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ESSENTIAL NORM OF THE DIFFERENTIAL OPERATOR

TESFA MENGESTIE

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Abstract. This paper is a follow-up contribution to our work [10] where we studied some spectral properties of the differential operator D acting between generalized Fock spaces $\mathscr{F}_{(m,p)}$ and $\mathscr{F}_{(m,q)}$ when both exponents p and q are finite. In this note we continue to study the properties for the case when at least one of the spaces is growth type. We also estimate the essential norm of $D:\mathscr{F}_{(m,p)} \to \mathscr{F}_{(m,q)}$ for all $1 \leqslant p,q \leqslant \infty$, and showed that if the operator fails to be compact, then its essential norm is comparable to the operator norm and $\|D\|_e \simeq \left|m^{2+p} - m^{1+p}\right|^{\frac{1}{p}} \simeq \|D\|$.

1. Introduction

The differential operator $Df = f'$ is one of the fundamental operators in function related operator theory. However, the operator is known to act in a discontinuous fashion on many Banach spaces including on the classical Fock spaces, weighted Fock spaces where the weight decays faster than the classical Gaussian weight [11], on Fock–Sobolev spaces which are typical examples of weighted Fock spaces where the weight decays slower than the Gaussian weight [13, 14]. In light of this, we studied the question of how slower must the weight function decay on generalized Fock spaces under which the operator D admits some basic spectral structures [10], and found out that the weight should in fact decay much slower than the classical Gaussian weight $e^{-|z|^2}$. More precisely, for $m > 0$, we considered a class of generalized Fock spaces $\mathscr{F}_{(m,p)}$ which consist of all entire functions f for which

$$\|f\|_{(m,p)}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^m} dA(z) < \infty,$$

where dA denotes the usual Lebesgue area measure on $\mathbb{C}$. Then for $0 < p \leqslant q < \infty$, it was proved that $D:\mathscr{F}_{(m,p)} \to \mathscr{F}_{(m,q)}$ is bounded if and only if

$$m \leqslant 2 - \frac{pq}{pq+q-p} \tag{1.1}$$

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and in this case the norm is estimated as

$$\|D\| \simeq \begin{cases} |m^{2+p}-m^{1+p}|^{\frac{1}{p}} \sup_{w\in\mathbb{C}} \big(1+|w|\big)^{(m-1)+\frac{(q-p)(m-2)}{qp}}, & m\neq 1.\\ 1, & m=1 \end{cases} \tag{1.2}$$

Compactness has been described by the strict inequality (1.1) while the corresponding equivalent condition for the case when $p>q$ has been found to be

$$m<1-2\Big(\frac{1}{q}-\frac{1}{p}\Big) \tag{1.3}$$

which is yet stronger than (1.1) and in addition, forces boundedness to imply compactness.

One of the main purposes of this note is to study the situation when one of the Fock type spaces $\mathscr{F}_{(m,p)}$ is replaced by the natural growth type space $\mathscr{F}_{(m,\infty)}$ which consist of entire functions f for which

$$\|f\|_{(m,\infty)}=\sup_{z\in\mathbb{C}}|f(z)|e^{-|z|^m}<\infty.$$

For this, our first result below shows that the weight function $|z|^m$ can grow at most as a complex polynomial of degree not exceeding $2-\frac{p}{p+1}$.

THEOREM 1.

(i) Let $0<p<\infty$ and $m>0$. Then $D:\mathscr{F}_{(m,p)}\to\mathscr{F}_{(m,\infty)}$ is

(a) bounded if and only if

$$m\leqslant 2-\frac{p}{p+1} \tag{1.4}$$

and the norm is estimated by

$$\|D\| \overset{1}{\simeq} \begin{cases} |m^{2+p}-m^{1+p}|^{\frac{1}{p}} \sup_{w\in\mathbb{C}}(1+|w|)^{\frac{m(p+1)-(p+2)}{p}}, & m\neq 1\\ 1, & m=1 \end{cases}. \tag{1.5}$$

(b) compact if and only if

$$m<2-\frac{p}{p+1}.$$

(ii) Let $0<p<\infty$ and $m>0$. Then the following statements are equivalent.

(a) $D:\mathscr{F}_{(m,\infty)}\to\mathscr{F}_{(m,p)}$ is bounded;

[1]The notation $U(z)\lesssim V(z)$ (or equivalently $V(z)\gtrsim U(z)$) means that there is a constant C such that $U(z)\leqslant CV(z)$ holds for all z in the set of a question. We write $U(z)\simeq V(z)$ if both $U(z)\lesssim V(z)$ and $V(z)\lesssim U(z)$.

(b) $D:\mathscr{F}_{(m,\infty)}\to\mathscr{F}_{(m,p)}$ *is compact;*

(c) It holds that

$$m<1-\frac{2}{p}. \tag{1.6}$$

Before going further, we want to remark that the study in [10] was initiated in a quest for answering the question of how fast should the associated weight function on generalized Fock spaces decay in order that the operator D admits some basic spectral structures. Now Theorem 1 and the corresponding result in [10] provide a clear description for its decay, namely that the weight should decay in all cases much slower than the classical Gaussian weight as precisely specified in (1.1), (1.3), (1.4), and (1.6). On the other hand, when $p=q=\infty$, as can be seen from (1.4), $D:\mathscr{F}_{(m,\infty)}\to\mathscr{F}_{(m,\infty)}$ is bounded if and only if $m\leqslant 1$ and compactness is described by the strict inequality $m<1$. This particular case follows also from many other related works for example in [1, 8].

Another purpose of this note is to estimate the essential norm of the differential operator on the generalized Fock spaces $\mathscr{F}_{(m,p)}$. Recall that for two Banach spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ the essential norm $\|T\|_e$ of a bounded linear operator $T:\mathscr{H}_1\to\mathscr{H}_2$ is defined as the distance from T to the space of compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$:

$$\|T\|_e=\inf_K\big\{\|T-K\|;\ \ K:\mathscr{H}_1\to\mathscr{H}_2 \text{ is a compact operator }\big\}. \tag{1.7}$$

In particular, (1.7) implies that T is compact if and only if its essential norm vanishes. Thus, the essential norm can be interpreted as a quantity that provides a useful measure for the noncompactness of operators. Several authors have studied such norms on various functional spaces including the Hardy spaces, Bergaman spaces, and Fock spaces; see for example [6, 7, 12, 17, 18, 19, 20]. We prove the following estimates for D acting between the spaces $\mathscr{F}_{(m,p)}$.

THEOREM 2. *Let* $1\leqslant p\leqslant q\leqslant\infty$ *and* $D:\mathscr{F}_{(m,p)}\to\mathscr{F}_{(m,q)}$ *is bounded. Then*

$$\|D\|_e\simeq\begin{cases}|m^{2+p}-m^{1+p}|^{\frac{1}{p}}, & q<\infty \text{ and } m=2-\frac{pq}{pq+q-p}\\ 1, & q=\infty=p,\ \text{and } m=1\\ |m^{2+p}-m^{1+p}|^{\frac{1}{p}}, & q=\infty,\ p<\infty,\ \text{and } m=2-\frac{p}{p+1}\\ 0, & \text{otherwise.}\end{cases} \tag{1.8}$$

We observe that if D fails to be compact, then from Theorem 2 and the relations in (1.2) and (1.5), its essential norm can be simply estimated by $\|D\|_e\simeq|m^{2+p}-m^{1+p}|^{\frac{1}{p}}\simeq\|D\|$.

