Operators
and
Matrices
Volume 13, Number 1 (2019), 19–34
doi:10.7153/oam-2019-13-02

NORMWISE, MIXED AND COMPONENTWISE CONDITION NUMBERS OF MATRIX EQUATION $X-\sum_{i=1}^{p}A_i^TXA_i+\sum_{j=1}^{q}B_j^TXB_j=Q$

JIE MENG AND HYUN-MIN KIM*

(*Communicated by R.-C. Li*)

Abstract. We consider a symmetric matrix equation $X-\sum_{i=1}^{p}A_i^TXA_i+\sum_{j=1}^{q}B_j^TXB_j=Q$, where $A_1, A_2, \ldots, A_p$, B_1, $B_2, \ldots, B_q \in \mathbb{R}^{n\times n}$, and Q is an $n\times n$ symmetric positive definite matrix. The explicit expressions of normwise, mixed and componentwise condition numbers of the matrix equation are investigated. Some numerical examples are given to show the sharpness of the three condition numbers.

1. Introduction

We consider the matrix equation

$$X-\sum_{i=1}^{p}A_i^TXA_i+\sum_{j=1}^{q}B_j^TXB_j=Q, \tag{1}$$

where p and q are nonnegative integers and at least one of p and q is a positive integer, $A_1,A_2,\ldots,A_p,B_1,B_2,\ldots,B_q\in\mathbb{R}^{n\times n}$, and Q is an $n\times n$ symmetric positive definite matrix.

Matrix equations of the form (1) have many applications in control theory [5, 16], dynamic programming [12, 21], ladder networks [2, 1], etc. For solving some nonlinear matrix equations like the one appearing in [25] by Newton method, an equation of the form (1) arises in each step of Newton's method. Moreover, finding the positive definite solution of equation (1) is also of practical importance since equation (1) with $Q=I$ is a general case of the generalized Lyapunov equation $MYS^*+SYM^*+\sum_{k=1}^{t}N_kYN_k^*+CC^*=0$, whose positive definite solution is the controllability of the bilinear control system

$$M\dot{x}(t)=S\dot{x}(t)+\sum_{k=1}^{t}N_kx(t)u_k(t)+Cu(t),$$

Mathematics subject classification (2010): 15A24, 15A12, 47A05.
Keywords and phrases: Matrix equation, perturbation analysis, condition number, mixed and componentwise.
This research is supported by China Scholarship Council (CSC, No. 201608260003).
* Corresponding author.

and we refer the reader to [7, 10, 30] for more details. It can be seen from [10] that if we set

$$E=\frac{1}{\sqrt{2}}(M-S-I),\ F=\frac{1}{\sqrt{2}}(M+S+I),\ G=M-I,\ R=CC^{*},$$

$$X=R^{-1/2}YR^{-1/2},\ A_i=R^{1/2}N_i^{*}R^{-1/2},\ i=1,2,\ldots,t,$$
$$A_{t+1}=R^{1/2}F^{*}R^{-1/2},\ A_{t+2}=R^{1/2}G^{*}R^{-1/2},$$
$$B_1=R^{1/2}E^{*}R^{-1/2},\ B_2=R^{1/2}M^{*}R^{-1/2},$$

then the above generalized Lyapunov equation can be written as $X-\sum_{i=1}^{t+2}A_iXA_i^{*}+\sum_{j=1}^{2}B_jYB_j^{*}=I$, which is equation (1) with complex coefficient matrices.

Recently, some special cases of equation (1) has been investigated [10, 11, 13, 20, 22, 23, 24]. Ran and Reurings [23] and Duan el at. [9] considered the classes of nonlinear matrix equations

$$X=Q\pm\sum_{i=1}^{m}A_i^{*}X^{\delta_i}A_i, \tag{2}$$

where $\delta_i=1$ or $0<\delta_i<1$. Under some hypotheses, they established the existence and uniqueness of positive definite solutions of (2). In [24], Ran and Reurings applied the fixed-point theory to the nonlinear matrix equation $X=Q\pm\sum_{j=1}^{m}A_j^{*}F(X)A_j$ and derived the existence of the positive definite solution.

Berzig [3] investigated the existence and uniqueness of the positive definite solution of equation (1). Based on the coupled fixed point results of Bhaskar and Lakshmikantham [4], he gave one sufficient condition under which equation (1) has a unique positive definite solution. He also proposed an algorithm for computing the solution. Duan and Wang [10] then studied the perturbation analysis for equation (1) with $Q=I$. They derived a perturbation bound with respect to 2-norm for the positive definite solution based on matrix differentiation.

The aim of this paper is to discuss the condition numbers of equation (1), which are of great importance in perturbation analysis. We investigate the normwise, mixed and componentwise condition numbers. The mixed and componentwise condition numbers are developed by Gohberg and Koltracht [14], and see also [8, 18, 19, 27, 28, 29, 31]. We also derive two upper linear perturbation bounds for the mixed and componentwise condition numbers.

This paper is organized as follows. In Section 2, we investigate three kinds of normwise condition numbers and derived their explicit expressions. In Section 3, we obtain explicit expressions and the upper bounds of mixed and componentwise condition numbers. In Section 4, we give some numerical examples to illustrate and compare theses three condition numbers.

We begin with the notation used throughout this paper. $\mathbb{R}^{n\times n}$ stands for the set of $n\times n$ matrices with elements on field $\mathbb{R}$. $\|\cdot\|_2$ and $\|\cdot\|_F$ are the spectral norm and the Frobenius norm, respectively. For $X=(x_{ij})\in\mathbb{R}^{n\times n}$, $\|X\|_{\max}=\max_{i,j}|x_{ij}|$ and $|X|$ is the matrix whose elements are $|x_{ij}|$. For a matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and a matrix B,

$\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A) = (a_1^T, \cdots, a_n^T)^T$ with a_i being the i-th column of A and $A \otimes B = (a_{ij}B)$ is the Kronecker product. For Hermitian matrices X and Y, $X \geqslant Y (X > Y)$ means that $X - Y$ is positive semidefinite (definite). For a matrix H, $\sigma_{\min}(H)$ and $\sigma_{\max}(H)$ denote the minimal and maximal singular values, respectively.

2. Preliminaries

We shall start this section by recalling some results concerning equation (1). In [3], Berzig gave the following result for the existence and uniqueness of the positive definite solution.

