\|T\|_{e,X\to Y} = 0$.

We have the following convenient compactness criterion for a linear combination of composition operators acting on the weighted Bergman spaces.

LEMMA 2. *Let $\alpha, \beta > -1$ and $0 < p, q < \infty$. Assume that Y is either $A^p_\alpha(\mathbb{B})$ or $A^q_\beta(\mathbb{B})$. Let $T : A^p_\alpha(\mathbb{B}) \to Y$ be a linear combination of composition operators and suppose that T is bounded on $A^p_\alpha(\mathbb{B})$. Then T is compact if and only if $Tf_k \to 0$ in Y for any bounded sequence $\{f_k\}$ in $A^p_\alpha(\mathbb{B})$ satisfying $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$.*

A proof can be found in [6, Proposition 3.11] for composition operators on a Hardy space over the unit disk and it can be easily modified for the operator T on $A^p_\alpha(\mathbb{B})$.

The following compact difference characterization is due to Choe et al. [3].

THEOREM 2. *Let $\alpha > \beta > -1$, $0 < p, q < \infty$ and $\varphi_1, \varphi_2 \in \mathrm{S}$. Suppose that C_{φ_1} and C_{φ_2} are bounded on $A^q_\beta(\mathbb{B})$. Then $C_{\varphi_1} - C_{\varphi_2} : A^p_\alpha(\mathbb{B}) \to A^p_\alpha(\mathbb{B})$ is compact if and only if*

$$\lim_{|z|\to 1} \rho_{12}(z)\left(\frac{1-|z|^2}{1-|\varphi_1(z)|^2} + \frac{1-|z|^2}{1-|\varphi_2(z)|^2}\right) = 0.$$

In addition, the following theorem for the sum operators is also proved in [3].

THEOREM 3. *Let $0 < p, q < \infty$ and $\alpha > \beta > -1$. Assume that $\varphi_i \in \mathrm{S}$ and C_{φ_i} are bounded on $A^q_\beta(\mathbb{B})$ for $i = 1, 2, \ldots, n$. Then $C_{\varphi_1} - \sum_{j=2}^n C_{\varphi_j} : A^p_\alpha(\mathbb{B}) \to A^p_\alpha(\mathbb{B})$ is compact if and only if*

(1) $F_2,\dots,F_n$ *are pairwise disjoint and* $F_1=\cup_{j=2}^{n}F_j$;

(2) $\lim_{z\to\xi}\rho_{1j}(z)\left(\frac{1-|z|^2}{1-|\varphi_1(z)|^2}+\frac{1-|z|^2}{1-|\varphi_j(z)|^2}\right)=0$, *for all* $\xi\in F_j$ *and for* $j=2,3,\dots,n$.

There is certain coefficient relation for T, a general linear sum of composition operators, to be compact on $A^p_\alpha(\mathbb{B})$. In order to describe the result, throughout the paper, we need to introduce the following notation. For each $\xi,\eta\in\partial\mathbb{B}$, let

$$F_i(\eta)=\{\xi\in\partial\mathbb{B}:\xi\in F_i \text{ and } \varphi_i(\xi)=\eta\},\tag{8}$$

and

$$\Gamma_{\xi,\eta}=\{j:\xi\in F_j(\eta)\}.\tag{9}$$

In addition, for $s>0$, put

$$\Gamma_{\xi,\eta,s}=\{j:\xi\in F_j(\eta),d_{\varphi_j}(\xi)=s\}.\tag{10}$$

The coefficient relation is as follows, see [3, Corollary 3.6].

THEOREM 4. *Let* $\alpha>-1$ *and* $0<p<\infty$. *Suppose that* $\varphi_i\in S$ *for* $i=1,2,\dots,n$. *If* $T=\sum_{i=1}^n a_iC_{\varphi_i}:A^p_\alpha(\mathbb{B})\to A^p_\alpha(\mathbb{B})$ *is compact, then*

$$\sum_{j\in\Gamma_{\xi,\eta,s}}a_j=0\tag{11}$$

for all $\zeta,\eta\in\partial\mathbb{B}$ *and* $s>0$.

3. Main results

3.1. Cancellation properties of composition operators

In this subsection we first characterize the compactness of a linear combination of three composition operators on $A^p_\alpha(\mathbb{B})$ which extends the result in the unit disk case [13]. Moreover, we investigate the cancellation property of composition operators on $A^p_\alpha(\mathbb{B})$.

THEOREM 5. *Let* $\alpha>\beta>-1$ *and* $0<p,q<\infty$. *Assume that* $a_i\in\mathbb{C}\setminus 0$, $\varphi_i\in S$ *and* C_{φ_i} *is bounded on* $A^q_\beta(\mathbb{B})$ *but not compact on* $A^p_\alpha(\mathbb{B})$ *for each* $i=1,2,3$. *If* $T:=\sum_{i=1}^3 a_iC_{\varphi_i}$ *is compact on* $A^p_\alpha(\mathbb{B})$, *then one of the following holds:*

(i) $T=a_i(C_{\varphi_i}-C_{\varphi_j}-C_{\varphi_k})$.

(ii) $T=a_i(C_{\varphi_i}-C_{\varphi_j})+a_k(C_{\varphi_k}-C_{\varphi_j})$.

Here, (i,j,k) *is some permutation of* $(1,2,3)$.

Proof. According to the assumption, if $T := \sum_{i=1}^{3} a_i C_{\varphi_i}$ is compact, then $\sum_{j\in\Gamma_{\xi,\eta,s}} a_j = 0$ for all $\zeta,\eta \in \partial\mathbb{B}$ by (11). Then,

$$\sum_{j\in\Gamma_{\xi,\eta}} a_j = 0. \tag{12}$$

We let $|\Gamma_{\xi,\eta}|$ be the number of elements of the set $\Gamma_{\xi,\eta}$.

First, we claim that $|\Gamma_{\xi,\eta}| \neq 1$ for all $\xi,\eta \in \partial\mathbb{B}$ by (12) because $a_j \neq 0$ for all $j = 1,2,3$. Thus, we conclude that $|\Gamma_{\xi,\eta}| \in \{0,2,3\}$ for all $\xi,\eta \in \partial\mathbb{B}$.