2. Preliminaries

In this section we collect some basic facts and preliminary results that will be used in the sequel. One of the important ingredients needed is the Littlewood–Paley type estimate from [4],

$$\|f\|_{(m,p)}^p \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}} dA(z), \tag{2.1}$$

which holds for all functions $f \in \mathscr{F}_{(m,p)}$. On the other hand, for $p = \infty$, from a simple modification of the arguments used in the proof of Lemma 2.1 of [11], it follows that f belongs to the spaces $\mathscr{F}_{(m,\infty)}$ if and only if

$$\sup_{z\in\mathbb{C}} \frac{|f'(z)|e^{-|z|^m}}{(1+|z|)^{m-1}} < \infty,$$

and in this case we estimate the norm by

$$\|f\|_{(m,\infty)} \simeq |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|e^{-|z|^m}}{(1+|z|)^{m-1}}. \tag{2.2}$$

Several properties of linear operators can often be described by their action on some special elements in the spaces. The reproducing kernels do often used for such purpose in many functional spaces. Since an explicit expression for the kernels in our current setting is still unknown, we will use another sequence of special test functions. Such a sequence was first constructed in [3] and has been since then used by several authors for example [5, 11, 16]. We introduce the sequence as follows. We may first set

$$\tau_m(z) = \begin{cases} 1, & 0 \leqslant |(m^2-m)z| < 1 \\ \frac{|z|^{\frac{2-m}{2}}}{|m^2-m|^{\frac{1}{2}}}, & |(m^2-m)z| \geqslant 1. \end{cases}$$

Then, for a sufficiently large positive number R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w,R)}$ such that

$$|f_{(w,R)}(z)|e^{-|z|^m} \leqslant C \min\left\{1, \left(\frac{\min\{\tau_m(w), \tau_m(z)\}}{|z-w|}\right)^{\frac{R^2}{2}}\right\} \tag{2.3}$$

for all $z \in \mathbb{C}$ and for some constant C that depends on $|z|^m$ and R. In particular when $z \in D(w, R\tau_m(w))$, the estimate becomes

$$|f_{(w,R)}(z)|e^{-|z|^m} \simeq 1, \tag{2.4}$$

where $D(w,r)$ denotes the Euclidean disk centered at w and radius $r > 0$. In addition, $f_{(w,R)}$ belongs to $\mathscr{F}_{(m,p)}$ with norm estimated by

$$\|f_{(w,R)}\|_{(m,p)}^p \simeq \tau_m^2(w), \quad \eta(R) \leqslant |w| \tag{2.5}$$

for all p in the range $0<p<\infty$. On the other hand, when $p=\infty$, from (2.3) and (2.4), we easily deduce that

$$\|f_{(w,R)}\|_{(m,\infty)} \simeq 1. \tag{2.6}$$

Another important fact is the pointwise estimate for subharmonic functions f, namely

$$|f(z)|^p e^{-p|z|^m} \lesssim \frac{1}{\sigma^2 \tau_m^2(z)} \int_{D(z,\sigma\tau_m(z))} |f(w)|^p e^{-p|w|^m} dA(w) \tag{2.7}$$

for all finite exponent p and a small positive number σ. The estimate follows from Lemma 2 of [16].

Next, we recall the following covering lemma which is essentially from [5, 15].

LEMMA 1. *Let τ_m be as above. Then, there exists a positive $\sigma>0$ and a sequence of points z_j in $\mathbb{C}$ satisfying the following conditions.*

(i) $z_j \notin D(z_k, \sigma\tau_m(z_k)),\ \ j\neq k$;

(ii) $\mathbb{C}=\bigcup_j D(z_j,\sigma\tau_m(z_j))$;

(iii) $\bigcup_{z\in D(z_j,\sigma\tau_m(z_j))} D(z,\sigma\tau_m(z)) \subset D(z_j, 3\sigma\tau_m(z_j))$;

(iv) The sequence $D(z_j,3\sigma\tau_m(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity $N_{\max}$.

The composition operator $C_\Phi f = f(\Phi)$ is one of the classical and well studied objects in function related operator theories. Our next result on C_Φ describes its ompactness property while acting on the spaces $\mathscr{F}_{(m,p)}$. The result will not only play a vital role to prove our second main result in the previous section but also is interest of its own.

PROPOSITION 3. Let $0<p\leqslant q\leqslant\infty$ and Φ be a nonconstant entire function on $\mathbb{C}$. Then the composition operator $C_\Phi : \mathscr{F}_{(m,p)} \to \mathscr{F}_{(m,q)}$ is compact if and only if $\Phi(z)=az+b$ for some complex numbers a and b such that $|a|<1$.

Proof. We begin with the proof of the necessity and assume that C_Φ is compact. We also bserve that the normalized sequence

$$f^*_{(w,R)} = \frac{f_{(w,R)}}{\|f_{(w,R)}\|_{(m,p)}} \simeq \begin{cases} \frac{f_{(w,R)}}{\tau_m(w)^{\frac{2}{p}}}, & 1\leqslant p<\infty \\ f_{(w,R)}, & p=\infty, \end{cases} \tag{2.8}$$

as described from (2.3)-(2.6), converges to zero as $|w|\to\infty$, and the convergence is uniform on compact subset of $\mathbb{C}$. Now if $0<p<q=\infty$, then our assumption and C_Φ applied to the sequence $f^*_{(w,R)}$ imply

$$0=\lim_{|w|\to\infty} \|C_\Phi f^*_{(w,R)}\|_{(m,\infty)} \simeq \lim_{|w|\to\infty} \sup_{z\in\mathbb{C}} \frac{|f_{(w,R)}(\Phi(z))|}{\tau_m(w)^{\frac{2}{p}} e^{|z|^m}} \geqslant \lim_{|w|\to\infty} \frac{\left|f_{(w,R)}(\Phi(z))\right|}{\tau_m(w)^{\frac{2}{p}} e^{|z|^m}}$$

for all $z,w\in\mathbb{C}$. In particular, setting $w=\Phi(z)$ and applying (2.4) give

$$\begin{aligned} 0 &= \lim_{|\Phi(z)|\to\infty}\left|f_{(\Phi(z),R)}(\Phi(z))\right|e^{-|\Phi(z)|^m}\frac{e^{|\Phi(z)|^m-|z|^m}}{\tau_m^{\frac{2}{p}}(\Phi(z))}\simeq \lim_{|\Phi(z)|\to\infty}\frac{e^{|\Phi(z)|^m-|z|^m}}{\tau_m^{\frac{2}{p}}(\Phi(z))}\\ &\simeq \lim_{|\Phi(z)|\to\infty} e^{|\Phi(z)|^m-|z|^m-\frac{2}{p}\log\left(\tau_m(|\Phi(z))\right)} \end{aligned} \tag{2.9}$$

from which we may first claim that $\Phi(z)=az+b$ for some complex numbers a and b. If not, there exists a sequence z_k such that $|z_k|\to\infty$ and $\left|\frac{\Phi(z_k)}{z_k}\right|^m\to\infty$ as $k\to\infty$. It follows from this that there exists an N_1 such that