LEMMA 1. (Theorem 3.1, [3]) *Suppose that*

$$\sum_{i=1}^{p} A_i^T Q A_i < \frac{Q}{2} \quad \text{and} \quad \sum_{j=1}^{q} B_j^T Q B_j < \frac{Q}{2}.$$

Then,
(i) Equation (1) has one and only one positive definite solution X.
(ii) The sequences $\{X_k\}$ and $\{Y_k\}$ defined by $X_0 = 0$, $Y_0 = 2Q$, and

$$\begin{cases} X_{k+1} = Q + \sum_{i=1}^{p} A_i^T X_k A_i - \sum_{j=1}^{q} B_j^T Y_k B_j, \\ Y_{k+1} = Q + \sum_{i=1}^{p} A_i^T Y_k A_i - \sum_{j=1}^{q} B_j^T X_k B_j, \end{cases}$$

converge to the unique positive definite solution X.

LEMMA 2. (Theorem 2.2, [23]) *Assume that there exists a positive definite $n \times n$ matrix $\tilde{Q}$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$. Then the matrix*

$$K = I_{n^2} - \sum_{j=1}^{m} A_j^T \otimes A_j^T$$

is invertible.

REMARK 1. According the proof of Theorem 2.2 in [23], the matrix $K = I_{n^2} - \sum_{j=1}^{m} A_j^T \otimes A_j^T$ in Lemma 2 is actually stable with respect to the unite circle, that is, $\rho(K) < 1$.

THEOREM 1. *Suppose that*

$$\sum_{i=1}^{p} A_i^T Q A_i < \frac{Q}{2} \quad \text{and} \quad \sum_{j=1}^{q} B_j^T Q B_j < \frac{Q}{2}. \tag{3}$$

Then, the matrix

$$P = I_{n^2} - \sum_{i=1}^{p} A_i^T \otimes A_i^T + \sum_{j=1}^{q} B_j^T \otimes B_j^T$$

is invertible.

Proof. Under the assumption (3), it is clearly that

$$Q - \sum_{i=1}^{p} A_i^T Q A_i - \sum_{j=1}^{q} B_j^T Q B_j > 0. \tag{4}$$

We first prove that if $I - \sum_{i=1}^{p} A_i^T A_i - \sum_{j=1}^{q} B_j^T B_j > 0$ holds, then matrix P is invertible. Let λ be an eigenvalue of matrix $\sum_{i=1}^{p} A_i^T \otimes A_i - \sum_{j=1}^{q} B_j^T \otimes B_j$ and x a corresponding eigenvector, i.e.,

$$\Big(\sum_{i=1}^{p} A_i^T \otimes A_i - \sum_{j=1}^{q} B_j^T \otimes B_j\Big)x = \lambda x.$$

Set $X = \mathrm{vec}^{-1}(x)$, then X solves the following equation

$$\sum_{i=1}^{p} A_i^T X A_i - \sum_{j=1}^{q} B_j^T X B_j = \lambda X. \tag{5}$$

Let

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_p \\ B_1 \\ \vdots \\ B_q \end{pmatrix} \quad \text{and} \quad \hat{X} = \mathrm{diag}(X_1, X_2) \in \mathbb{R}^{(p+q)n \times (p+q)n},$$

where

$$X_1 = \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & X \end{pmatrix} \in \mathbb{R}^{pn \times pn} \quad \text{and} \quad X_2 = \begin{pmatrix} -X & 0 & \dots & 0 \\ 0 & -X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -X \end{pmatrix} \in \mathbb{R}^{qn \times qn}.$$

Equation (5) can be rewritten as

$$\lambda X = A^T \hat{X} A.$$

Note that $\|X\|_2 = \|\hat{X}\|_2$ and $\|A\|_2^2 = \lambda_{\max}(A^T A) = \|A^T A\|_2$, it yields

$$|\lambda| \|X\|_2 = \|A^T \hat{X} A\|_2 \leqslant \|A\|_2^2 \|\hat{X}\|_2 = \|A^T A\|_2 \|X\|_2.$$

Since $A^T A = \sum_{i=1}^{p} A_i^T A_i + \sum_{j=1}^{q} B_j^T B_j < I_n$, we have $|\lambda| \leqslant \|A^T A\|_2 < 1$, which shows that $\rho(\sum_{i=1}^{p} A_i^T \otimes A_i - \sum_{j=1}^{q} B_j^T \otimes B_j) < 1$. Thus matrix P is invertible.

Since Q is positive definite, it follows from (4) that

$$I_n - \sum_{i=1}^{p}(Q^{-\frac{1}{2}}A_i^TQ^{\frac{1}{2}})(Q^{\frac{1}{2}}A_iQ^{-\frac{1}{2}}) - \sum_{j=1}^{q}(Q^{-\frac{1}{2}}B_j^TQ^{\frac{1}{2}})(Q^{\frac{1}{2}}B_jQ^{-\frac{1}{2}}) > 0.$$

According to the first part of the proof it can be shown that

$$\rho\Big(\sum_{i=1}^{p}(Q^{-\frac{1}{2}}A_i^TQ^{\frac{1}{2}})\otimes(Q^{-\frac{1}{2}}A_i^TQ^{\frac{1}{2}}) - \sum_{j=1}^{q}(Q^{-\frac{1}{2}}B_j^TQ^{\frac{1}{2}})\otimes(Q^{-\frac{1}{2}}B_j^TQ^{\frac{1}{2}})\Big) < 1.$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{p}(Q^{-\frac{1}{2}}A_i^TQ^{\frac{1}{2}})\otimes(Q^{-\frac{1}{2}}A_i^TQ^{\frac{1}{2}}) - \sum_{j=1}^{q}(Q^{-\frac{1}{2}}B_j^TQ^{\frac{1}{2}})\otimes(Q^{-\frac{1}{2}}B_j^TQ^{\frac{1}{2}}) \qquad (6)\\
&= \sum_{i=1}^{p}(Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}})^{-1}(A_i^T\otimes A_i^T)(Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}}) - \sum_{j=1}^{q}(Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}})^{-1}(B_j^T\otimes B_j^T)(Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}})\\
&= (Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}})^{-1}\Big(\sum_{i=1}^{p}(A_i^T\otimes A_i^T) - \sum_{j=1}^{q}(B_j^T\otimes B_j^T)\Big)(Q^{\frac{1}{2}}\otimes Q^{\frac{1}{2}}),
\end{aligned}$$

which shows that matrix $\sum_{i=1}^{p}(A_i^T\otimes A_i^T) - \sum_{j=1}^{q}(B_j^T\otimes B_j^T)$ has the same eigenvalues as the matrix in (6). Hence

$$\rho\Big(\sum_{i=1}^{p}(A_i^T\otimes A_i^T) - \sum_{j=1}^{q}(B_j^T\otimes B_j^T)\Big) < 1.$$

