Operators
and
Matrices
Volume 13, Number 1 (2019), 61–83 doi:10.7153/oam-2019-13-04

HYPONORMAL TOEPLITZ OPERATORS WITH NON–HARMONIC SYMBOL ACTING ON THE BERGMAN SPACE

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(*Communicated by R. Curto*)

Abstract. The Toeplitz operator acting on the Bergman space $A^2(\mathbb{D})$, with symbol φ is given by $T_\varphi f = P(\varphi f)$, where P is the projection from $L^2(\mathbb{D})$ onto the Bergman space. We present some history on the study of hyponormal Toeplitz operators acting on $A^2(\mathbb{D})$, as well as give results for when φ is a non-harmonic polynomial. We include a first investigation of Putnam's inequality for hyponormal operators with non-analytic symbols. Particular attention is given to unusual hyponormality behavior that arises due to the extension of the class of allowed symbols. For instance, in a peculiar example, perturbation of a self-adjoint operator by a subnormal operator of arbitrarily small (though not arbitrarily large!) norm yields an operator that is not hyponormal.

1. Introduction

Let H be a complex Hilbert space and T be a bounded linear operator acting on H with adjoint T^*. Operator T is said to be *hyponormal* if $[T^*,T] := T^*T - TT^* \geqslant 0$. That is, if for all $u \in H$

$$\langle [T^*,T]u,u\rangle \geqslant 0.$$

The study of hyponormal operators is strongly related to the spectral and perturbation theories of Hilbert space operators, singular integral equations, and scattering theory. The interested reader is referred to the monograph [10] by M. Martin and M. Putinar. One particularly interesting result for hyponormal operators, Putnam's inequality, states that if T is hyponormal, then

$$\|[T^*,T]\| \leqslant \frac{\text{Area}(\sigma(T))}{\pi},$$

where $\sigma(T)$ denotes the spectrum of T (cf. [2]).

We study the hyponormality of certain operators acting on the Bergman space

$$A^2(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\}.$$

Mathematics subject classification (2010): 47B35, 47B20.
Keywords and phrases: Toeplitz operator, Bergman space, hyponormality, non-harmonic symbol.
The work of C. Liaw was supported by NSF-DMS1802682.

Paper OaM-13-04

Let $\varphi \in L^\infty(\mathbb{D})$. The *Toeplitz operator* T_φ is given by

$$T_\varphi f = P(\varphi f) \qquad f \in A^2(\mathbb{D}),$$

where P is the orthogonal projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$.

In the Hardy space setting the question of when T_φ is hyponormal for $\varphi \in L^\infty(\mathbb{T})$ was answered by C. Cowen in [4], who proved the following theorem:

THEOREM 1. *Let $\varphi \in L^\infty(\mathbb{T})$ be given by $\varphi = f + \bar{g}$, with $f, g \in H^2$. Then T_φ is hyponormal if and only if*

$$g = c + T_{\bar{h}} f,$$

for some constant c and some $h \in H^\infty(\mathbb{D})$, with $\|h\|_\infty \leqslant 1$.

This result completely characterized hyponormal Toeplitz operators acting on the Hardy space. Cowen's proof relies on a dilation theorem of D. Sarason [14, Theorem 1], and the fact that $\left(H^2\right)^\perp$ is just the conjugates of H^2 functions which vanish at the origin.

In the Bergman space setting, where we lack an analog to Sarason's dilation theorem, and where $\left(A^2\right)^\perp$ is a much larger space, a similar characterization is lacking. One of the principle difficulties in exploring questions of hyponormality originates from the behavior of the self-commutator under operator addition. In particular, if we let u be in a complex Hilbert space H, and T and S be operators on H, then we find

$$\begin{aligned} &\langle [(T+S)^*, T+S]u, u\rangle \\ &= \langle Tu, Tu\rangle - \langle T^*u, T^*u\rangle + 2\mathrm{Re}\left[\langle Tu, Su\rangle - \langle T^*u, S^*u\rangle\right] + \langle Su, Su\rangle - \langle S^*u, S^*u\rangle . \end{aligned} \tag{1}$$

As we shall see, the "cross-terms" $2\mathrm{Re}\left[\langle Tu, Su\rangle - \langle T^*u, S^*u\rangle\right]$ lead to many somewhat unexpected results which reveals a subtlety in the study of hyponormal operators. The explicit expressions in (1) lead to involved series computations. Our primary effort consists of extracting reasonable necessary and/or sufficient conditions from series corresponding to several different types of non-harmonic symbols. It is worth noting that if both T and S are Toeplitz operators with harmonic symbols, then these cross terms vanish, which leads to a smoother study of such operators, e.g. in [1], [8], and [13].