Now we assume that $|\Gamma_{\xi,\eta}| = 3$ for some $\xi,\eta \in \partial\mathbb{B}$, then $\Gamma_{\xi,\eta} = \{i,j,k\}$ which implies $a_i + a_j + a_k = 0$ by (12). Thus, $T = a_i(C_{\varphi_i} - C_{\varphi_j}) + a_k(C_{\varphi_k} - C_{\varphi_j})$ is obtained.

Next, suppose $|\Gamma_{\xi,\eta}| \in \{0,2\}$ for all $\xi,\eta \in \partial\mathbb{B}$. If $|\Gamma_{\xi,\eta}| = 0$ for all ξ,η which means the angular derivative of φ_j does not exist at all $\xi \in \partial\mathbb{B}$ for all $j = 1,2,3$. Since C_{φ_j} are bounded on $A^q_\beta(\mathbb{B})$ for $\alpha > \beta > -1$, it follows from [25] that each C_{φ_j} is compact on $A^p_\alpha(\mathbb{B})$ for all $j = 1,2,3$. This contradicts our assumption. Consequently, there exist some $\xi,\eta \in \partial\mathbb{B}$ such that

$$|\Gamma_{\xi,\eta}| = 2.$$

Without loss of generality, we may suppose that $\Gamma_{\xi,\eta} = \{i,j\}$. Then, by (12) we have

$$a_i + a_j = 0. \tag{13}$$

Since $k \notin \Gamma_{\xi,\eta}$, there are two possibilities:

(a) the angular derivative of φ_k does not exist at ξ;

(b) φ_k has finite angular derivative at ξ but $\varphi_k(\xi) \neq \eta$.

If (b) holds, we have that $a_k = 0$ by (12) and this is a contradiction to the fact that $a_k \neq 0$. Thus, (a) holds, i.e., the angular derivative of φ_k does not exist at ξ. Since C_{φ_k} is not compact, the angular derivative of φ_k exists at some other point ξ'. Then, it follows that $k \in \Gamma_{\xi',\varphi_k(\xi')}$ and $|\Gamma_{\xi',\varphi_k(\xi')}| = 2$. Thus, we deduce from (12) that $a_k + a_i = 0$ or $a_k + a_j = 0$.

Consequently, we have $T = a_i(C_{\varphi_i} - C_{\varphi_j} - C_{\varphi_k})$ or $T = a_j(C_{\varphi_j} - C_{\varphi_i} - C_{\varphi_k})$. This completes the proof. □

The compactness of the case (i) of Theorem 5 is characterized by Theorem 3. The following and Theorem 2 completely characterize the compactness of the case (ii) of Theorem 5.

THEOREM 6. *Let $\alpha > \beta > -1$, and $0 < p,q < \infty$. Assume that $\varphi_i \in S$, C_{φ_i} is bounded on $A^q_\beta(\mathbb{B})$ and not compact on $A^p_\alpha(\mathbb{B})$ for each $i = 1,2,3$. Let $a,b \in \mathbb{C}\setminus 0$ and $a+b \neq 0$. Then $T := a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact on $A^p_\alpha(\mathbb{B})$ if and only if both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact on $A^p_\alpha(\mathbb{B})$.*

Proof. The sufficiency is trivial and it only needs to show the necessity. Suppose $T := a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact. We will obtain a contradiction if either $C_{\varphi_2} - C_{\varphi_1}$ or $C_{\varphi_3} - C_{\varphi_1}$ is not compact. Without loss of generality, we assume $C_{\varphi_2} - C_{\varphi_1}$ is not compact.

Since $C_{\varphi_2} - C_{\varphi_1}$ is not compact, due to Theorem 2, there exist $\varepsilon > 0$ and a sequence $\{z_n\} \subset \mathbb{B}$ such that $z_n \to \partial\mathbb{B}$ $(n \to \infty)$ and

$$\left(\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} + \frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}\right)\rho_{12}(z_n) \geqslant \varepsilon. \tag{14}$$

For each $i = 1,2,3,$ Since

$$\frac{1-|\varphi_i(z)|^2}{1-|z|^2} \geqslant \frac{1-|\varphi_i(0)|}{1+|\varphi_i(0)|}, \quad z \in \mathbb{B} \tag{15}$$

by the Schwarz-Pick Lemma (see [16, Theorem 8.1.4]) the sequence $\{\frac{1-|z_n|^2}{1-|\varphi_i(z_n)|^2}\}$ is bounded. Due to the fact that $\rho_{12} \leqslant 1$, taking ε small enough if necessary, then it follows from (14) that

$$\rho_{12}(z_n) \gtrsim \varepsilon \tag{16}$$

and

$$M_1(z_n) := \frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} \gtrsim \varepsilon$$

or

$$M_2(z_n) := \frac{1-|z_n|^2|}{1-|\varphi_2(z_n)|^2} \gtrsim \varepsilon.$$

Without loss of generality, suppose that $M_2(z_n) \gtrsim \varepsilon$ (the proof for the case $M_1(z_n) \gtrsim \varepsilon$ is similar). Then $1-|z_n|^2 \approx 1-|\varphi_2(z_n)|^2$. For $j = 1,2,3,$ note that $g_{j,n}(z) := \tau^k_{\varphi_j(z_n)}(z) = \frac{1}{(1-\langle z,\, \varphi_j(z_n)\rangle)^k}$ with $pk > \alpha + N + 1$. For $i, j = 1,2,3, i \neq j,$ put

$$x^i_{j,n} = \frac{1-|\varphi_i(z_n)|^2}{1-\langle \varphi_j(z_n), \varphi_i(z_n)\rangle}.$$