Therefore, matrix $P = I_{n^2} - \sum_{i=1}^{p}A_i^T\otimes A_i^T + \sum_{j=1}^{q}B_j^T\otimes B_j^T$ is invertible. $\square$

3. Three kinds of condition numbers

3.1. Normwise condition number

In this section, we investigate the normwise condition number of equation (1). In the sequel, we always suppose $\sum_{i=1}^{p}A_i^TQA_i < \frac{Q}{2}$ and $\sum_{j=1}^{q}B_j^TQB_j < \frac{Q}{2}$. Under this assumption, matrix equation (1) always has a unique positive definite solution.

Define the mapping

$$\phi : (A_1,\ldots,A_p,B_1,\ldots,B_q,Q) \mapsto \mathrm{vec}(X), \qquad (7)$$

where X is the unique positive definite solution of equation (1). We define three kinds of normwise condition numbers by

$$k_i(\phi) = \lim_{\varepsilon\to 0}\sup_{\Delta_i\leqslant\varepsilon}\frac{\|\Delta X\|_F}{\varepsilon\|X\|_F}, \quad i = 1,2,3, \qquad (8)$$

where

$$\Delta_1 = \Big\| \Big[\frac{\|\Delta A_1\|_F}{\delta_1}, \ldots, \frac{\|\Delta A_p\|_F}{\delta_p}, \frac{\|\Delta B_1\|_F}{\theta_1}, \ldots, \frac{\|\Delta B_q\|_F}{\theta_q}, \frac{\|\Delta Q\|_F}{\sigma} \Big] \Big\|_2,$$

$$\Delta_2 = \max\Big\{ \frac{\|\Delta A_1\|_F}{\delta_1}, \ldots, \frac{\|\Delta A_p\|_F}{\delta_p}, \frac{\|\Delta B_1\|_F}{\theta_1}, \ldots, \frac{\|\Delta B_q\|_F}{\theta_q}, \frac{\|\Delta Q\|_F}{\sigma} \Big\},$$

$$\Delta_3 = \frac{\big\| [\|\Delta A_1\|_F, \ldots, \|\Delta A_p\|_F, \|\Delta B_1\|_F, \ldots, \|\Delta B_q\|_F, \|\Delta Q\|_F] \big\|_2}{\| [\|A_1\|_F, \ldots, \|A_p\|_F, \|B_1\|_F, \ldots, \|B_q\|_F, \|Q\|_F] \|_2}.$$

The nonzero parameters δ_i, θ_j and σ in Δ_1 and Δ_2 provide some freedom in how to measure the perturbations. Generally, δ_i, θ_j and σ are chosen, respectively, as functions of $\|A_i\|_F$, $\|B_j\|_F$ and $\|Q\|_F$, and the most natural choice is $\delta_i = \|A_i\|_F$, $\theta_j = \|B_j\|_F$ and $\sigma = \|Q\|_F$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$.

The perturbed equation of (1) is

$$\begin{aligned} &(X + \Delta X) - \sum_{i=1}^{p} (A_i + \Delta A_i)^T (X + \Delta X)(A_i + \Delta A_i) \\ &\quad + \sum_{j=1}^{q} (B_j + \Delta B_j)^T (X + \Delta X)(B_j + \Delta B_j) = Q + \Delta Q. \end{aligned} \tag{9}$$

Dropping the second and higher-order terms in (9) yields

$$\begin{aligned} 0 &= X - \sum_{i=1}^{p} A_i^T X A_i + \sum_{j=1}^{q} B_j^T X B_j - Q \\ &\approx -\Delta X + \sum_{i=1}^{p} (A_i^T X \Delta A_i + \Delta A_i^T X A_i + A_i^T \Delta X A_i) \\ &\quad - \sum_{j=1}^{q} (B_j^T X \Delta B_j + \Delta B_j^T X B_j + B_j^T \Delta X B_j) + \Delta Q. \end{aligned} \tag{10}$$

Applying the vec operator, we get

$$\begin{aligned} &\left(I_{n^2} - \sum_{i=1}^{p} A_i^T \otimes A_i^T + \sum_{j=1}^{q} B_j^T \otimes B_j^T \right) \mathrm{vec}(\Delta X) \\ &\approx \mathrm{vec}(\Delta Q) + \sum_{i=1}^{p} \Big(I \otimes (A_i^T X) + \big((A_i^T X) \otimes I\big)\Pi \Big) \mathrm{vec}(\Delta A_i) \\ &\quad - \sum_{j=1}^{q} \Big(I \otimes (B_j^T X) + \big((B_j^T X) \otimes I\big)\Pi \Big) \mathrm{vec}(\Delta B_j), \end{aligned} \tag{11}$$

where Π is the vec-permutation matrix satisfying $\Pi \mathrm{vec}(A) = \mathrm{vec}(A^T)$,

$$\Pi = \sum_{i=1}^{n} \sum_{j=1}^{n} E_{ij}(n \times n) \otimes E_{ji}(n \times n),$$

where $E_{ij} = e_i^{(n)}(e_j^{(n)})^T \in \mathbb{R}^{n\times n}$ and $e_i^{(n)}$ is an n-dimensional column vector which has 1 in the i-th position and 0's elsewhere. For more details about the permutation matrix Π, see [15, 17, 26].