One of the central questions this paper explores is the following:

Given a hyponormal Toeplitz operator T_φ acting on $A^2(\mathbb{D})$ and a symbol $\psi \in L^\infty(\mathbb{D})$, when is $T_{\varphi+\psi}$ hyponormal?

When ψ is not harmonic, this question turns out to be particularly elusive. As we shall see in Section 3, even requiring that T_ψ be self-adjoint is not enough to guarantee the hyponormality of $T_{\varphi+\psi}$.

We are also interested in some spectral properties of hyponormal T_φ, especially because the commutator has interesting interactions with the geometry of the image $\varphi(\mathbb{D})$. It is an immediate consequence of Putnam's inequality and the spectral mapping theorem (cf. [12, p. 263]) that the norm of the commutator of T_φ^* and T_φ is bounded above by $\mathrm{Area}(\varphi(\mathbb{D}))/\pi$ for *analytic* φ, and in [11] it was shown that this bound can be improved to $\text{Area}(\varphi(\mathbb{D}))/(2\pi)$ for analytic and univalent φ. In [7], it was conjectured that the hypothesis "univalent" is superfluous for this stronger bound. We extend this conjecture to non-analytic symbols.

The paper proceeds as follows: In Section 2, we give an overview of some known results for the hyponormality results of Toeplitz operators with harmonic symbols. This overview is by no means exhaustive, but gives a flavor for the types of results in this area to date. Of particular note is that questions of hyponormality even of operators with harmonic polynomials as symbols have still not been completely answered, as well as the elusiveness of both necessary and sufficient conditions for hyponormality. In Section 3, we focus on operators with symbols which are not harmonic. We give several sufficient conditions for the hyponormality of certain operators whose symbol is a non-harmonic polynomial, as well as several examples which indicate that the situation is rather subtle. Finally, in Section 4, we look at operators whose symbols satisfy $\varphi(z) = a_1 z^{m_1}\bar{z}^{n_1} + \ldots + a_k z^{m_k}\bar{z}^{n_k}$, with $m_1 - n_1 = \ldots = m_k - n_k = \delta \geqslant 0$. In particular we observe that the arguments of the coefficients of φ may play a non-trivial role in the hyponormality of T_φ.

Acknowledgement. Many thanks to B. Simanek and D. Khavinson for many inspiring and insightful discussions, and to C. Cowen for his very helpful correspondance and encouragement.

2. Toeplitz operators with harmonic symbol

The study of hyponormal operators with harmonic symbols is greatly simplified by the lack of cross-terms. In particular, if $\varphi = f + \bar{g}$ where f and g are holomorphic and bounded in $\mathbb{D}$ then one may show that the cross-term $2\text{Re}\left[\left\langle T_f u, T_{\bar{g}} u\right\rangle - \left\langle T_{\bar{f}} u, T_g u\right\rangle\right]$ vanishes. Thus, one can show the hyponormality of T_φ by showing that $\left\|H_{\bar{f}} u\right\|^2 \geqslant \left\|H_{\bar{g}} u\right\|^2$ for all u in the Bergman space, where $H_{\bar{\varphi}}$ is the *Hankel operator* $I - T_{\bar{\varphi}}$.

In [13], H. Sadraoui examined the hyponormality of Toeplitz operators T_φ acting on the Bergman space when φ is harmonic. One of his first results, [13, Prop. 1.4.3], gave a necessary boundary condition for f and g whenever f' is in the Hardy space. This result is particularly interesting because in the Bergman space, boundary value results are so rare.

THEOREM 2. *Let f and g be bounded analytic functions, such that $f' \in H^2$. If $T_{f+\bar{g}}$ is hyponormal, then $g' \in H^2$ and $|g'| \leqslant |f'|$ almost everywhere on $\mathbb{T}$.*

He also showed that this result is sharp, but not in general sufficient. In particular, he proved the following theorem [13, Prop. 1.4.4] for harmonic polynomials.