$$\left|\frac{\Phi(z_k)}{z_k}\right|^m-1-\frac{2}{p|z_k|^m}\log\left(\tau_m(|\Phi(z_k))\right)\geqslant 1$$

for all $k\geqslant N_1$. To this end, we have

$$\begin{aligned} e^{|\Phi(z_k)|^m-|z|^m-\frac{2}{p}\log\left(\tau_m(\Phi(z_k))\right)} &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(|\Phi(z_k)|^m-|z_k|^m-\frac{2}{p}\log\left(\tau_m(\Phi(z_k))\right)\Big)^n\\ &= \sum_{n=0}^{\infty}\frac{|z_k|^{nm}}{n!}\left(\frac{|\Phi(z_k)|^m}{|z_k|^m}-1-\frac{2}{p|z_k|^m}\log\left(\tau_m(\Phi(z_k))\right)\right)^n\\ &\geqslant \frac{|z_k|^{nm}}{n!}\left(\frac{|\Phi(z_k)|^m}{|z_k|^m}-1-\frac{2}{p|z_k|^m}\log\left(\tau_m(\Phi(z_k))\right)\right)^n,\ k\geqslant N_1 \end{aligned}$$

from which we deduce

$$e^{|\Phi(z-k)|^m-|z|^m-\frac{2}{p}\log\left(\tau_m(\Phi(z))\right)}\gtrsim\frac{|z_k|^{nm}}{n!}\to\infty \text{ as } |z_k|\to\infty \tag{2.10}$$

that contradicts (2.9) and hence $\Phi(z)=az+b$. We further claim that $|a|<1$. If not, observe that the estimate in (2.9) does in addition imply

$$\begin{aligned} &\lim_{|\Phi(z)|\to\infty}|\Phi(z)|^m-|z|^m-\frac{2}{p}\log\left(\tau_m(\Phi(z))\right)\\ &\qquad= \lim_{|az+b|\to\infty}|az+b|^m-|z|^m-\frac{2}{p}\log\left(\tau_m(|az+b|)\right)<0 \end{aligned} \tag{2.11}$$

which holds only if $|a|<1$.

If $p=q=\infty$, then following the same arguments as those leading to (2.9), we obtain

$$0=\lim_{|\Phi(z)|\to\infty}\left|f_{(\Phi(z),R)}(\Phi(z))\right|e^{-|\Phi(z)|^m}e^{|\Phi(z)|^m-|z|^m}\simeq\lim_{|\Phi(z)|\to\infty}e^{|\Phi(z)|^m-|z|^m} \tag{2.12}$$

from which and repeating the arguments leading to (2.10) and (2.11), we easily arrive at the desired conclusion.

In a similar manner, when $0 < p \leqslant q < \infty$, then applying (2.7) we estimate

$$
\begin{aligned}
0 = \lim_{|w|\to\infty} \|C_\Phi f^*_{(w,R)}\|_{(m,q)} &\geqslant \lim_{|w|\to\infty} \frac{1}{\tau_m(w)^{\frac{2q}{p}}} \int_{D(w,\sigma\tau_m(w))} \frac{|f_{(w,R)}(\Phi(z))|^q}{e^{q|z|^m}} dA(z)\\
&= \lim_{|w|\to\infty} \frac{1}{\tau_m(w)^{\frac{2q}{p}}} \int_{D(w,\sigma\tau_m(w))} \frac{|f_{(w,R)}(z)|^q e^{-q|z|^m}}{e^{q|\Phi^{-1}(z)|^m - q|z|^m}} dA(\Phi^{-1}(z))\\
&\gtrsim \lim_{|w|\to\infty} \frac{1}{\tau_m(w)^{\frac{2q}{p}}} \tau_m(w)^2 e^{-q|\Phi^{-1}(w)|^m + q|w|^m}\\
&= \lim_{|\Phi(w)|\to\infty} \tau_m(\Phi(w))^{2-\frac{2q}{p}} e^{q|\Phi(w)|^m - q|w|^m}\\
&\simeq \lim_{|\Phi(w)|\to\infty} e^{q|\Phi(w)|^m - q|w|^m + \frac{2(p-q)}{p}\log\left(\tau_m(\Phi(w))\right)}
\end{aligned}
$$

from which and repeating the arguments leading to (2.10) and (2.11) again, we arrive at our assertion.

To prove the sufficiency of the condition, we let f_n be a uniformly bounded sequence of functions in $\mathscr{F}_{(m,p)}$ that converge uniformly to zero on compact subsets of $\mathbb{C}$. Then we consider three different cases.

Case 1: if $q = \infty$ and $0 < p < \infty$, then for a positive number r and eventually applying (2.7), we estimate

$$
\begin{aligned}
\|C_\Phi f_n\|_{(m,\infty)} &= \sup_{z\in\mathbb{C}} |f_n(\Phi(z))| e^{-|z|^m} = \sup_{z\in\mathbb{C}} |f_n(az+b)| e^{-|z|^m}\\
&\lesssim \sup_{|az+b|>r} |f_n(az+b)| e^{-|az+b|^m} \frac{e^{|az+b|^m}}{e^{|z|^m}} + \sup_{|az+b|\leqslant r} |f_n(az+b)| e^{-|z|^m}\\
&\lesssim \|f_n\|_{(m,p)} \sup_{|az+b|>r} \frac{e^{|az+b|^m - |z|^m}}{\tau_m(az+b)^{\frac{2}{p}}} + \sup_{|az+b|\leqslant r} |f_n(az+b)|\\
&\simeq \|f_n\|_{(m,p)} \sup_{|az+b|>r} \frac{e^{|az+b|^m - |z|^m}}{\tau_m(az+b)^{\frac{2}{p}}} + \sup_{|az+b|\leqslant r} |f_n(az+b)|,
\end{aligned}
$$

where in the last inequality we used the pointwise estimate (2.7). Since $\|f_n\|_{(m,\infty)}$ is uniformly bounded and $|a| < 1$, the first summand above goes to zero as $r \to \infty$ and the second goes to zero when $n \to \infty$. This implies $\|C_\Phi f_n\|_{\mathscr{F}_{(m,\infty)}} \to 0$ as $n \to \infty$ from which our assertion follows in this case.

Case 2: if $q = \infty = p$, then for a positive number r, we also have

$$
\begin{aligned}
\|C_\Phi f_n\|_{(m,\infty)} &= \sup_{z\in\mathbb{C}} |f_n(\Phi(z))| e^{-|z|^m} = \sup_{z\in\mathbb{C}} |f_n(az+b)| e^{-|z|^m}\\
&\simeq \sup_{|az+b|>r} |f_n(az+b)| e^{-|az+b|^m} \frac{e^{|az+b|^m}}{e^{|z|^m}} + \sup_{|az+b|\leqslant r} |f_n(az+b)| e^{-|z|^m}\\
&\lesssim \|f_n\|_{(m,\infty)} \sup_{|az+b|>r} \frac{e^{|az+b|^m}}{e^{|z|^m}} + \sup_{|az+b|\leqslant r} |f_n(az+b)|
\end{aligned}
$$

from which the claim follows.