Theorem 16.3.2 in [17] shows that for any $n\times n$ matrices A and B,

$$(A\otimes B)\Pi = \Pi(B\otimes A).$$

Then, (11) can be written as

$$\begin{aligned}
&\left(I_{n^2} - \sum_{i=1}^{p} A_i^T \otimes A_i^T + \sum_{j=1}^{q} B_j^T \otimes B_j^T\right) \mathrm{vec}(\Delta X) \\
&\approx \mathrm{vec}(\Delta Q) + \sum_{i=1}^{p} \Big[(I_{n^2}+\Pi)\big(I\otimes (A_i^T X)\big)\Big]\mathrm{vec}(\Delta A_i) \\
&\quad - \sum_{j=1}^{q} \Big[(I_{n^2}+\Pi)\big(I\otimes (B_j^T X)\big)\Big]\mathrm{vec}(\Delta B_j),
\end{aligned} \tag{12}$$

Let

$$\begin{aligned}
P &= I_{n^2} - \sum_{i=1}^{p} A_i^T \otimes A_i^T + \sum_{j=1}^{q} B_j^T \otimes B_j^T, \\
L &= (I_{n^2}+\Pi)\Big[I\otimes (A_1^T X),\ldots, I\otimes (A_p^T X), -I\otimes (B_1^T X),\ldots, -I\otimes (B_q^T X), (I_{n^2}+\Pi)^{-1}\Big], \\
r &= [\mathrm{vec}(\Delta A_1)^T,\ldots,\mathrm{vec}(\Delta A_p)^T, \mathrm{vec}(\Delta B_1)^T,\ldots,\mathrm{vec}(\Delta B_q)^T, \mathrm{vec}(\Delta Q)^T]^T.
\end{aligned}$$

It follows from (12) that

$$P\mathrm{vec}(\Delta X) \approx Lr. \tag{13}$$

Under the assumption $\sum_{i=1}^{p} A_i^T Q A_i < \frac{Q}{2}$ and $\sum_{j=1}^{q} B_j^T Q B_j < \frac{Q}{2}$, it follows from Theorem 1 that matrix P is invertible. Hence, $\mathrm{vec}(\Delta X) \approx P^{-1}Lr$.

THEOREM 2. *Using the notations given above, the explicit expressions or an upper bound of the three condition numbers defined in* (8) *are*

$$k_1(\phi) \approx \frac{\|P^{-1}L_1\|_2}{\|X\|_F}, \tag{14}$$

$$k_2(\phi) \lesssim \min\{\sqrt{p+q+1}\,k_1(\phi), \mu/\|X\|_F\}, \tag{15}$$

$$k_3(\phi) \approx \frac{\|P^{-1}L\|_2\sqrt{\sum_{i=1}^{m}\|A_i\|_F^2 + \sum_{j=1}^{n}\|B_j\|_F^2 + \|Q\|_F^2}}{\|X\|_F}, \tag{16}$$

where

$$L_1 = L\,\mathrm{diag}([\delta_1,\ldots,\delta_p,\theta_1,\ldots,\theta_q,\sigma]^T),$$

$$\mu = \sum_{i=1}^{p}\delta_i \left\|P^{-1}(I_{n^2}+\Pi)(I\otimes(A_i^TX))\right\|_2 + \sum_{j=1}^{q}\theta_j \left\|P^{-1}(I_{n^2}+\Pi)(I\otimes(B_j^TX))\right\|_2 + \sigma\left\|P^{-1}\right\|_2.$$

Proof. Equation (13) can be written as

$$P\mathrm{vec}(\Delta X) \approx L_1 r_1, \tag{17}$$

where

$$L_1 = (I_{n^2}+\Pi)\Big[\delta_1 I\otimes(A_1^TX),\ldots,\delta_p I\otimes(A_p^TX),-\theta_1 I\otimes(B_1^TX),\ldots, -\theta_q I\otimes(B_q^TX),\sigma(I_{n^2}+\Pi)^{-1}\Big],$$

$$r_1 = \left[\frac{\mathrm{vec}(\Delta A_1)^T}{\delta_1},\ldots,\frac{\mathrm{vec}(\Delta A_p)^T}{\delta_p},\frac{\mathrm{vec}(\Delta B_1)^T}{\theta_1},\ldots,\frac{\mathrm{vec}(\Delta B_q)^T}{\theta_q},\frac{\mathrm{vec}(\Delta Q)^T}{\sigma}\right]^T.$$

It follows form (17) that

$$\|\Delta X\|_F \lesssim \|P^{-1}L_1\|_2\|r_1\|_2. \tag{18}$$

Since $\|r_1\|_2 = \Delta_1 \leqslant \varepsilon$, according to (8) we arrive at (14).

We now show (16) is true. It follows from (13) that

$$\begin{aligned}\|\Delta X\|_F &\lesssim \|P^{-1}L\|_2\|r\|_2 \\ &\leqslant \|P^{-1}L\|_2\Delta_3\cdot\sqrt{\sum_{i=1}^{m}\|A_i\|_F^2+\sum_{j=1}^{n}\|B_j\|_F^2+\|Q\|_F^2} \\ &\leqslant \varepsilon\|P^{-1}L\|_2\sqrt{\sum_{i=1}^{m}\|A_i\|_F^2+\sum_{j=1}^{n}\|B_j\|_F^2+\|Q\|_F^2}.\end{aligned}$$

By (8), a direct computation yields (16).

Let $\varepsilon = \Delta_2$. According to inequality (18) we obtain

$$\begin{aligned}\|\Delta X\|_F &\lesssim \|P^{-1}L_1\|_2\sqrt{\sum_{i=1}^{p}\frac{\|\Delta A_i\|_F^2}{\delta_i^2}+\sum_{j=1}^{q}\frac{\|\Delta B_j\|_F^2}{\theta_j^2}+\frac{\|\Delta Q\|_F^2}{\sigma^2}} \\ &\leqslant \varepsilon\sqrt{p+q+1}\|P^{-1}L_1\|_2 \\ &\lesssim \varepsilon k_1(\phi)\sqrt{p+q+1}\|X\|_F.\end{aligned} \tag{19}$$

On the other hand, it follows form (17) that

$$\begin{aligned}\operatorname{vec}(\Delta X) \approx &\sum_{i=1}^{p} \delta_i P^{-1}(I_{n^2}+\Pi)(I\otimes(A_i^T X))\frac{\operatorname{vec}(\Delta A_i)}{\delta_i}\\ &-\sum_{j=1}^{q}\theta_j P^{-1}(I_{n^2}+\Pi)(I\otimes(B_j^T X))\frac{\operatorname{vec}(\Delta B_j)}{\theta_j}+\sigma P^{-1}\frac{\operatorname{vec}(\Delta Q)}{\sigma},\end{aligned}$$

which yields

$$\begin{aligned}\|\Delta X\|_F \lesssim &\sum_{i=1}^{p}\delta_i \left\|P^{-1}(I_{n^2}+\Pi)(I\otimes(A_i^T X))\right\|_2\frac{\|\Delta A_i\|_F}{\delta_i}\\ &+\sum_{j=1}^{q}\theta_j\left\|P^{-1}(I_{n^2}+\Pi)(I\otimes(B_j^T X))\right\|_2\frac{\|\Delta B_j\|_F}{\theta_j}+\sigma\left\|P^{-1}\right\|_2\frac{\|\Delta Q\|_F}{\sigma}\\ \leqslant{} &\varepsilon\mu. \end{aligned}\tag{20}$$

Finally, from inequalities (19) and (20), we obtain (15). □

REMARK 2. Theorem 2.2 shows that the normwise condition numbers will be large if matrix P is ill-conditioned. Hence the condition number of matrix P is a good indication of the magnitudes of the normwise condition numbers $k_i(\phi)$ for $i=1,2,3$.