It is clear to notice that

$$\begin{aligned} |x^i_{j,n}| &= \left|\frac{1-|\varphi_i(z_n)|^2}{1-\langle \varphi_j(z_n), \varphi_i(z_n)\rangle}\right| \\ &\leqslant \frac{1-|\varphi_i(z_n)|^2}{1-|\varphi_i(z_n)|} \\ &\leqslant 2. \end{aligned} \tag{17}$$

Meanwhile, from (4) we have

$$
\begin{aligned}
|x^i_{j,n}|^2 &= \frac{(1-|\varphi_i(z_n)|^2)^2}{|1-\langle\varphi_j(z_n),\varphi_i(z_n)\rangle|^2} \\
&= (1-\rho^2_{ij}(z_n))\frac{1-|\varphi_i(z_n)|^2}{1-|\varphi_j(z_n)|^2} \\
&\leqslant \frac{1-|\varphi_i(z_n)|^2}{1-|\varphi_j(z_n)|^2}.
\end{aligned} \tag{18}
$$

By the submean value property, for $j=1,2,3,$ we have

$$
\begin{aligned}
|Tg_{j,n}(z_n)|^p &\lesssim \frac{1}{(1-|z_n|^2)^{\alpha+N+1}}\int_{D_r(z_n)}|Tg_{j,n}(z)|^p dv_\alpha(z) \\
&\leqslant \frac{\|Tg_{j,n}(z)\|^p_{A^p_\alpha}}{(1-|z_n|^2)^{\alpha+N+1}}.
\end{aligned} \tag{19}
$$

Accordingly, using the fact that $1-|z_n|^2 \approx 1-|\varphi_2(z_n)|^2$ and (18), we get that

$$
\begin{aligned}
&\frac{\|Tg_{2,n}(z)\|^p_{A^p_\alpha}}{\|g_{2,n}(z)\|^p_{A^p_\alpha}} \gtrsim (1-|\varphi_2(z_n)|^2)^{pk-\alpha-N-1}(1-|z_n|^2)^{\alpha+N+1}|Tg_{2,n}(z_n)|^p \\
&\gtrsim |a|^p\left|1-\frac{a+b}{a}\left(\frac{1-\langle\varphi_2(z_n),\varphi_2(z_n)\rangle}{1-\langle\varphi_1(z_n),\varphi_2(z_n)\rangle}\right)^k+\frac{b}{a}\left(\frac{1-\langle\varphi_2(z_n),\varphi_2(z_n)\rangle}{1-\langle\varphi_3(z_n),\varphi_2(z_n)\rangle}\right)^k\right|^p \qquad (20) \\
&\gtrsim \left[1-\left|\frac{a+b}{a}\right|\left|\frac{1-\langle\varphi_2(z_n),\varphi_2(z_n)\rangle}{1-\langle\varphi_1(z_n),\varphi_2(z_n)\rangle}\right|^{2\cdot\frac{k}{2}}-\left|\frac{b}{a}\right|\left|\frac{1-\langle\varphi_2(z_n),\varphi_2(z_n)\rangle}{1-\langle\varphi_3(z_n),\varphi_2(z_n)\rangle}\right|^{2\cdot\frac{k}{2}}\right]^p \\
&\gtrsim \left[1-\left|\frac{a+b}{a}\right|\left|\frac{1-|\varphi_2(z_n)|^2}{1-|\varphi_1(z_n)|^2}\right|^{\frac{k}{2}}-\left|\frac{b}{a}\right|\left|\frac{1-|\varphi_2(z_n)|^2}{1-|\varphi_3(z_n)|^2}\right|^{\frac{k}{2}}\right]^p. \qquad (21)
\end{aligned}
$$

We now claim that

$$
|\varphi_1| > |\varphi_2|. \tag{22}
$$

Indeed, if (22) fails, it follows from (3) and (4) that

$$
\begin{aligned}
\rho^2_{12}(z_n) &\leqslant 1-\left(\frac{(1-|\varphi_2(z_n)|^2}{|1-\langle\varphi_1(z_n),\varphi_2(z_n)\rangle|}\right)^2 \\
&= 1-|x^2_{1,n}|^2.
\end{aligned}
$$

Then

$$
|x^2_{1,n}| \leqslant 1.
$$

Therefore, if $\limsup_{n\to\infty}|x^2_{1,n}| = 1$ then

$$
\liminf_{n\to\infty}\rho_{12}(z_n) \leqslant 0
$$

which contradicts (16). Thus, we obtain $\limsup_{n\to\infty}|x_{1,n}^2| \lneqq 1$. Therefore, we deduce from (20) that $\frac{b}{a}=-1$, namely, $a+b=0$, which contradicts our assumption $a+b\neq 0$. So our claim is true.

Thus, by (22) we have that

$$\begin{aligned}\rho_{12}^2(z_n) &\leqslant 1-\left(\frac{(1-|\varphi_1(z_n)|^2}{|1-\langle\varphi_1(z_n),\varphi_2(z_n)\rangle|}\right)^2\\ &=1-|x_{2,n}^1|^2.\end{aligned}$$

If $\limsup_{n\to\infty}|x_{2,n}^1|=1$ holds, then there exists some subsequence $\{z_{n_k}\}$ such that

$$\liminf_{n\to\infty}\rho_{12}(z_{n_k})\leqslant 0$$

which contradicts (16). Thus, we obtain $\limsup_{n\to\infty}|x_{2,n}^1| \lneqq 1$.

By (7) and $M_2(z_n)\gtrsim\varepsilon$, we obtain that

$$\frac{|g_{2,n}(z)|}{\|g_{2,n}(z)\|_{A_\alpha^p}}\approx\frac{(1-|\varphi_2(z_n)|^2)^{\frac{pk-\alpha-N-1}{p}}}{|1-\langle z,\varphi_2(z_n)\rangle|^k}\lesssim\frac{(1-|z_n|^2)^{\frac{pk-\alpha-N-1}{p}}}{|1-\langle z,\varphi_2(z_n)\rangle|^k}. \tag{23}$$

Hence,

$$\frac{g_{2,n}}{\|g_{2,n}\|_{A_\alpha^p}}\to 0 \text{ uniformly on compact subsets of } \mathbb{B} \text{ as } n\to\infty.$$

Since T is compact, then we get $\frac{\|Tg_{2,n}(z)\|_{A_\alpha^p}^p}{\|g_{2,n}(z)\|_{A_\alpha^p}^p}\to 0$, $(n\to\infty)$. Therefore, due to (21) and the fact that $1-|z_n|^2\approx 1-|\varphi_2(z_n)|^2$, at least one of $M_1(z_n)$ and $M_3(z_n)$ does not converge to 0.