THEOREM 3. *Consider the operator $T_{z^n + \alpha\bar{z}^m}$.*

1. If $m \leqslant n$, then $T_{z^n + \alpha\bar{z}^m}$ is hyponormal if and only if $|\alpha| \leqslant \sqrt{\frac{m+1}{n+1}}$.

2. If $m \geqslant n$, $T_{z^n + \alpha\bar{z}^m}$ is hyponormal if and only if $|\alpha| \leqslant \frac{n}{m}$.

This leads to a host of examples where $|g'| \leqslant |f'|$ on $\mathbb{T}$, but $T_{f+\bar{g}}$ is not hyponormal. In [1, Theorem 4], P. Ahern and Z. Čučković showed the following result giving another necessary, but not sufficient, condition for the hyponormality of T_φ when φ is harmonic.

THEOREM 4. *Suppose f and g are holomorphic in $\mathbb{D}$ and $\varphi = f + \bar{g} \in L^\infty(\mathbb{D})$. If T_φ is hyponormal then $Tu \geqslant u$ in $\mathbb{D}$ where $u = |f|^2 - |g|^2$ and T is the Berezin transform*

$$Tu(z) = \frac{1}{\pi}\int_{\mathbb{D}} u\left(\frac{z-\zeta}{1-\bar{z}\zeta}\right) dA(\zeta),$$

defined for any $u \in L^1(\mathbb{D})$.

Using this, they were able to show, as a corollary, a more general version of Sadraoui's result.

COROLLARY 1. *Suppose f and g are holomorphic in $\mathbb{D}$, that $\varphi = f + \bar{g}$ is bounded in $\mathbb{D}$, and that T_φ is hyponormal. Then $\overline{\lim}_{z\to\zeta}\left(|f'(z)|^2 - |g'(z)|^2\right) \geqslant 0$ for all $\zeta \in \mathbb{T}$. In particular, if f' and g' are continuous at $\zeta \in \mathbb{T}$, then $|f'(\zeta)| \geqslant |g'(\zeta)|$.*

Finally, in [8], I. S. Hwang proved the following theorem as part of his study of hyponormal operators whose symbol is a harmonic polynomial. We note here that the condition deals only with the modulus of the coefficients of the given harmonic polynomial.

THEOREM 5. *Let $f(z) = a_m z^m + a_n z^n$ and $g(z) = a_{-m} z^m + a_{-n} z^n$, with $0 < m < n$. If $T_{f+\bar{g}}$ is hyponormal and $|a_n| \leqslant |a_{-n}|$, then we have*

$$n^2\left(|a_{-n}|^2 - |a_n|^2\right) \leqslant m^2\left(|a_m|^2 - |a_{-m}|^2\right).$$

Work continues to this day on the study of hyponormal Toeplitz operators whose symbol is a harmonic polynomial. It is a testament to the subtlety of the topic that even in this case there is still much to be said about such symbols. Recently, in [5], Z. Čučković and R. Curto proved the following result.

THEOREM 6. *Suppose T_φ is hyponormal on $A^2(\mathbb{D})$ with $\varphi(z) = \alpha z^m + \beta z^n + \gamma \bar{z}^p + \delta \bar{z}^q$, where $m < n$ and $p < q$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Assume also that $n - m = q - p$. Then*

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geqslant 2\left|\bar{\alpha}\beta mn - \bar{\gamma}\delta pq\right|.$$

Note that in the above Theorems, only the moduli of the coefficients are taken into account. As we shall see in Section 4, this is not necessarily the case when φ is not harmonic. We now turn our attention to such operators.

3. Toeplitz operators with non-harmonic symbol

So far, all of these results deal with Toeplitz operators whose symbol is harmonic. The study of operators whose symbol is not harmonic turns out to be more complicated because the cross-terms in equation (1) do not vanish.

3.1. Simple non-harmonic symbols

We begin our own investigations by looking at some simple examples. We did not have to look far for some results which we found surprising.

It seemed heuristically plausible that adding a symbol corresponding to a hyponormal Toeplitz operator to a symbol corresponding to a self-adjoint Toeplitz operator should generate a hyponormal Toeplitz operator. But this is not the case.

EXAMPLE 1. The operator $T_{z+C|z|^2}$ is not hyponormal when $|C|>2\sqrt{2}$.