3.2. Mixed and componentwise condition numbers

In this section, we investigate the mixed and componentwise condition numbers of equation (1). The explicit expressions of these two kinds of condition numbers are derived. To define mixed and componentwise condition numbers, the following distance function is useful. For any $a,b\in\mathbb{R}^n$, we define $\frac{a}{b}=[c_1,c_2,\ldots,c_n]^T$ as

$$c_i=\begin{cases} a_i/b_i, & \text{if } b_i\neq 0,\\ 0, & \text{if } a_i=b_i=0,\\ \infty, & \text{otherwise.}\end{cases}$$

Then we define

$$d(a,b)=\left\|\frac{a-b}{b}\right\|_\infty=\max_{i=1,2,\ldots,n}\left\{\left|\frac{a_i-b_i}{b_i}\right|\right\}.$$

For matrices $A,B\in\mathbb{R}^{n\times n}$, we define

$$d(A,B)=d(\operatorname{vec}(A),\operatorname{vec}(B)).$$

Note that if $d(a,b)<\infty$, $d(a,b)=\min\{\nu\geqslant 0 \,|\, |a_i-b_i|\leqslant\nu|b_i| \text{ for } i=1,2,\ldots,n\}$. Throughout this paper, we will only consider pairs (a,b) for which $d(a,b)<\infty$. For $\varepsilon>0$, we set $B^0(a,\varepsilon)=\{x \,|\, d(x,a)\leqslant\varepsilon\}$. For a vector-valued function $F:\mathbb{R}^p\to\mathbb{R}^q$, we denote $\operatorname{Dom}(F)$ as the domain of F.

The mixed and componentwise condition numbers introduced by Gohberg and Koltracht [14] are listed as follows:

DEFINITION 1. ([14]) Let $F:\mathbb{R}^p\to\mathbb{R}^q$ be a continuous mapping defined on an open set $\mathrm{Dom}(F)\subset\mathbb{R}^p$ such that $\mathbf{0}\notin\mathrm{Dom}(F)$ and $F(a)\neq 0$ for a given $a\in\mathbb{R}^p$.
(1) The mixed condition number of F at a is defined by

$$m(F,a)=\lim_{\varepsilon\to 0}\sup_{\substack{x\in B^0(a,\varepsilon)\\ x\neq a}}\frac{\|F(x)-F(a)\|_\infty}{\|F(a)\|_\infty}\frac{1}{d(x,a)}.$$

(2) Suppose $F(a)=\left[f_1(a),f_2(a),\ldots,f_q(a)\right]^T$ such that $f_j(a)\neq 0$ for $j=1,2,\ldots,q$. The componentwise condition number of F at a is defined by

$$c(F,a)=\lim_{\varepsilon\to 0}\sup_{\substack{x\in B^0(a,\varepsilon)\\ x\neq a}}\frac{d(F(x),F(a))}{d(x,a)}.$$

The explicit expressions of the mixed and componentwise condition numbers of F at a are given by the following lemma [6, 14].

LEMMA 3. *Suppose F is Fréchet differentiable at a. We have*
(1) if $F(a)\neq 0$, then

$$m(F,a)=\frac{\|F'(a)\mathrm{diag}(a)\|_\infty}{\|F(a)\|_\infty}=\frac{\||F'(a)||a|\|_\infty}{\|F(a)\|_\infty};$$

(2) if $F(a)=\left[f_1(a),f_2(a),\ldots,f_q(a)\right]^T$ such that $f_j(a)\neq 0$ for $j=1,2,\ldots,q$, then

$$c(F,a)=\|\mathrm{diag}^{-1}(F(a))F'(a)\mathrm{diag}(a)\|_\infty=\left\|\frac{|F'(a)||a|}{|F(a)|}\right\|_\infty.$$

THEOREM 3. *Let $m(\phi)$ and $c(\phi)$ be the mixed and componentwise condition numbers of equation (1), we have*

$$m(\phi)\approx\frac{\|T\|_\infty}{\|X\|_{\max}}\quad\text{and}\quad c(\phi)\approx\left\|\frac{T}{|\mathrm{vec}(X)|}\right\|,$$

where ϕ is defined in (7),

$$T=\sum_{i=1}^{p}\left|P^{-1}(I_{n^2}+\Pi)(I\otimes(A_i^TX))\right|\mathrm{vec}(|A_i|)$$
$$+\sum_{j=1}^{q}\left|P^{-1}(I_{n^2}+\Pi)(I\otimes(B_j^TX))\right|\mathrm{vec}(|B_j|)+|P^{-1}|\mathrm{vec}(|Q|).$$

Furthermore, we have two upper bounds for $m(\phi)$ and $c(\phi)$ as follows:

$$m_U(\phi):=\frac{\|P^{-1}\|_\infty\|M\|_{\max}}{\|X\|_{\max}}\gtrsim m(\phi),$$

and

$$c_U(\phi) := \|\mathrm{diag}^{-1}(\mathrm{vec}(X))P^{-1}\|_\infty \|M\|_{\max} \gtrsim c(\phi).$$

where

$$M = \sum_{i=1}^{p} \left(|A_i^T X||A_i| + |A_i|^T |XA_i|\right) + \sum_{j=1}^{q} \left(|B_j^T X||B_j| + |B_j|^T |XB_j|\right) + |Q|.$$