Assume $M_3(z_n)\nrightarrow 0$ but $M_1(z_n)\to 0$. Then $M_2(z_n), M_3(z_n)\geqslant C$ but $M_1(z_n)\to 0$ for some subsequence, which we still denote by $\{z_n\}$. Then $1-|z_n|^2\approx 1-|\varphi_3(z_n)|^2$ and $\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2}\to 0$. Similar to (23), we obtain that

$$\frac{g_{3,n}}{\|g_{3,n}\|_{A_\alpha^p}}\to 0 \text{ uniformly on compact subsets of } \mathbb{B} \text{ as } n\to\infty.$$

By (19) and $M_3(z_n)\geqslant C$, we get

$$\begin{aligned}\frac{\|Tg_{3,n}(z)\|_{A_\alpha^p}^p}{\|g_{3,n}(z)\|_{A_\alpha^p}^p}&\gtrsim(1-|\varphi_3(z_n)|^2)^{pk-\alpha-N-1}(1-|z_n|^2)^{\alpha+N+1}|Tg_{3,n}(z_n)|^p\\ &\gtrsim |b|^p\left|1-\frac{a+b}{b}\left(\frac{1-|\varphi_3(z_n)|^2}{1-\langle\varphi_1(z_n),\varphi_3(z_n)\rangle}\right)^k+\frac{a}{b}\left(\frac{1-|\varphi_3(z_n)|^2}{1-\langle\varphi_2(z_n),\varphi_3(z_n)\rangle}\right)^k\right|^p\\ &\gtrsim\left|1-\frac{a+b}{b}(x_{1,n}^3)^k+\frac{a}{b}(x_{2,n}^3)^k\right|^p.\end{aligned} \tag{24}$$

Since T is compact, we obtain $\frac{\|Tg_{3,n}(z)\|^p_{A^p_\alpha}}{\|g_{3,n}(z)\|^p_{A^p_\alpha}} \to 0$, $(n\to\infty)$. From (18), we have

$$|x^3_{1,n}|^2 \leqslant \frac{1-|\varphi_3(z_n)|^2}{1-|\varphi_1(z_n)|^2} \approx \frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} = M_1(z_n),$$

which implies $x^3_{1,n} \to 0$ as $n\to\infty$. Then, it follows from (24) that

$$1+\frac{a}{b}(x^3_{2,n})^k = 1+\frac{a}{b}\left(\frac{1-|\varphi_3(z_n)|^2}{1-\langle\varphi_2(z_n),\varphi_3(z_n)\rangle}\right)^k \to 0.$$

Note that this holds for any $pk>\alpha+N+1$ which implies $a+b=0$. This contradicts the assumption $a+b\neq 0$. Consequently, $M_1(z_n)\nrightarrow 0$. By the same argument we have a contradiction if $M_1(z_n)\nrightarrow 0$ but $M_3(z_n)\to 0$. Thus, we conclude that there exists a subsequence of $\{z_n\}$, for convenience, we still use the same notation $\{z_n\}$ such that $M_i(z_n)\geqslant C$ for all $i=1,2,3$.

Together (19) with $M_1(z_n)\geqslant C$, we have that

$$\begin{aligned}
&\frac{\|Tg_{1,n}(z)\|^p_{A^p_\alpha}}{\|g_{1,n}(z)\|^p_{A^p_\alpha}} \gtrsim (1-|\varphi_1(z_n)|^2)^{pk-\alpha-N-1}(1-|z_n|^2)^{\alpha+N+1}|Tg_{1,n}(z_n)|^p\\
&\gtrsim \left|\frac{a}{a+b}\left(\frac{1-|\varphi_1(z_n)|^2}{1-\langle\varphi_2(z_n),\varphi_1(z_n)\rangle}\right)^k+\frac{b}{a+b}\left(\frac{1-|\varphi_1(z_n)|^2}{1-\langle\varphi_3(z_n),\varphi_1(z_n)\rangle}\right)^k-1\right|^p\\
&=\left|\frac{a}{a+b}(x^1_{2,n})^k+\frac{b}{a+b}(x^1_{3,n})^k-1\right|^p.
\end{aligned}$$

Taking a subsequence of $\{z_n\}$ if necessary, we obtain that

$$\frac{g_{1,n}}{\|g_{1,n}\|_{A^p_\alpha}} \to 0 \text{ uniformly on compact subsets of } \mathbb{B} \text{ as } n\to\infty.$$

Since T is compact, we have $\frac{\|Tg_{1,n}(z)\|^p_{A^p_\alpha}}{\|g_{1,n}(z)\|^p_{A^p_\alpha}} \to 0$, $(n\to\infty)$ for all $pk>\alpha+N+1$. Then

$$\frac{a}{a+b}(x^1_{2,n})^k+\frac{b}{a+b}(x^1_{3,n})^k-1\to 0.$$

Since $\limsup_{n\to\infty}|x^1_{2,n}|\lneqq 1$, it follows that $\frac{b}{a+b}-1=0$. But then $a=0$ which contradicts our assumption. Therefore, the compactness of $T=a(C_{\varphi_2}-C_{\varphi_1})+b(C_{\varphi_3}-C_{\varphi_1})$ implies that both $C_{\varphi_2}-C_{\varphi_1}$ and $C_{\varphi_3}-C_{\varphi_1}$ are compact. We completes the proof. □

Especially, the following useful corollary shows that there is no cancellation property for the compactness of double difference of composition operators under a suitable restriction on inducing maps, which is automatically satisfied in the case of the disk [13].