Case 3: if $0<p\leqslant q<\infty$, then applying (2.7)

$$\begin{aligned}\|C_\Phi f_n\|_{(m,q)}^q &= \int_{\mathbb{C}} |f_n(az+b)|^q e^{-q|az+b|^m}\Big(e^{q|az+b|^m-q|z|^m}\Big)dA(z)\\ &\lesssim \int_{\mathbb{C}}\bigg(\frac{1}{\tau_m(az+b)^2}\int_{D(az+b,\sigma\tau_m(az+b))}\frac{|f_n(w)|^p}{e^{p|w|^m}}dA(w)\bigg)^{\frac{q}{p}}\frac{e^{q|az+b|^m}}{e^{q|z|^m}}dA(z).\end{aligned}$$

Now if $|az+b|>r$ for some positive number r, then the part of the integral on $\{z\in\mathbb{C}:|az+b|>r\}$ is bounded by

$$\|f_n\|_{(m,p)}^q\int_{|az+b|>r}\frac{e^{q|az+b|^m-q|z|^m}}{\tau_m(az+b)^{\frac{2q}{p}}}dA(z)\lesssim\int_{|az+b|>r}\frac{e^{q|az+b|^m-q|z|^m}}{\tau_m(az+b)^{\frac{2q}{p}}}dA(z)$$

which is finite as $|a|<1$ and tends to zero as $r\to\infty$. On the other hand, if $|az+b|\leqslant r$, then using the fact that $|a|<1$ we find that the remaining part of the integral is bounded by

$$\sup_{|az+b|\leqslant r}|f_n(az+b)|^q\int_{|az+b|\leqslant r}e^{q|az+b|^m-q|z|^m}dA(z)\lesssim\sup_{|az+b|\leqslant r}|f_n(az+b)|^q\to 0$$

as $n\to\infty$ and completes the proof of the proposition. □

3. Proof of the main results

3.1. Proof of theorem 1

In this section we prove our first main result.

Part i): Let $0<p<\infty$ and $D:\mathscr{F}_{(m,p)}\to\mathscr{F}_{(m,\infty)}$ is bounded. Then, applying the sequence of function in (2.8) we estimate

$$\|D\|\gtrsim\|Df^*_{(w,R)}\|_{(m,\infty)}\simeq\frac{\sup_{z\in\mathbb{C}}|f'_{(w,R)}(z)|e^{-|z|^m}}{\tau_m^{\frac{2}{p}}(w)}\geqslant\frac{|f'_{(w,R)}(w)|e^{-|w|^m}}{\tau_m^{\frac{2}{p}}(w)}\simeq\frac{m|w|^{m-1}}{\tau_m^{\frac{2}{p}}(w)}$$

for all $w\in\mathbb{C}$. This happens to hold only if

$$\|D\|\gtrsim\sup_{w\in\mathbb{C}}\frac{m|w|^{m-1}}{\tau_m^{\frac{2}{p}}(w)}\simeq\begin{cases}\dfrac{\sup_{w\in\mathbb{C}}(1+|w|)^{\frac{m(p+1)-(p+2)}{p}}}{m^{-1}|m^2-m|^{-\frac{1}{p}}}, & m\neq 1\\ 1, & m=1\end{cases}\tag{3.1}$$

from which our assertion and one side of the norm estimate for D follow.

Conversely, applying (2.1) and (2.7), we also have

$$\|Df\|_{(m,\infty)}=\sup_{z\in\mathbb{C}}\frac{|f'(z)|}{e^{|z|^m}}\lesssim\sup_{z\in\mathbb{C}}\bigg(\frac{1}{\tau_m^2(z)}\int_{D(z,\sigma\tau_m(z))}\frac{|f'(w)|^p}{e^{p|w|^m}}dA(w)\bigg)^{\frac{1}{p}}dA(z)$$

Now for each point $z \in D(w, \sigma\tau_m(w))$, observe that $1+|z| \simeq 1+|w|$. Taking this into account, we further estimate the above by

$$\sup_{z\in\mathbb{C}} \Bigg(\frac{m^p(1+|z|)^{p(m-1)}}{\tau_m^2(z)} \int_{D(z,\sigma\tau_m(z))} \frac{|f'(w)|^p e^{-p|w|^m}}{m^p(1+|w|)^{p(m-1)}} dA(w) \Bigg)^{\frac{1}{p}} dA(z)$$
$$\lesssim \begin{cases} \|f\|_{(m,p)} \frac{\sup_{w\in\mathbb{C}}(1+|w|)^{\frac{m(p+1)-(p+2)}{p}}}{|m^{2+p}-m^{1+p}|^{-\frac{1}{p}}}, & m \neq 1 \\ \|f\|_{(m,p)}, & m = 1 \end{cases}$$

from which the sufficiency of the condition and the reverse side of the estimate in (3.1) follow.

We now turn to the proof of the compactness part and first assume that $m < 2 - \frac{p}{p+1}$. Then for each positive ε, there exists N_1 such that

$$\sup_{|w|>N_1} |m^{2+p} - m^{1+p}|^{\frac{1}{p}} (1+|w|)^{\frac{m(p+1)-(p+2)}{p}} \simeq \sup_{|w|>N_1} \frac{m^p(1+|w|)^{p(m-1)}}{\tau_m^2(w)} < \varepsilon. \qquad (3.2)$$

Next, we let f_n to be a uniformly bounded sequence of functions in $\mathscr{F}_{(m,p)}$ that converges uniformly to zero on compact subsets of $\mathbb{C}$. Then applying (2.1) and arguing in the same way as in the series of estimations made above, and invoking eventually (3.2) it follows that

$$\begin{aligned} \frac{|f_n'(z)|^p}{e^{p|z|^m}} &\lesssim \frac{1}{\tau_m^2(z)} \int_{D(z,\sigma\tau_m(z))} \frac{|f_n'(w)|^p}{e^{p|w|^m}} dA(w) = \frac{1}{\tau_m^2(z)} \int_{\substack{w\in D(z,\sigma\tau_m(z)) \\ |w|\leqslant N_1}} \frac{|f_n'(w)|^p}{e^{p|w|^m}} dA(w) \\ &\quad + \int_{\substack{w\in D(z,\sigma\tau_m(z)) \\ |w|>N_1}} \frac{|f_n'(w)|^p e^{-p|w|^m}}{\tau_m^2(w)} dA(w) \\ &\lesssim \sup_{|w|\leqslant N_1} |f_n(w)|^p + \|f_n\|_{(m,p)}^p \sup_{|w|>N_1} \frac{m^p(1+|w|)^{p(m-1)}}{\tau_m^2(w)} \\ &\lesssim \sup_{|w|\leqslant N_1} |f_n(w)|^p + \sup_{|w|>N_1} (1+|w|)^{m(p+1)-(p+2)} \\ &\lesssim \sup_{|w|>N_1} \frac{m^p(1+|w|)^{p(m-1)}}{\tau_m^2(w)} \lesssim \varepsilon^p \text{ as } n\to\infty \end{aligned}$$

and from which we have that

$$\|Df_n\|_{(m,\infty)} = \sup_{z\in\mathbb{C}} |f_n'(z)| e^{-|z|^m} \lesssim \varepsilon \text{ as } n \to \infty.$$