Proof. According to (13), we get the Fr*é*chet derivative of ϕ defined in (7)

$$\phi'(A_1, \ldots, A_p, B_1, \ldots, B_q, Q) \approx P^{-1}L.$$

Let $a = [\mathrm{vec}(A_1)^T, \ldots, \mathrm{vec}(A_p)^T, \mathrm{vec}(B_1)^T, \ldots, \mathrm{vec}(B_q)^T, \mathrm{vec}(Q)^T]^T$. It follows from (1) of Lemma 3.2 that

$$m(\phi) \approx \frac{\||P^{-1}L||a|\|_\infty}{\|\mathrm{vec}(X)\|_\infty} = \frac{\||P^{-1}L||a|\|_\infty}{\|X\|_{\max}} = \frac{\|T\|_\infty}{\|X\|_{\max}},$$

where

$$\begin{aligned}
T &= |P^{-1}L||a| \\
&= \sum_{i=1}^{p} \left|P^{-1}(I_{n^2} + \Pi)(I \otimes (A_i^T X))\right| \mathrm{vec}(|A_i|) \\
&\quad + \sum_{j=1}^{q} \left|P^{-1}(I_{n^2} + \Pi)(I \otimes (B_j^T X))\right| \mathrm{vec}(|B_j|) + |P^{-1}|\mathrm{vec}(|Q|).
\end{aligned}$$

It holds that

$$\begin{aligned}
\|T\|_\infty &\leqslant \||P^{-1}||L||a|\|_\infty \\
&\leqslant \|P^{-1}\|_\infty \||L||a|\|_\infty \\
&\leqslant \|P^{-1}\|_\infty \Big\| \sum_{i=1}^{p} \left(I \otimes |A_i^T X| + (|A_i^T X| \otimes I)\Pi\right) \mathrm{vec}(|A_i|) \\
&\quad + \sum_{j=1}^{q} \left(I \otimes |B_j^T X| + (|B_j^T X| \otimes I)\Pi\right) \mathrm{vec}(|B_j|) + \mathrm{vec}(|\mathrm{Q}|) \Big\|_\infty \\
&\leqslant \|P^{-1}\|_\infty \Big\| \sum_{i=1}^{p} \left(|A_i^T X||A_i| + |A_i|^T |XA_i|\right) \\
&\quad + \sum_{j=1}^{q} \left(|B_j^T X||B_j| + |B_j|^T |XB_j|\right) + |Q| \Big\|_{\max}.
\end{aligned}$$

Therefore,

$$m(\phi) \lesssim \frac{\|P^{-1}\|_\infty \|M\|_{\max}}{\|X\|_{\max}}.$$

According to (2) of Lemma 3.2, we obtain

$$c(\phi) \approx \left\| \frac{|P^{-1}L||a|}{|\mathrm{vec}(X)|} \right\|_\infty = \left\| \frac{T}{|\mathrm{vec}(X)|} \right\|_\infty .$$

It holds that

$$\begin{aligned} c(\phi) &\lesssim \left\| \frac{|P^{-1}||L||a|}{|\mathrm{vec}(X)|} \right\|_\infty \\ &= \left\| \mathrm{diag}^{-1}(\mathrm{vec}(X))|P^{-1}||L||a| \right\|_\infty \\ &\leqslant \left\| \mathrm{diag}^{-1}(\mathrm{vec}(X))|P^{-1}| \right\|_\infty \||L||a|\|_\infty \\ &= \|\mathrm{diag}^{-1}(\mathrm{vec}(X))P^{-1}\|_\infty \|M\|_{\max}. \quad \square \end{aligned}$$

4. Numerical examples

In this section, we give two numerical examples to illustrate the effectiveness of our results about normwise, mixed and componentwise condition numbers. All computations are made in MATLAB 7.10.0 with the unit roundoff being $u \approx 2.2 \times 10^{-16}$.

EXAMPLE 1. (Example 3.1, [10]) We consider the matrix equation

$$X = I + A_1^T X A_1 + A_2^* X A_2 - B_1^* X B_1 - B_2^* X B_2,$$

where

$$A_1 = \begin{pmatrix} 0.02 & -0.01 & -0.02 \\ 0.08 & -0.01 & 0.02 \\ -0.06 & -0.12 & 0.14 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.08 & -0.10 & -0.02 \\ 0.08 & -0.10 & 0.02 \\ -0.06 & -0.12 & 0.14 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.47 & 0.02 & 0.04 \\ -0.10 & 0.36 & -0.02 \\ -0.04 & 0.01 & 0.47 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.10 & 0.10 & 0.05 \\ 0.15 & 0.275 & 0.075 \\ 0.05 & 0.05 & 0.175 \end{pmatrix}.$$

Suppose that coefficient matrices A_1, A_2, B_1 and B_2 are perturbed by

$$\Delta A_1 = \begin{pmatrix} 0.5 & 0.1 & -0.2 \\ -0.4 & 0.2 & 0.6 \\ -0.2 & 0.1 & -0.1 \end{pmatrix} \times 10^{-j}, \quad \Delta A_2 = \begin{pmatrix} -0.4 & 0.10 & -0.2 \\ 0.5 & 0.7 & -1.3 \\ 1.1 & 0.9 & 0.6 \end{pmatrix} \times 10^{-j},$$

$$\Delta B_1 = \begin{pmatrix} 0.8 & 0.2 & 0.05 \\ -0.2 & 0.12 & 0.14 \\ -0.25 & -0.2 & 0.26 \end{pmatrix} \times 10^{-j}, \quad \Delta B_2 = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ -0.3 & 0.15 & -0.15 \\ 0.1 & -0.1 & 0.25 \end{pmatrix} \times 10^{-j},$$

where $j \geqslant 2$ is a parameter.

j	$\|\Delta X\|/\|X\|_F$	$k_1(\phi)\Delta_1$	$k_2^U(\phi)\Delta_2$	$k_2^M(\phi)\Delta_2$	$k_3(\phi)\Delta_3$	cond$(P)\Delta_1$
2	1.6332e-003	9.7027 e-003	1.6115e-002	1.0851e-002	3.6007e-003	9.9618e-003
3	1.6319e-004	9.7028e-004	1.600e-003	1.100e-003	3.6007e-004	9.8618e-004
4	1.6317e-005	9.7028e-005	1.6115e-004	1.0851e-004	2.5459e-005	9.8618e-005
5	1.6317e-006	9.7028e-006	1.6115e-005	1.0851e-005	3.6007e-006	9.8618e-006
6	1.6317e-007	9.7028e-007	1.6115e-006	1.0851e-006	3.6007e-007	9.8618e-007

Table 1: *Comparison of the relative error* $\|\Delta X\|_F/\|X\|_F$ *with our estimates and the values of* cond$(P)\Delta_1$

Using the iterative method in Lemma 2.1

$$\begin{cases} X_0 = 0, Y_0 = 2I, \\ X_{k+1} = I + \sum_{i=1}^{2} A_i^T X_k A_i - \sum_{j=1}^{2} B_j^T Y_k B_j, \\ Y_{k+1} = I + \sum_{i=1}^{2} A_i^T Y_k A_i - \sum_{j=1}^{2} B_j^T X_k B_j, \end{cases}$$

and letting the iterations terminate if

$$\rho(X_k) = \|X - A_1^* X A_1 - A_2^* X A_2 + B_1^* X B_1 + B_2^* X B_2 - I\|_F \leqslant nu,$$

we can get the positive definite solution X and the corresponding positive definite solution $\tilde{X}$ of the perturbed equation.