COROLLARY 1. *Let $\alpha > \beta > -1$, and $0 < p, q < \infty$. Suppose that C_{φ_i} is bounded on $A^q_\beta(\mathbb{B})$ and not compact on $A^p_\alpha(\mathbb{B})$ for each $i = 1,2,3$. Then, $T := (C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ is compact on $A^p_\alpha(\mathbb{B})$ if and only if both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact on $A^p_\alpha(\mathbb{B})$.*

3.2. Difference of weighted composition operators

Recently, Jiang-Stevic in [11], characterized the compact differences of two weighted composition operators from the weighted Bergman space to the weighted type space in the unit disk. Motivated by the idea of [11], in this section, we discuss the compactness and essential norm estimate of the differences of weighted composition operators between the weighted Bergman spaces on the unit ball.

THEOREM 7. *Let $0 < p, q < \infty$, α, $\beta > -1$, φ_1, $\varphi_2 \in S$ and u_1, $u_2 \in H(\mathbb{B})$. Assume that $u_1 C_{\varphi_1}, u_2 C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ are bounded. The operator $u_1 C_{\varphi_1} - u_2 C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ is compact, then the following conditions hold*

$$\lim_{|\varphi_1(z)|\to 1} \frac{|u_1(z)|(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z)|^2)^{\frac{\alpha+N+1}{p}}} \rho_{12}(z) = 0; \tag{25}$$

$$\lim_{|\varphi_2(z)|\to 1} \frac{|u_2(z)|(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_2(z)|^2)^{\frac{\alpha+N+1}{p}}} \rho_{12}(z) = 0; \tag{26}$$

$$\lim_{|\varphi_1(z)|,|\varphi_2(z)|\to 1} \left| \frac{u_1(z)(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z)|^2)^{\frac{\alpha+N+1}{p}}} - \frac{u_2(z)(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_2(z)|^2)^{\frac{\alpha+N+1}{p}}} \right| = 0. \tag{27}$$

Proof. Let $\{z_n\}$ be a sequence of points in $\mathbb{B}$ such that $\varphi_1(z_n) \to 1$ as $n \to \infty$. Define the test function

$$f_n(z) = \frac{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}{(1-\langle z, \varphi_1(z_n)\rangle)^{\frac{2(\alpha+N+1)}{p}}} \cdot \frac{\langle \sigma_{\varphi_2(z_n)}(z), \sigma_{\varphi_2(z_n)}(\varphi_1(z_n))\rangle}{|\sigma_{\varphi_2(z_n)}(\varphi_1(z_n))|}$$

when $\varphi_1(z_n) \neq \varphi_2(z_n)$. Note that

$$f_n(\varphi_1(z_n)) = \frac{\rho_{12}(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}, \quad f_n(\varphi_2(z_n)) = 0. \tag{28}$$

We can easily prove that $f_n(z) \in A^p_\alpha$ with $\|f_n\|_{A^p_\alpha} \leqslant 1$ for all n. Clearly, f_n converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $n \to \infty$. By the compactness of $u_1 C_{\varphi_1} - u_2 C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ and Lemma 2, it follows that $\|(u_1 C_{\varphi_1} - u_2 C_{\varphi_2}) f_n\|_{A^q_\beta} \to 0$, $n \to \infty$. On the other hand, by the submean value property and (28) we have

$$\begin{aligned}\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})f_n\|_{A^q_\beta} &\geqslant \left(\int_{D_r(z_n)}|(u_1C_{\varphi_1}-u_2C_{\varphi_2})f_n(z)|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\ &\gtrsim (1-|z_n|^2)^{\frac{\beta+N+1}{q}}|(u_1C_{\varphi_1}-u_2C_{\varphi_2})f_n(z_n)|\\ &=(1-|z_n|^2)^{\frac{\beta+N+1}{q}}|u_1(z_n)f_n(\varphi_1(z_n))-u_2(z_n)f_n(\varphi_2(z_n))|\\ &=\frac{|u_1(z_n)|(1-|z_n|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}\rho_{12}(z_n).\end{aligned}\tag{29}$$

Letting $n\to\infty$ in (29), it follows that (25) holds. The condition (26) holds by similar arguments.

Now we need only show that the condition (27) holds. Assume that $\{z_n\}$ is a sequence of points in $\mathbb{B}$ such that $|\varphi_1(z_n)|\to 1$ and $|\varphi_2(z_n)|\to 1$ as $n\to\infty$. Take the test function

$$g_n(z)=\frac{(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}}}{(1-\langle z,\varphi_2(z_n)\rangle)^{\frac{2(\alpha+N+1)}{p}}}.$$

Note that

$$g_n(\varphi_2(z_n))=\frac{1}{(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}}}.\tag{30}$$

It is easy to check that $g_n(z)$ converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $n\to\infty$ and $g_n(z)\in A^p_\alpha$ with $\|g_n\|_{A^p_\alpha}\leqslant 1$ for all n. By Lemma 2, we obtain that $\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})g_n\|_{A^q_\beta}\to 0$, $n\to\infty$. Then by the submean value property and (30) we have

$$\begin{aligned}\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})g_n\|_{A^q_\beta} &\geqslant \left(\int_{D_r(z_n)}|(u_1C_{\varphi_1}-u_2C_{\varphi_2})g_n(z)|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\ &\gtrsim (1-|z_n|^2)^{\frac{\beta+N+1}{q}}|(u_1C_{\varphi_1}-u_2C_{\varphi_2})g_n(z_n)|\\ &=(1-|z_n|^2)^{\frac{\beta+N+1}{q}}|u_1(z_n)g_n(\varphi_1(z_n))-u_2(z_n)g_n(\varphi_2(z_n))|\\ &=|I(z_n)+J(z_n)|\end{aligned}\tag{31}$$

where

$$\begin{aligned}I(z_n)&=(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}}g_n(\varphi_2(z_n))\left[\frac{u_1(z_n)(1-|z_n|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}-\frac{u_2(z_n)(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}}}\right]\\ &=\frac{u_1(z_n)(1-|z_n|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}-\frac{u_2(z_n)(1-|z|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}}}\end{aligned}$$

and

$$J(z_n) = \frac{u_1(z_n)(1-|z_n|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}\big[(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}} g_n(\varphi_1(z_n)) - (1-|\varphi_2(z_n)|^2)^{\frac{\alpha+N+1}{p}} g_n(\varphi_2(z_n))\big].$$