Proof. We verify the statement in Example 1. Let $\psi_n(z)=\sqrt{\frac{n+1}{\pi}}z^n$. The collection $\{\psi_n\}_{n=0}^{\infty}$ is the standard orthonormal basis of $A^2(\mathbb{D})$. Given $u(z)=\sum_{n=0}^{\infty}u_n\psi_n\in A^2(\mathbb{D})$, where $\{u_n\}\in\ell^2$ we have that

$$T_zu=\sum_{n=0}^{\infty}\sqrt{\frac{n+1}{n+2}}u_n\psi_{n+1},\qquad\text{and}\qquad T_{|z|^2}u=\sum_{n=0}^{\infty}\frac{n+1}{n+2}u_n\psi_n.$$

Thus, we have that the cross-terms are

$$\begin{aligned}&2\mathrm{Re}\left[\left\langle T_{|z|^2}T_zu,u\right\rangle-\left\langle T_zT_{|z|^2}u,u\right\rangle\right]\\&=2\mathrm{Re}\left[\left\langle T_zu,T_{|z|^2}u\right\rangle-\left\langle T_{\bar{z}}u,T_{|z|^2}u\right\rangle\right]\\&=2\mathrm{Re}\left[\left\langle\sum_{n=0}^{\infty}\sqrt{\frac{n+1}{n+2}}\left(\frac{n+2}{n+3}-\frac{n+1}{n+2}\right)u_n\psi_{n+1},\sum_{n=0}^{\infty}u_n\psi_n\right\rangle\right]\\&=2\mathrm{Re}\sum_{n=0}^{\infty}\sqrt{\frac{n+1}{n+2}}\left(\frac{n+2}{n+3}-\frac{n+1}{n+2}\right)u_n\overline{u_{n+1}}.\end{aligned}$$

Now, by [7] and [11] we have

$$\langle T_zu,T_zu\rangle-\langle T_{\bar{z}}u,T_{\bar{z}}u\rangle\leqslant\frac{1}{2}\|u\|^2,$$

and since $T_{C|z|^2}$ is normal we have

$$\left\langle T_{C|z|^2}u,T_{C|z|^2}u\right\rangle-\left\langle T_{C|z|^2}u,T_{C|z|^2}u\right\rangle=0.$$

We then have the cross-terms

$$2\mathrm{Re}\left[\left\langle T_zu,T_{C|z|^2}u\right\rangle-\left\langle T_{\bar{z}}u,T_{C|z|^2}u\right\rangle\right]=2\mathrm{Re}C\sum_{n=0}^{\infty}\sqrt{\frac{n+1}{n+2}}\left(\frac{n+2}{n+3}-\frac{n+1}{n+2}\right)u_n\overline{u_{n+1}}.$$

Thus we may choose $u \in A^2(\mathbb{D})$ and $C \in \mathbb{C}$, such that

$$\frac{1}{2}\|u\|^2 + 2\mathrm{Re}C \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n+2}} \left(\frac{n+2}{n+3} - \frac{n+1}{n+2}\right) u_n \overline{u_{n+1}} < 0.$$

For such a choice of C then, operator $T_{z+C|z|^2}$ would not be hyponormal. In particular if we choose $u(z) = \frac{1}{2}e^{i\theta_0}\varphi_0 + \frac{1}{2}e^{i\theta_1}\varphi_1$, where $\pi + \theta_0 - \theta_1 = \arg C$, then

$$\left\langle \left[T^*_{z+C|z|^2}, T_{z+C|z|^2}\right] u, u \right\rangle = \frac{1}{6} - \frac{|C|}{12\sqrt{2}},$$

which will be negative whenever we choose $|C| > 2\sqrt{2}$. Thus, for any such choice of C, we have that $T_{z+C|z|^2}$ is not hyponormal. $\square$

At this point it is not known whether $2\sqrt{2}$ is sharp. This example came as a surprise to us. We had conjectured that the sum of a self-adjoint plus a hyponormal symbol would always correspond to a hyponormal operator, and the above simple counterexample was striking. Brian Simanek pointed out to us the following, very interesting observation: Since hyponormality does not change when you multiply the symbol by a constant, Example 1 also shows that $T_{\frac{z}{C}+|z|^2}$ is not hyponormal whenever $|C| > 2\sqrt{2}$. This shows that you may perturb a self-adjoint operator by a subnormal operator of arbitrarily small norm, and yet the result will not be hyponormal.