Set $\delta_i = \|A_i\|_F$, $\theta_j = \|B_j\|_F$, $\sigma = \|Q\|_F$ and denote $k_2^U = \sqrt{n} k_1(\phi)$ and $k_2^M(\phi) = \mu/\|X\|_F$. From Theorem 2.2, we can obtain three first order normwise perturbation bounds (first order bounds) $\|\Delta X\|_F/\|X\|_F \lesssim k_i(\phi)\Delta_i$. For different j, we compare the relative error $\|\Delta X\|_F/\|X\|_F$ with the three approximation bounds. The last column of Table 1 corresponds to $k_1(\phi)\Delta_1$ and the result is similar for $k_2(\phi)\Delta_2$ and $k_3(\phi)\Delta_3$. We can see that the condition number of P is a good indiction of $k_i(\phi)$ as pointed out in Remark 2.3.

EXAMPLE 2. ([3]) Consider the matrix equation

$$X = Q + A_1^* X A_1 + A_2^* X A_2 - B_1^* X B_1 - B_2^* X B_2,$$

where

$$A_1 = \begin{pmatrix} 0.1 & 0.05 & 0.05 \\ 0.05 & 0.1 & 0.05 \\ 0.05 & 0.05 & 0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & -0.02 & -0.02 \\ -0.02 & 0.5 & -0.02 \\ -0.02 & -0.02 & 0.5 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.01 & 0.001 & 0.01 \\ 0.001 & 0.01 & 0.001 \\ 0.01 & 0.001 & 0.01 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.1413 & 0.008294 & 0.1413 \\ 0.008294 & 0.1997 & 0.008294 \\ 0.1413 & 0.008294 & 0.1413 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix}.$$

j	6	8	10
γ_k	8.3398e-007	9.3512e-009	7.0116e-011
$k_1(\phi)\Delta_1$	2.0335e-006	2.6917e-008	2.4987e-010
$k_2^U(\phi)\Delta_2$	4.5470e-006	6.0187e-008	5.5872e-010
$k_2^M(\varphi)\Delta_2$	1.3895e-006	1.9198e-008	1.8097e-010
$k_3(\varphi)\Delta_3$	2.3089e-006	2.4992e-008	1.5083e-010
γ_m	1.1743e-006	1.0039e-008	1.0246e-010
$m(\phi)\varepsilon_0$	2.1418e-006	2.0670e-008	2.1614e-010
γ_c	1.1743e-006	1.0039e-008	1.0306e-010
$c(\phi)\varepsilon_0$	1.5549e-005	1.5007e-007	1.5692e-009

Table 2: *Linear asymptotic bounds*

Let $\tilde{A}_k = A_k + \text{rand}(3) \circ A_k \times 10^{-j}$, $\tilde{B}_k = B_k + \text{rand}(3) \circ B_k \times 10^{-j}$, $\tilde{Q} = Q + \text{rand}(3) \circ Q \times 10^{-j}$, where $k = 1,2,\ \circ$ is the Hadamard product.

Using the iterative method in Lemma 2.1

$$\begin{cases} X_0 = 0, \ \ Y_0 = 2Q, \\ X_{k+1} = Q + \sum_{i=1}^2 A_i^T X_k A_i - \sum_{j=1}^2 B_j^T Y_k B_j, \\ Y_{k+1} = Q + \sum_{i=1}^2 A_i^T Y_k A_i - \sum_{j=1}^2 B_j^T X_k B_j, \end{cases}$$

and letting the iterations terminate if the relative residual satisfies

$$\rho(X_k) = \|X - A_1^* X A_1 - A_2^* X A_2 + B_1^* X B_1 + B_2^* X B_2 - Q\|_F \leqslant nu,$$

we can get the positive definite solution X and the corresponding positive definite solution $\tilde{X}$ of the perturbed equation.

Similarly to Example 4.1, we obtain three first order normwise perturbation bounds: $\|\Delta X\|_F / \|X\|_F \lesssim k_i(\phi)\Delta_i$. Let $|\Delta A_i| \leqslant \varepsilon |A_i|$, $|\Delta B_j| \leqslant \varepsilon |B_j|$ and $|\Delta Q| \leqslant \varepsilon |Q|$ for $i = 1, \ldots p$ and $j = 1, \ldots, q$, we obtain the first order mixed and componentwise perturbation bounds $\|\Delta X\|_{\max} / \|X\|_{\max} \lesssim \varepsilon m(\phi)$ and $\|\text{vec}(\Delta X) ./ \text{vec}(X)\|_\infty \lesssim \varepsilon c(\phi)$.
Denote

$$\gamma_k = \frac{\|\Delta X\|_F}{\|X\|_F}, \quad \gamma_m = \frac{\|\Delta X\|_{\max}}{\|X\|_{\max}}, \quad \gamma_c = \left\| \frac{\Delta X}{X} \right\|_{\max},$$

and

$$\varepsilon_0 = \min\{\varepsilon : |\Delta A_i| \leqslant \varepsilon |A_i|, |\Delta B_j| \leqslant \varepsilon |B_j|, |\Delta Q| \leqslant \varepsilon |Q|\}$$

for $i = 1,2$ and $j = 1,2$.

Table 2 shows that the first order bounds given by the three condition numbers are almost tight.

5. Conclusion

In this paper, we investigate normwise, mixed and componentwise condition numbers of matrix equation $X-\sum_{i=1}^{p}A_i^TXA_i+\sum_{j=1}^{q}B_j^TXB_j=Q$. The explicit expressions for the three condition numbers are derived. The upper bounds for the mixed and componentwise condition numbers are presented. The first order asymptotic bounds given by the three condition numbers are almost tight.