By Lemma 1 and the condition (25) that has been proved, we obtain

$$|J(z_n)| \lesssim \|g_n\|_{A^p_\alpha} \frac{|u_1(z_n)|(1-|z_n|^2)^{\frac{\beta+N+1}{q}}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+N+1}{p}}}\rho_{12}(z_n) \to 0,\ |\varphi_1(z_n)| \to 1. \tag{32}$$

Together (31) with (32), we obtain that $I(z_n) \to 0$ as $n \to \infty$. This shows that (27) holds. The proof is complete. $\square$

THEOREM 8. *Let* $0 < p,q < \infty$, $\alpha,\ \beta > -1$, $\varphi_1,\ \varphi_2 \in S$ *and* $u_1,\ u_2 \in H(\mathbb{B})$. *Assume that* $u_1C_{\varphi_1},\ u_2C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ *are bounded. Then the operator* $u_1C_{\varphi_1} - u_2C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ *is compact, if the following conditions hold*

(a) $$\lim_{r\to 1}\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z) = 0;$$

(b) $$\lim_{r\to 1}\int_{|\varphi_2(z)|>r}\left|\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z) = 0;$$

(c) $$\lim_{r\to 1}\int_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z) = 0.$$

Proof. Suppose that the conditions (a)–(c) hold. For any $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z) \leqslant \varepsilon; \tag{33}$$

$$\int_{|\varphi_2(z)|>r}\left|\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z) \leqslant \varepsilon; \tag{34}$$

$$\int_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z) \leqslant \varepsilon. \tag{35}$$

Now, let $\{f_n\}$ be a sequence in A^p_α such that $\|f_n\|_{A^p_\alpha} \leqslant 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{B}$. By using Lemma 2, we only need to show that

$$\|(u_1C_{\varphi_1} - u_2C_{\varphi_2})f_n\|_{A^q_\beta} \to 0 \text{ as } n \to \infty.$$

Since

$$
\begin{aligned}
&\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})f_n\|_{A^q_\beta}\\
=&\left(\int_{\mathbb{B}}|u_1(z)f_n(\varphi_1(z))-u_2(z)f_n(\varphi_2(z))|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
\lesssim&\left(\int_{|\varphi_1(z)|>r}|u_1(z)f_n(\varphi_1(z))-u_2(z)f_n(\varphi_2(z))|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
&+\left(\int_{|\varphi_2(z)|>r}|u_1(z)f_n(\varphi_1(z))-u_2(z)f_n(\varphi_2(z))|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
&+\left(\int_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\leqslant r}|u_1(z)f_n(\varphi_1(z))-u_2(z)f_n(\varphi_2(z))|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
=:&\ I_{n,1}(r)+I_{n,2}(r)+I_{n,3}(r).
\end{aligned}
$$

We first estimate $I_{n,1}(r)$ and $I_{n,2}(r)$. By (5) and Lemma 1, we have that

$$
\begin{aligned}
I_{n,1}(r)=&\left(\int_{|\varphi_1(z)|>r}|u_1(z)f_n(\varphi_1(z))-u_2(z)f_n(\varphi_2(z))|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
\lesssim&\left(\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}[(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}f_n(\varphi_1(z))\right.\right.\\
&\left.\left.-(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}f_n(\varphi_2(z))]\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}}+\left(\int_{|\varphi_1(z)|>r}\left|\left[\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\right.\right.\right.\\
&\left.\left.\left.-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right](1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}f_n(\varphi_2(z))\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
\lesssim&\left(\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}}\\
&+\left(\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}}.
\end{aligned}
$$

We divide it into two subcases:

Case 1: $|\varphi_2(z)|\leqslant r$. Note that $\{f_n\}$ converges to zero uniformly on $E=\{w: |w|\leqslant r\}$ as $n\to\infty$. It follows from the boundedness of $u_iC_{\varphi_i}: A^p_\alpha\to A^q_\beta$ for $i=1,2$ that

$$
\begin{aligned}
\int_{|\varphi_1(z)|>r}&\left|\left(\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right)\right.\\
&\left.\times(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}f_n(\varphi_2(z))\right|^q d\nu_\beta(z)\lesssim\varepsilon.
\end{aligned}
$$

Thus, by (34), we obtain that

$$\lim_{n\to\infty} I_{n,1}(r) \lesssim \left(\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}} \lesssim \varepsilon. \tag{36}$$

Case 2: $|\varphi_2(z)|>r$. By (33) and (35) we have

$$\lim_{n\to\infty} I_{n,1}(r) \lesssim \left(\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}}$$
$$+\left(\int_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z)\right)^{\frac{1}{q}} \lesssim \varepsilon. \tag{37}$$

Then, by (36) and (37), we obtain that $\lim_{n\to\infty} I_{n,1}(r)=0$. By a similar argument, we have $\lim_{n\to\infty} I_{n,2}(r)=0$.

We now estimate $I_{n,3}(r)$. By the assumption, $u_1C_{\varphi_1}, u_2C_{\varphi_2}: A^p_\alpha \to A^q_\beta$ are bounded, we have that $u_1\in A^q_\beta$ and $u_2\in A^q_\beta$. Note that $\{f_n\}$ converges to zero uniformly on $E=\{w:|w|\leqslant r\}$ as $n\to\infty$, thus it is easy to check that $I_{n,3}(r)\to 0, n\to\infty$ uniformly for all z with $|\varphi_1(z)|\leqslant r$ and $|\varphi_2(z)|\leqslant r$.