On the other hand, if D is compact, applying the sequence of functions $f^*_{(w,R)}$ in (2.8), (2.7) and (2.4), we find

$$\begin{aligned}\frac{m^p(1+|w|)^{p(m-1)}}{\tau_m^2(w)} &\simeq m^p(1+|w|)^{p(m-1)}e^{-q|w|^m}|f^*_{(w,\eta(R))}(w)|^p\\ &\lesssim \int_{D(w,\sigma\tau_m(w))} m^p(1+|z|)^{p(m-1)}|f^*_{(w,\eta(R))}(z)|^p e^{-p|z|^m}dA(z)\\ &\leqslant \int_{D(w,\sigma\tau_m(w))}\Big(\sup_{w\in\mathbb{C}} m(1+|z|)^{(m-1)}|f^*_{(w,\eta(R))}(z)|e^{-|z|^m}\Big)^p dA(z)\\ &\simeq \|Df^*_{(w,\eta(R))}\|^p_{(m,\infty)}\end{aligned}$$

from which we have that

$$|m^{2+p}-m^{1+p}|^{\frac{1}{p}}(1+|w|)^{m-1+\frac{m-2}{p}} \simeq \frac{(1+|w|)^{(m-1)}}{\tau_m^{\frac{2}{p}}(w)} \lesssim \|Df^*_{(w,\eta(R))}\|_{(m,\infty)} \to 0,$$

as $|w|\to\infty$ which holds only when $m-1+\frac{m-2}{p}<0$ as asserted.

Part ii): Since (b) $\Rightarrow$ (a), we will verify that (a)$\Rightarrow$ (c) and (c)$\Rightarrow$ (b). For the first we argue as follows. Let $0<p<\infty$ and R be a sufficiently large number and (z_k) be the covering sequence as in Lemma 1. Then by Lemma 2.4 of [11], the function

$$F=\sum_{z_k:|z_k|\geqslant\eta(R)} a_k f_{(z_k,R)}\in\mathscr{F}_{(m,\infty)} \text{ and } \|F\|_{(m,\infty)}\lesssim\|(a_k)\|_{\ell^\infty}$$

for every ℓ^∞ sequence (a_k). If $(r_k(t))_k$ is the Radmecher sequence of function on $[0,1]$ chosen as in [9], then the sequence $(a_kr_k(t))\in\ell^\infty$ with $\|(a_kr_k(t))\|_{\ell^\infty}=\|(a_k)\|_{\ell^\infty}$ for all t. This implies that the function

$$F_t=\sum_{z_k:|z_k|\geqslant\eta(R)} a_kr_k(t)f_{(z_k,R)}\in\mathscr{F}_{(m,\infty)} \text{ and } \|F_t\|_{(m,\infty)}\lesssim\|(a_k)\|_{\ell^\infty}.$$

Then, an application of Khinchine's inequality [9] yields

$$\Big(\sum_{z_k:|z_k|\geqslant\eta(R)}|a_k|^2|f'_{(z_k,R)}(z)|^2\Big)^{\frac{p}{2}}\lesssim\int_0^1\Big|\sum_{z_k:|z_k|\geqslant\eta(R)}a_kr_k(t)f'_{(z_k,R)}(z)\Big|^q dt. \tag{3.3}$$

Making use of (3.3), and subsequently Fubini's theorem, we have

$$\begin{aligned}&\int_{\mathbb{C}}\Big(\sum_{z_k:|z_k|\geqslant\eta(R)}|a_k|^2|f'_{(z_k,R)}(z)|^2\Big)^{\frac{p}{2}}e^{-p|z|^m}dA(z)\\ &\lesssim\int_{\mathbb{C}}\int_0^1\Big|\sum_{z_k:|z_k|\geqslant\eta(R)}a_kr_k(t)f'_{(z_k,R)}(z)\Big|^p dt\frac{dA(z)}{e^{p|z|^m}}\\ &=\int_0^1\int_{\mathbb{C}}\Big|\sum_{z_k:|z_k|\geqslant\eta(R)}a_kr_k(t)f'_{(z_k,R)}(z)\Big|^p\frac{dA(z)dt}{e^{p|z|^m}}\\ &\simeq\int_0^1\|DF_t\|^p_{(m,p)}dt\lesssim\|(a_k)\|^p_{\ell^\infty}.\end{aligned} \tag{3.4}$$

Then, using (2.4) we get

$$
\begin{aligned}
&\sum_{z_k:|z_k|\geqslant\eta(R)} |a_k|^p \int_{D(z_k,3\sigma\tau_m(z_k))} (1+|z|)^{-p(m-1)} dA(z)\\
\simeq &\sum_{z_k:|z_k|\geqslant\eta(R)} |a_k|^p \int_{D(z_k,3\sigma\tau_m(z_k))} (1+|z|)^{p(m-1)} \frac{|f'_{(z_k,R)}(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}} dA(z)\\
\simeq &\int_{\mathbb{C}} \sum_{z_k:|z_k|\geqslant\eta(R)} |a_k|^p \chi_{D(z_k,3\sigma\tau_m(z_k))}(z) |f'_{(z_k,R)}(z)|^p e^{-p|z|^m} dA(z)\\
\lesssim &\max\{1, N_{\max}^{1-p/2}\} \int_{\mathbb{C}} \left(\sum_{z_k:|z_k|\geqslant\eta(R)} |a_k|^2 |f'_{(z_k,R)}(z)|^2 \right)^{\frac{p}{2}} e^{-p|z|^m} dA(z) \lesssim \|(a_k)\|_{\ell^\infty}^p.
\end{aligned}
$$

Setting, in particular, $a_k = 1$ for all k in the above series of estimates results in

$$
\sum_{z_k:|z_k|\geqslant\eta(R)} \int_{D(z_k,3\sigma\tau_m(z_k))} (1+|z|)^{-p(m-1)} dA(z) < \infty.
$$

Now we take a positive number $r \geqslant \eta(R)$ such that whenever z_k of the covering sequence belongs to $\{|z| < \eta(R)\}$, then $D(z_k, \sigma\tau_m(z_k))$ belongs to $\{|z| < \eta(R)\}$. Thus,

$$
\begin{aligned}
&\int_{|w|\geqslant r} m^p (1+|w|)^{p(m-1)} dA(w)\\
\simeq &\int_{|w|\geqslant r} \frac{1}{\tau_m^2(w)} \int_{D(w,3\sigma\tau_m(w))} \frac{dA(z)dA(w)}{(1+|z|)^{-p(m-1)}}\\
\leqslant &\sum_{|z_k|\geqslant\eta(R)} \int_{D(z_k,\sigma\tau_m(z_k))} \frac{1}{\tau_m^2(w)} \int_{D(w,3\sigma\tau_m(w))} \frac{dA(z)dA(w)}{m^{-p}(1+|z|)^{-p(m-1)}}\\
\lesssim &\sum_{z_k:|z_k|\geqslant\eta(R)} \int_{D(z_k,3\sigma\tau_m(z_k))} \frac{dA(z)}{(1+|z|)^{-p(m-1)}} < \infty.
\end{aligned}
\tag{3.5}
$$

It follows that

$$
\int_{|w|<r} \frac{1}{\tau_m^2(w)} \int_{D(w,3\sigma\tau_m(w))} m^p (1+|z|)^{p(m-1)} dA(z)dA(w) < \infty
$$

from which and taking into account (3.5), we obtain

$$
\int_{\mathbb{C}} m^p (1+|z|)^{p(m-1)} dA(z) < \infty
$$

which holds only if $p(m-1) < -2$ as asserted.