REFERENCES

[1] W. N. ANDERSON, T. D. MORLEY, G. E. TRAPP, *Ladder networks, fixed points and the geometric mean*, Circuits Systems Signal Process. **3**, 3 (1983), 259–268.

[2] T. ANDO, *Limit of Iterates of Cascade Addition of Matrices*, Numer. Funct. Anal. Optim. **21**, 7–8 (1980), 579–589.

[3] M. BERZIG, *Solving a class of matrix equation via Bhaskar-Lakshmikantham coupled fixed point theorem*, Appl. Math. Lett. **25**, 11 (2012), 1638–1643.

[4] T. G. BHASKAR, V. LAKSHMIKANTHAM, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65**, 7 (2006), 1379–1393.

[5] B. L. BUZBZZ, G. H. GOLUB, C. W. NIELSON, *On direct methods for solving Poisson's equations*, SIAM J. Numer. Anal. **7**, 4 (1970), 627–656.

[6] F. CUCKER, H. DIAO, Y. WEI, *On mixed and componentwise condition mumbers for Moore-Penrose inverse and linear least squares problems*, Math. Comp. **76**, 258 (2007), 947–963.

[7] T. DAMM, *Direct methods and ADI-preconditioned Krylov subspace methods for generalized Lyapunov equations*, Numer. Linear Algebra Appl. **15**, 9 (2008), 853–871.

[8] H.-A. DIAO, Y. WEI, S. QIAO, *Structured condition numbers of structured Tikhonov regularization problem and their estimations*, J. Comput. Appl. Math. **308**, (2016), 276–300.

[9] X. DUAN, A. LIAO, B. TANG, *On the nonlinear matrix equation* $X-\sum_{i=1}^{m}A_i^*X^{\delta_i}A_i=Q$, Linea Algebra Appl. **429**, 1 (2008), 110–121.

[10] X.-F. DUAN, Q.-W. WANG, *Perturbation analysis for the matrix equation* $X-\sum_{i=1}^{m}A_i^*XA_i+\sum_{j=1}^{n}B_j^*XB_j=I$, J. Appl. Math. (2012), Art. ID 784620, 13 pp.

[11] S. M. EL-SAYED, A. C. M. RAN, *On an iteration method for solving a class of nonlinear matrix eqautions*, SIAM J. Matrix Anal. Appl. **23**, 3 (2002), 632–645.

[12] J. C. ENGWERDA, *On the existence of a positive solution of the matrix equation* $X+A^TX^{-1}A=I$, Linear Algebra Appl. **194**, (1993), 91–108.

[13] Z. GAJIC, M. T. J. QURESHI, *Lyapunov matrix equation in system stability and control*, Academic Press, London, UK, 1995.

[14] I. GOHBERG, I. KOLTRACHT, *Mixed, componentwise and structured condition numbers*, SIAM J. Matrix Anal. Appl. **14**, 3 (1993), 688–704.

[15] A. GRAHAM, *Kronecker Products and Matrix Calculus with Applications*, Ellis Horwood Ltd: New York, NY, USA, 1981.

[16] W.L. GREEN, E. KAMEN, *Stabilization of linear systems over a commutative normed algebra with allications to spatially-distributed parameter-dependent systems*, SIAM J. Control Optim. **23**, 1 (1985), 1–18.

[17] D. A. HARVILLE, *Matrix algebra from a statistician's perspective*, Springer-Verlag, New York, 1997.

[18] N. J. HIGHAM, *A survey of componentwise perturbation theory in numerical linear algebra*, Proc. Sympos. Appl. Math. **48**, Amer. Math. Soc., Providence, RI, 1994.

[19] L. D. LIU, *Mixed and componentwise condition numbers of nonsymmetric algebraic Riccati equation*, Appl. Math. Comput. **218**, 14 (2012), 7595–7601.

[20] J. MENG, H.-M. KIM, *The positive definite solution to a nonlinear matrix equation*, Linear Multilinear Algebra **64**, 4 (2016), 653–666.

[21] W. PUSZ, S. L. WORONOWITZ, *Functional calsulus for sequiliear forms and the purification map*, Rep. Math. Phys. **8**, 2 (1975), 159–170.

[22] A. C. M. RAN, M. C. B. REURINGS, *On the nonlinear matrix equation* $X+A^*\mathscr{F}(X)A=Q$*: solution and perturbation theory*, Linear Algebra Appl. **346**, 1–3 (2002), 15–26.

[23] A. C. M. RAN, M. C. B. REURINGS, *The symmetric linear matrix equation*, Electron. J. Linear Algebra **9**, (2002), 93–107.

[24] A. C. M. RAN, M. C. B. REURINGS, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Pro. Amer. Math. Soc. **132**, 5 (2004), 1435–1443.

[25] L. A. SAKHNOVICH, *Interpolation theorey and its applicaitons, mathematics and its applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[26] J. R. SCHOTT, *Matrix Analysis for Statistics*, 2nd Edition. John Wiley Sons, Inc: Hoboken, NJ, USA, 2005.

[27] J.-G. XUE, S.-F. XU, R.-C. LI, *Accurate solutions of M-matrix sylvester equations*, Numer. Math. **120**, 4 (2012), 639–670.

[28] J.-G. XUE, S.-F. XU, R.-C. LI, *Accurate solutions of M-matrix algebraic Riccati equations*, Numer. Math. **120**, 4 (2012), 671–700.

[29] W.-G. WANG, C.-S. WANG, Y.-M. WEI, P.-P. XIE, *Mixed, componentwise condition numbers and small sample statistical condition estimation for generalized spectral projections and matrix sign functions*, Taiwanese J. Math. **20**, 2 (2016), 333–363.

[30] L. ZHANG, J. LAM, *On H_2 model reduction of bilinear systems*, Automatica. **38**, 2 (2002), 205–216.

[31] L.-M. ZHOU, Y.-Q. LIN, Y.-M. WEI, S.-Z. QIAO, *Perturbation analysis and condition numbers of symmetric algebraic Riccati equations*, Automatica **45**, 4 (2009), 1005–1011.

(Received September 19, 2017)

Jie Meng
Department of Mathematics
Pusan National University
Busan, 46241, Republic of Korea
e-mail: mengjiehw@163.com

Hyun-Min Kim
Department of Mathematics
Pusan National University
Busan, 46241, Republic of Korea
e-mail: hyunmin@pusan.ac.kr