Together with the above estimates, we conclude that $\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})f_n\|_{A^q_\beta}\lesssim\varepsilon$ for sufficiently large n which implies that $u_1C_{\varphi_1}-u_2C_{\varphi_2}: A^p_\alpha\to A^q_\beta$ is compact. □

The following gives an essential norm estimate for the difference of weighted composition operators.

THEOREM 9. *Let* $0<p,q<\infty$, α, $\beta>-1$, φ_1, $\varphi_2\in S$ *and* u_1, $u_2\in H(\mathbb{B})$. *Assume that* $u_1C_{\varphi_1}$, $u_2C_{\varphi_2}: A^p_\alpha\to A^q_\beta$ *are bounded. Then we have the following estimate:*

$$\|u_1C_{\varphi_1}-u_2C_{\varphi_2}\|^q_{e,A^p_\alpha\to A^q_\beta}\lesssim \max\{\text{(i)},\text{(ii)},\text{(iii)}\}$$

where

(i) : $\lim\limits_{r\to 1}\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)$;

(ii) : $\lim\limits_{r\to 1}\int_{|\varphi_2(z)|>r}\left|\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)$;

(iii) : $\lim\limits_{r\to 1}\int_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z)$.

Proof. Consider the operators on $H(\mathbb{B})$ defined by

$$P_k(f)(z)=f\left(\frac{k}{k+1}z\right),\ k\in\mathbb{N}.$$

It is well known that they are continuous with respect to the compact open topology and that $P_k(f) \to f$ pointwise on compact sets of $\mathbb{B}$ as $k \to \infty$. Since the integral

$$M_p(f,r) = \left(\int_{\partial\mathbb{B}} |f(r\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}}$$

increases in r and

$$\int_{\mathbb{B}} f(z)dv(z) = 2n \int_0^1 r^{2n-1} dr \int_{\partial\mathbb{B}} f(r\xi)d\sigma(\xi),$$

it follows that

$$\begin{aligned}
\|P_k(f)\|_{A^p_\alpha}^p &= \int_{\mathbb{B}} |P_k f(z)|^p d\nu_\alpha(z) \\
&= \int_{\mathbb{B}} c_\alpha |f(\frac{k}{k+1}z)|^p (1-|z|^2)^\alpha d\nu(z) \\
&= c_\alpha 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha dr \int_{\partial\mathbb{B}} |f(r\frac{k}{k+1}\xi)|^p d\sigma(\xi) \\
&\leqslant c_\alpha 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha dr \int_{\partial\mathbb{B}} |f(r\xi)|^p d\sigma(\xi) \\
&= \|f\|_{A^p_\alpha}^p.
\end{aligned}$$

Thus, we obtain that $\|P_k(f)\|_{A^p_\alpha} \leqslant \|f\|_{A^p_\alpha}$, $k \in \mathbb{N}$, which means that $\sup_{k\in\mathbb{N}} \|P_k\|_{A^p_\alpha \to A^p_\alpha} \leqslant 1$.

Moreover, it is clear that the operators $(P_k)_{k\in\mathbb{N}}$ are also compact on A^p_α. Let $f \in A^p_\alpha$ such that $\|f\|_{A^p_\alpha} \leqslant 1$. Put

$$g_k := (I - P_k)f,\ k \in \mathbb{N}.$$

Then clearly $g_k \in A^p_\alpha$, $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} \|g_k\|_{A^p_\alpha} \leqslant 2$.

It is easy to see that for each $f \in H(\mathbb{B})$, $\lim_{k\to\infty}(I-P_k)f = 0$ and the space $H(\mathbb{B})$ endowed with compact open topology c_0 is a Fréchet space. Therefore, it follows by Banach-Steinhaus theorem, $(I-P_k)f$ converges to zero uniformly on compact sets of $(H(\mathbb{B}), c_0)$. Let $r \in (0,1)$ be fixed. Since the unit ball A^p_α is a compact subset of $(H(\mathbb{B}), c_0)$, then

$$\lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{|\xi|\leqslant r} |(I-P_k)(f)(\xi)| = 0. \tag{38}$$

We have that

$$\begin{aligned}
&\|u_1C_{\varphi_1} - u_2C_{\varphi_2}\|_{e,A^p_\alpha \to A^q_\beta}^q \\
&\leqslant \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \|(u_1C_{\varphi_1} - u_2C_{\varphi_2})g_k\|_{A^q_\beta}^q \\
&\leqslant \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \int_{|\varphi_1(z)|>r} |u_1(z)g_k(\varphi_1(z)) - u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z)
\end{aligned}$$

$$\begin{aligned}
&+\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\int_{|\varphi_2(z)|>r}|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z)\\
&+\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\int_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\leqslant r}|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z)\\
=\,&I_{k,1}(r)+I_{k,2}(r)+I_{k,3}(r).
\end{aligned}$$

First we estimate $I_{k,1}(r)$. By (5) and Lemma 1, with the fact that $\sup_{k\in\mathbb{N}}\|g_k\|_{A^p_\alpha}\leqslant 2,$ we obtain

$$\begin{aligned}
&\int_{|\varphi_1(z)|>r}|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z)\\
&\lesssim\int_{|\varphi_1(z)|>r}\Bigg|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\Big[(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}g_k(\varphi_1(z))\\
&\quad-(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}g_k(\varphi_2(z))\Big]\Bigg|^q d\nu_\beta(z)\\
&\quad+\int_{|\varphi_1(z)|>r}\Bigg|(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}g_k(\varphi_2(z))\\
&\quad\times\left[\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right]\Bigg|^q d\nu_\beta(z) \qquad (39)\\
&\lesssim\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)\\
&\quad+2\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z). \qquad (40)
\end{aligned}$$

By the boundedness of $u_1C_{\varphi_1}$, $u_2C_{\varphi_2}:A^p_\alpha\to A^q_\beta$ and (38), (39), for each $r\in(0,1)$ and $|\varphi_2(z)|\leqslant r$, we have that

$$\limsup_{k\to\infty}I_{k,1}(r)\lesssim\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z). \qquad (41)$$

On the other hand, if $|\varphi_2(z)|>r,$ then from (40) we have

$$\begin{aligned}
&\limsup_{k\to\infty}I_{k,1}(r)\\
&\lesssim\int_{|\varphi_1(z)|>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho_{12}(z)\right|^q d\nu_\beta(z)\\
&\quad+\int_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|^q d\nu_\beta(z). \qquad (42)
\end{aligned}$$

Letting $r\to 1$ in (41) and (42), we can obtain that $\limsup_{r\to 1}\limsup_{k\to\infty}I_{k,1}(r)\lesssim\max\{\text{(i)},\text{(iii)}\}$.