It remains to show that condition (c) implies (b). To this end, let f_n be a uniformly bounded sequence of functions in $\mathscr{F}_{(m,\infty)}$ that converges uniformly to zero on compact subsets of $\mathbb{C}$, and by the given condition, for each $\varepsilon > 0$, there exists a positive number r_1 such that

$$
\int_{|z|>r_1} (1+|z|)^{p(m-1)} dA(z) < \varepsilon.
$$

It follows from this and (2.2) that

$$\begin{aligned}\int_{|z|>r_1} |f_n'(z)|^p e^{-p|z|^m} dA(z) &= \int_{|z|>r_1} \frac{|f_n'(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}}(1+|z|)^{p(m-1)} dA(z)\\ &\lesssim \|f\|_{(m,\infty)}^p \int_{|z|>r_1} (1+|z|)^{p(m-1)} dA(z) \lesssim \|f\|_{(m,\infty)}^p \varepsilon \lesssim \varepsilon.\end{aligned} \tag{3.6}$$

On the other hand, when $|z| \leqslant r_1$ we find

$$\begin{aligned}\int_{|z|\leqslant r_1} |f_n'(z)|^p e^{-p|z|^m} dA(z) &\lesssim \int_{|z|\leqslant r_1} |f_n(z)|^p (1+|z|)^p e^{-q|z|^m} dA(z)\\ &\lesssim \sup_{|z|\leqslant r_1} |f_n(z)|^p \int_{|z|\leqslant r_1} (1+|z|)^p e^{-p|z|^m} dA(z)\\ &\lesssim \sup_{|z|\leqslant r_1} |f_n(z)|^p \to 0\end{aligned}$$

as $n \to \infty$ and, from which and (3.6) our claim $\|Df_n\|_{(m,p)} \to 0$ as $n \to \infty$ follows.

3.2. Proof of Theorem 2

In this section we prove our main result on essential norm. Assume that $1 \leqslant p \leqslant \infty$ and $D : \mathscr{F}_{(m,p)} \to \mathscr{F}_{(m,q)}$ is bounded. If $q < \infty$ and $m < 2 - \frac{pq}{pq+q-p}$ or $q = \infty$ and $m < 2 - \frac{p}{p+1}$, then as noticed before D becomes a compact operator and its essential norm vanishes. Thus, our aim here is to establish the result only for the two remaining cases namely for $q < \infty$ and $m = 2 - \frac{pq}{pq+q-p}$ and for $q = \infty$ and $m = 2 - \frac{p}{p+1}$.

3.3. Proof of the lower estimates in (1.8)

To prove the lower bounds we will again use the sequence of functions in (2.8), and applying D to such a sequence we find

$$\|D\|_e \geqslant \limsup_{|w|\to\infty} \|Df^*_{(w,R)}\|_{(m,q)}.$$

Now if $p = q = \infty$, then making use of (2.4) we obtain

$$\begin{aligned}\|D\|_e &\geqslant \limsup_{|w|\to\infty} \|Df^*_{(w,R)}\|_{(m,\infty)} \simeq \limsup_{|w|\to\infty} \sup_{z\in\mathbb{C}} |f'_{(w,R)}(z)| e^{-|z|^m}\\ &\geqslant \limsup_{|w|\to\infty} |f'_{(w,R)}(w)| e^{-|w|^m} \geqslant \limsup_{|w|\to\infty} m(1+|z|)^{m-1} \simeq 1,\end{aligned}$$

where we set m= 1 and from which the assertion follows. If we instead consider $1 \leqslant p < q = \infty$, then it follows from (2.4) and eventually setting $m = 2 - \frac{p}{p+1}$ that

$$\|D\|_e \geqslant \limsup_{|w|\to\infty} \|Df^*_{(w,R)}\|_{(m,\infty)} \geqslant \limsup_{|w|\to\infty} \frac{|f'_{(w,R)}(w)|e^{-|w|^m}}{\tau_m^{\frac{2}{p}}(w)} \simeq \limsup_{|w|\to\infty} \frac{m(1+|w|)^{m-1}}{\tau_m^{\frac{2}{p}}(w)}$$
$$\simeq m|m^2-m|^{\frac{1}{p}} \limsup_{|w|\to\infty}(1+|w|)^{m-1+\frac{m-2}{p}} = \left|m^{2+p} - m^{1+p}\right|^{\frac{1}{p}}.$$

On the other hand, if $1 \leqslant p \leqslant q < \infty$, then making use of (2.5) we again estimate

$$\|D\|_e \geqslant \limsup_{|w|\to\infty} \left\|Df^*_{(w,R)}\right\|_{(m,q)} \simeq \limsup_{|w|\to\infty} \frac{1}{\tau_m^{\frac{2}{p}}(w)} \left(\int_{\mathbb{C}} \frac{|f'_{(w,R)}(z)|^q}{e^{q|z|^m}} dA(z)\right)^{\frac{1}{q}}$$
$$\geqslant \limsup_{|w|\to\infty} \frac{1}{\tau_m^{\frac{2}{p}}(w)} \left(\int_{D(w,\sigma\tau_m(w))} |f'_{(w,R)}(z)|^q e^{-q|z|^m} dA(z)\right)^{\frac{1}{q}}$$

for some small positive number σ. An application of (2.4) and also setting $m = 2 - \frac{pq}{pq+q-p}$ imply that the last term above is comparable to

$$\limsup_{|w|\to\infty} \frac{1}{\tau_m(w)^{\frac{2}{p}}} \left(\int_{D(w,\sigma\tau_m(w))} m^q(1+|z|)^{q(m-1)} dA(z)\right)^{\frac{1}{q}}$$
$$\simeq m|m^2-m|^{\frac{1}{p}} \limsup_{|w|\to\infty}(1+|w|)^{m-1+\frac{(m-2)(q-p)}{pq}} \simeq \left|m^{2+p} - m^{1+p}\right|^{\frac{1}{p}}$$

which completes the proof of the lower estimate in (1.8).

3.4. Proof of the upper estimates in (1.8)

For this, we may apply Proposition 3 and consider a sequence of compact composition operators C_{Φ_k} where $\Phi_k(z) = \frac{k}{k+1}z$ for each $k \in \mathbb{N}$. Since D is bounded, then $D \circ C_{\Phi_k} : \mathscr{F}_{(m,p)} \to \mathscr{F}_{(m,q)}$ also constitutes a sequence of compact operators. Then we may consider two different cases.

Case 1: If $q = \infty$, then we have

$$\|D\|_e \leqslant \|D - D\circ C_{\Phi_k}\| = \sup_{\|f\|_{(m,p)}\leqslant 1} \|(D - D\circ C_{\Phi_k})f\|_{(m,\infty)}$$
$$\simeq \sup_{\|f\|_{(m,p)}\leqslant 1} \sup_{|z|>r} \left|f'(z) - \frac{k}{k+1} f'(\Phi_k(z))\right| e^{-|z|^m}$$
$$+ \sup_{\|f\|_{(m,p)}\leqslant 1} \sup_{|z|\leqslant r} \left|f'(z) - \frac{k}{k+1} f'(\Phi_k(z))\right| e^{-|z|^m} \tag{3.7}$$

for a certain fixed positive number r. If in addition $p=\infty$, the first summand above is bounded by

$$\begin{aligned}
&\sup_{\|f\|_{(m,p)}\leqslant 1}\ \sup_{|z|>r}\Big(\frac{k}{k+1}\Big|f'(z)-f'(\Phi_k(z))\Big|e^{-|z|^m}+\frac{1}{k+1}|f'(z)|e^{-|z|^m}\Big)\\
&\leqslant \sup_{\|f\|_{(m,\infty)}\leqslant 1}\ \sup_{|z|>r} m(1+|z|)^{m-1}\Big(\frac{\big|f'(z)-f'(\Phi_k(z))\big|e^{-\psi(z)}}{m(1+|z|)^{m-1}}+\frac{1}{k+1}\frac{|f'(z)|e^{-|z|^m}}{m(1+|z|)^{m-1}}\Big)\\
&\leqslant \sup_{|z|>r} m(1+|z|)^{m-1}+\frac{1}{k+1}\sup_{|z|>r} m(1+|z|)^{m-1}\lesssim \sup_{|z|>r} m(1+|z|)^{m-1}=1,
\end{aligned}$$

where the last equality follows when we set $m=1$.

Similarly, if $1\leqslant p<\infty$, then it follows from (2.7) and eventually setting $m=2-\frac{p}{p+1}$ that the first summand in (3.7) is bounded by

$$\begin{aligned}
&\sup_{\|f\|_{(m,p)}\leqslant 1}\ \sup_{|z|>r}\frac{1}{\tau_m^{\frac{2}{p}}(z)}\bigg(\int_{D(z,\sigma\tau_m(z))}\bigg(\frac{k^p\big|f'(w)-f'(\Phi_k(w))\big|^p+|f'(w)|^p}{e^{p|z|^m}(k+1)^p}\bigg)dA(w)\bigg)^{\frac{1}{p}}\\
&\lesssim \sup_{\|f\|_{(m,p)}\leqslant 1}\ \sup_{|z|>r}\|f\|_{(m,p)}\bigg(\frac{m(1+|z|)^{m-1}(k+1)}{\tau_m^{\frac{2}{p}}(z)}\bigg)\\
&\leqslant |m^{2+p}-m^{1+p}|^{\frac{1}{p}}\sup_{|z|>r}(1+|z|)^{m-1+\frac{m-2}{p}}\Big(\frac{k+2}{k+1}\Big)\\
&\lesssim |m^{2+p}-m^{1+p}|^{\frac{1}{p}}\sup_{|z|>r}(1+|z|)^{m-1+\frac{m-2}{p}}\simeq |m^{2+p}-m^{1+p}|^{\frac{1}{p}}.
\end{aligned}\tag{3.8}$$

As for the second summand in (3.7), we observe that by integrating the function f'' along the radial segment $[\frac{kz}{k+1}z,z]$ we find

$$\Big|f'(z)-f'\Big(\frac{k}{k+1}z\Big)\Big|\leqslant\frac{|z||f''(z^*)|}{k+1}$$

for some z^* in the radial segment $[\frac{kz}{k+1}z,z]$. By Cauchy estimate's for f'', we also have

$$|f''(z^*)|\leqslant\frac{1}{r}\max_{|z|=2r}|f'(z)|,$$

and hence

$$\begin{aligned}
\Big|f'(z)-\frac{k}{k+1}f'\Big(\frac{k}{k+1}z\Big)\Big|e^{-|z|^m}&\leqslant\frac{k}{k+1}\Big|f'(z)-f'\Big(\frac{k}{k+1}z\Big)\Big|e^{-|z|^m}+\frac{|f'(z)|e^{-|z|^m}}{k+1}\\
&\lesssim\frac{|z|e^{-|z|^m}}{r(k+1)}\max_{|z|=2r}|f'(z)|+\frac{|f'(z)|e^{-|z|^m}}{k+1}
\end{aligned}\tag{3.9}$$

from which and if $p=\infty$ and applying (2.2), then the second summand in (3.7) is bounded by

$$\sup_{\|f\|_{(m,\infty)}\leqslant 1}\ \sup_{|z|\leqslant r}\left(\frac{|z|e^{-|z|^m}}{r(k+1)}\max_{|z|=2r}|f'(z)|+\frac{|f'(z)|e^{-|z|^m}}{k+1}\right)\lesssim\frac{m(1+|r|)^{m-1}}{k+1}=\frac{1}{k+1}\to 0$$

as $k\to\infty$ and when we set $m=1$ here again.

On the other hand, if $1\leqslant p<\infty$, we may further make some estimations in (3.9). By (2.7) and (2.1) and eventually setting $m=2-\frac{p}{p+1}=\frac{p+2}{p+1}$, we have

$$\begin{aligned}\max_{|z|=2r}|f'(z)| &\lesssim \max_{|z|=2r}\frac{e^{|z|^m}m(1+|z|)}{\tau_m^{\frac{2}{p}}(z)}\left(\int_{D(z,\sigma\tau_m(z))}\frac{|f'(w)|^pe^{-p|w|^m}}{m^p(1+|w|)^p}dA(w)\right)^{\frac{1}{p}}\\ &\lesssim \|f\|_{(m,p)}\max_{|z|=2r}\frac{e^{|z|^m}m|m^2-m|^{\frac{1}{p}}(1+|z|)^{m-1}}{\tau_m^{\frac{2}{p}}(z)}\\ &\lesssim \|f\|_{(m,p)}e^{(2r)^m}m(1+|r|)^{m-1+\frac{m-2}{p}}\\ &= \|f\|_{(m,p)}e^{(2r)^m}|m^{2+p}-m^{1+p}|^{\frac{1}{p}}.\end{aligned}$$

Now combining all the above estimates, we see that the second piece of the sum in (3.7) is bounded by

$$\begin{aligned}\sup_{\|f\|_{(m,p)}\leqslant 1}\ \sup_{|z|\leqslant r}\left|f'(z)-\frac{k}{k+1}f'(\Phi_k(z))\right|e^{-|z|^m} &\lesssim \frac{p+2}{(k+1)(p+1)}\frac{\sup_{\|f\|_{(m,p)}\leqslant 1}\|f\|_{(m,p)}}{e^{-(2r)^m}}\\ &\leqslant \frac{1}{k+1}e^{(2r)^m}\to 0 \text{ as } k\to\infty,\end{aligned}$$

from which, (3.8), $m=\frac{p+2}{p+1}$ and since r is arbitrary, we deduce

$$\begin{aligned}\|D\|_e &\lesssim \sup_{|z|>r}|m^{2+p}-m^{1+p}|^{\frac{1}{p}}(1+|z|)^{m-1+\frac{m-2}{p}}\\ &= |m^{2+p}-m^{1+p}|^{\frac{1}{p}}\limsup_{|z|\to\infty}(1+|z|)^{m-1+\frac{m-2}{p}}=|m^{2+p}-m^{1+p}|^{\frac{1}{p}}\end{aligned}$$

and completes the first case.