Similarly, we have that $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,2}(r) \lesssim \max\{(\text{ii}\,),(\text{iii})\}$.

Since the operators $u_1C_{\varphi_1}$, $u_2C_{\varphi_2} : A^p_\alpha \to A^q_\beta$ are bounded, choosing $f(z) = 1 \in A^p_\alpha$, it follows that $u_1 \in A^q_\beta$ and $u_2 \in A^q_\beta$. From (38) that $\{g_k\}$ converges to zero uniformly on $E = \{w : |w| \leqslant r\}$ as $k \to \infty$, then

$$\begin{aligned}
&\lim_{k\to\infty} I_{k,3}(r)\\
&= \lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \int_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\leqslant r} |u_1(z)g_k(\varphi_1(z)) - u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z)\\
&\leqslant \lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \int_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\leqslant r} |u_1(z)g_k(\varphi_1(z))|^q d\nu_\beta(z)\\
&\quad + \lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \int_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\leqslant r} |u_2(z)g_k(\varphi_2(z))|^q d\nu_\beta(z) = 0.
\end{aligned}$$

Thus, $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,3}(r) = 0$, which completes the proof. □

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REFERENCES

[1] H. CHO, B. CHOE AND H. KOO, *Linear combinations of composition operators on the Fock-Sobolev spaces*, Potential Anal. **41**, 4 (2014), 1223–1246.

[2] B. CHOE, H. KOO AND I. PARK, *Compact differences of composition operators over polydisks*, Integral Equations Operator Theory **73**, 1 (2012), 57–91.

[3] B. CHOE, H. KOO AND I. PARK, *Compact differences of composition operators on the Bergman spaces over the ball*, Potential Anal. **40**, 1 (2014), 81–102.

[4] B. CHOE, H. KOO, M. WANG AND J. YANG, *Compact linear combinations of composition operators induced by linear fractional maps*, Math. Z. **280**, 3–4 (2015), 807–824.

[5] B. CHOE, H. KOO AND M. WANG, *Compact double differences of composition operators on the Bergman spaces*, J. Funct. Anal. **272**, 6 (2017), 2273–2307.

[6] C. COWEN AND B. MACCLUER, *Composition operators on spaces of analytic functions*, CRC Press, New York, 1995.

[7] X. GUO, M. WANG, *Difference of weighted composition operators on the space of Cauchy integral transforms*, Taiwanese J. Math. **22**, 6 (2018), 1435–1450.

[8] K. HELLER, B. MACCLUER AND P. WEIR, *Compact differences of composition operators in several variables*, Integral Equations Operator Theory **69**, 2 (2011), 247–268.

[9] T. HOSOKAWA AND S. OHNO, *Differences of composition oprtators on the Bloch spaces*, J. Operator Theory **57**, 2 (2007), 229–242.

[10] L. JIANG AND C. OUYANG, *Compact differences of composition operators on holomorphic function spaces in the unit ball*, Acta Math. Sci. Ser. B Engl. Ed. **31**, 5 (2011), 1679–1693.

[11] Z. JIANG AND S. STEVIĆ, *Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces*, Appl. Math. Comput. **217**, 7 (2010), 3522–3530.

[12] H. KOO AND M. WANG, *Joint Carleson measure and the difference of composition operatos on* $A^p_\alpha(\mathbb{B}_n)$, J. Math. Anal. Appl. **419**, 2 (2014), 1119–1142.

[13] H. KOO AND M. WANG, *Cancellation properties of composition operators on Bergman spaces*, J. Math. Anal. Appl. **432**, 2 (2015), 1174–1182.

[14] T. KRIETE AND J. MOORHOUSE, *Linear relations in the Calkin algebra for composition operators*, Trans. Amer. Math. Soc. **359**, 6 (2007), 2915–2944.

[15] J. MOORHOUSE, *Compact difference of composition operators*, J. Funct. Anal. **219**, 1 (2005), 70–92.

[16] W. RUDIN, *Function theory in the unit ball of C^n*, Springer-Verlag, New York-Berlin, 1980.

[17] E. SAUKKO, *Difference of composition operators between standard weighted Bergman spaces*, J. Math. Anal. Appl. **381**, 2 (2011), 789–798.

[18] E. SAUKKO, *An application of atomic decomposition in Bergman spaces to the study of differences of composition operators*, J. Funct. Anal. **262**, 9 (2012), 3872–3890.

[19] J. SHARPIRO, *Composition operators and classical function theory*, Springer-Verlag, New York, 1993.

[20] S. STEVIĆ AND Z. JIANG, *Compactness of the difference of weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball*, Taiwanese J. Math. **15**, 6 (2011), 2647–2665.

[21] M. WANG, X. GUO, *Difference of generalized integration operators on the space of Cauchy transforms*, Complex Var. Elliptic Equ., https://doi.org/10.1080/17476933.2018.1450394.

[22] M. WANG, X. GUO, *Difference of differentiation composition operators on the fractional Cauchy transforms spaces*, Numer. Funct. Anal. Optim., https://doi.org/10.1080/01630563.2018.1477798.

[23] M. WANG AND C. PANG, *Compact double differences of composition operators over the half-plane*, Complex Anal. Oper. Theory **12**, 1 (2018), 261–292.

[24] K. ZHU, *Spaces of holomorphic functions in the unit ball*, Springer-Verlag, New York, 2005.

[25] K. ZHU, *Compact compositions operators on Bergman spaces of the unit ball*, Houston J. Math. **33**, 1 (2007), 273–283.

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