Case 2: When $1\leqslant p\leqslant q<\infty$, then we argue as follows. We may first estimate

$$\begin{aligned}\|D\|_e \leqslant \|D-D\circ C_{\Phi_k}\| &= \sup_{\|f\|_{(m,p)}\leqslant 1}\|(D-D\circ C_{\Phi_k})f\|_{(m,q)}\\ &\simeq \sup_{\|f\|_{(m,p)}\leqslant 1}\left(\int_{\mathbb{C}}\left|f'(z)-f'(\Phi_k(z))\Phi_k'(z)\right|^qe^{-q|z|^m}dA(z)\right)^{\frac{1}{q}}. \qquad (3.10)\end{aligned}$$

Applying Lemma 1 and estimate (2.7), we get

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{\left|f'(z)-\frac{k}{k+1}f'(\Phi_k(z))\right|^q}{e^{q|z|^m}}dA(z)\\
&\leqslant \sum_j \int_{D(z_j,\sigma\tau_m(z_j))} \frac{\left|f'(z)-\frac{k}{k+1}f'(\Phi_k(z))\right|^q}{e^{q|z|^m}}dA(z)\\
&\lesssim \sum_j \int_{D(z_j,\sigma\tau_m(z_j))} \left(\int_{D(z,\sigma\tau_m(z))} \frac{\left|f'(w)-\frac{k}{k+1}f(\Phi_k(w))\right|^p}{e^{p|w|^m}}dA(w)\right)^{\frac{q}{p}} \frac{dA(z)}{\tau_m^{\frac{2q}{p}}(z)}\\
&\lesssim \sum_j \left(\int_{D(z_j,3\sigma\tau_m(z_j))} \frac{\left|f'(w)-\frac{k}{k+1}f'(\Phi_k(w))\right|^p}{e^{p|w|^m}m^p(1+|w|)^{p(m-1)}}dA(w)\right)^{\frac{q}{p}}\\
&\quad\times \int_{D(z_j,\sigma\tau_m(z_j))} \frac{m^q(1+|z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)}dA(z).
\end{aligned}$$

We spilt now the above sum as

$$\sum_j = \sum_{j:|z_j|>r} + \sum_{j:|z_j|\leqslant r} \tag{3.11}$$

for some fixed positive number r again. Then applying Minkowski inequality (since $q \geqslant p$), and the finite multiplicity N of the covering sequence $D(z_j, 3\sigma\tau(z_j))$ and setting $m = 2-\frac{pq}{pq+q-p}$, the first sum is bounded by

$$\begin{aligned}
&\sup_{j:|z_j|>r}\left(\int_{D(z_j,\sigma\tau_m(z_j))} \frac{m^q(1+|z|)^{q(m-1)}}{\tau_m(z)^{\frac{2q}{p}}}dA(z)\right)\\
&\quad\times\left(\int_{D(z_j,3\sigma\tau_m(z_j))} \frac{\left|f'(w)-\frac{k}{k+1}f'(\Phi_k(w))\right|^p}{e^{p|w|^m}m^p(1+|w|)^{p(m-1)}}dA(w)\right)^{\frac{q}{p}}\\
&\lesssim \|f\|_{(m,p)}^q\left(1+\frac{1}{k+1}\right)\sup_{|z_j|>r}\int_{D(z_j,\sigma\tau_m(z_j))} \frac{m^q(1+|z|)^{q(m-1)}}{\tau_m(z)^{\frac{2q}{p}}}dA(z).\\
&\simeq \|f\|_{(m,p)}^q \sup_{j:|z_j|>r} m^q|m^2-m|^{\frac{q}{p}}(1+|z_j|)^{q(m-1)+(m-2)\frac{q-p}{p}}\\
&\lesssim m^q|m^2-m|^{\frac{q}{p}}\sup_{j:|z_j|>r}(1+|z_j|)^{q(m-1)+(m-2)\frac{q-p}{p}} = m^q|m^2-m|^{\frac{q}{p}}
\end{aligned}$$

where we, in particular, used that $\|f\|_{\mathscr{F}_p^\psi} \leqslant 1$.

We plan to show that the second sum in (3.11) tends to zero when $k\to\infty$. Then since r is arbitrary, our upper estimate will follow from the series of estimates we made starting from (3.10). To this end, as done before, making use of (3.9) and Minkowski inequality again, we estimate

$$\sum_{j:|z_j|\leqslant r}\left(\int_{D(z_j,3\sigma\tau_m(z_j))}\frac{\left|f'(w)-\frac{k}{k+1}f'(\Phi_k(w))\right|^p}{m^p(1+|w|)^{p(m-1)}e^{p|w|^m}}dA(w)\right)^{\frac{q}{p}}$$
$$\times\int_{D(z_j,\sigma\tau_m(z_j))}\frac{m^q(1+|z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)}dA(z)$$
$$\lesssim\left(\sum_{j:|z_j|\leqslant r}\int_{D(z_j,3\sigma\tau_m(z_j))}\frac{|w|^p\left(\max_{|w|=2r}|f'(w)|\right)^p+|f'(w)|^p}{r(k+1)^p(m^p(1+|w|)^{p(m-1)})e^{p|w|^m}}dA(w)\right)^{\frac{q}{p}}$$
$$\times\int_{D(z_j,\sigma\tau_m(z_j))}\frac{m^q(1+|z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)}dA(z).$$

Now we also have

$$|w|\leqslant|w-z_j|+|z_j|\leqslant r+\sigma\tau_m(z_j)\leqslant r+\delta\sup_{|z_j|\leqslant r}\tau_m(z_j)\lesssim r+\delta r^{\frac{2-m}{2}}\leqslant 2r$$

from which we have that the preceding sum is bounded by

$$\frac{\|f\|_{(m,p)}^q}{(1+k)^q}\sup_{|z_j|\leqslant r}\int_{D(z_j,\sigma\tau_m(z_j))}\frac{m^q(1+|z|)^{q(m-1)}}{\tau_m(z)^{\frac{2q}{p}}}dA(z)$$
$$\lesssim\frac{m^q|m^2-m|^{\frac{q}{p}}}{(1+k)^q}\sup_{j:|z_j|\leqslant r}(1+|z_j|)^{q(m-1)+(m-2)\frac{q-p}{p}}\lesssim\frac{m^q|m^2-m|^{\frac{q}{p}}}{(1+k)^q}\to 0\ \text{ as }\ k\to\infty,$$

where the last inequality follows after setting $m=2-\frac{pq}{pq+q-p}$ again. From this and series of estimates made above we deduce that

$$\|D\|_e\lesssim|m^{2+p}-m^{1+p}|^{\frac{1}{p}}\sup_{j:|z_j|>r}(1+|z_j|)^{(m-1)+(m-2)\frac{q-p}{qp}}=|m^{2+p}-m^{1+p}|^{\frac{1}{p}}.$$

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Tesfa Mengestie
Western Norway University of Applied Sciences
Mathematics Section
Klingenbergvegen 8, N-5414 Stord, Norway
e-mail: Tesfa.Mengestie@hvl.no