THEOREM 7. *Let $\varphi(z) = a_{m,n}z^m\bar{z}^n$, with $m \geqslant n$ and $a_{m,n} \in \mathbb{C}$. Then T_φ is hyponormal. Further*

$$\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\| = \max\left\{\frac{2(m-n)^2}{(2m-n)^2}, \frac{(2(m-n)+1)(m-n+1)}{(2m-n+1)^2} - \frac{m-n+1}{(m+1)^2}\right\}.$$

Proof. It is a well known fact (cf. [6, Chapter 2, Lemma 6]) that

$$P(z^m\bar{z}^n) = \begin{cases} \frac{m-n+1}{m+1} z^{m-n} & m \geqslant n \\ 0 & m < n. \end{cases}$$

Thus, if we let $u(z) = \sum_{k=0}^{\infty} u_k z^k \in A^2(\mathbb{D})$, then we have

$$P(z^m\bar{z}^n u) = \begin{cases} \sum_{k=0}^{\infty} \frac{m+k-n+1}{m+k+1} u_k z^{m+k-n} & m \geqslant n \\ \sum_{k=n-m}^{\infty} \frac{m+k-n+1}{m+k+1} u_k z^{m+k-n} & m < n. \end{cases}$$

Taking into account that $T^*_\varphi = T_{\bar{\varphi}}$, we find that

$$
\begin{aligned}
&\left\langle [T_\varphi^*, T_\varphi]u, u\right\rangle \\
&= \left\langle T_\varphi u, T_\varphi u\right\rangle - \left\langle T_\varphi^* u, T_\varphi^* u\right\rangle \\
&= |a_{m,n}|^2 \left(\sum_{k=0}^{\infty} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 - \sum_{k=m-n}^{\infty} \frac{n+k-m+1}{(n+k+1)^2} |u_k|^2 \right) \\
&= |a_{m,n}|^2 \left(\sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 + \sum_{k=m-n}^{\infty} \left[\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \right] |u_k|^2 \right).
\end{aligned} \tag{2}
$$

Now,

$$
\begin{aligned}
&\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \\
&= \frac{(n+k+1)^2(m+k-n+1) - (m+k+1)^2(n+k-m+1)}{(m+k+1)^2(n+k+1)^2} \\
&= \frac{(m^2-n^2)k + (m-n+1)(n+1)^2 + (m-n-1)(m+1)^2}{(m+k+1)^2(n+k+1)^2}.
\end{aligned} \tag{3}
$$

This is clearly positive when $k = m-n \geqslant 1$.

Further, when we take the derivative of the numerator with respect to k, we find that it is positive whenever $m > n$, and so the numerator is increasing and thus always positive. Therefore we may conclude that

$$
\sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 + \sum_{k=m-n}^{\infty} \left(\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \right) |u_k|^2 \geqslant 0
$$

for all $u(z) = \sum_{k=0}^{\infty} u_k z^k \in A^2(\mathbb{D})$, and so T_φ is hyponormal.

Now, the above calculations show that $[T_{z^m\bar{z}^n}^*, T_{z^m\bar{z}^n}]$ is a diagonal operator on the basis of monomials, and that the standard orthonormal basis $\left\{ \psi_k(z) = \sqrt{\frac{k+1}{\pi}} z^k \right\}_{k=0}^{\infty}$ forms an eigenbasis for the selfcommutator with associated eigenvalues

$$
\lambda_k = \begin{cases} \left(\frac{k+m-n+1}{(k+m+1)^2} \right)(k+1) & 0 \leqslant k \leqslant m-n-1 \\ \left(\frac{k+m-n+1}{(k+m+1)^2} - \frac{k+n-m+1}{(k+n+1)^2} \right)(k+1) & k \geqslant m-n. \end{cases}
$$

That being the case, to find $\|[T_{z^m\bar{z}^n}^*, T_{z^m\bar{z}^n}]\|$, one need only find the maximum λ_k. Since, for $k \geqslant 0$, we have that $\left(\frac{k+m-n+1}{(k+m+1)^2} \right)(k+1)$ is a monotonically increasing function in k, and $\left(\frac{k+m-n+1}{(k+m+1)^2} - \frac{k+n-m+1}{(k+n+1)^2} \right)(k+1)$ is a monotonically decreasing function in k, the maximum eigenvalue will either be at

$$
\lambda_{m-n-1} = \frac{2(m-n)^2}{(2m-n)^2} \quad \text{or} \quad \lambda_{m-n} = \frac{(2(m-n)+1)(m-n+1)}{(2m-n+1)^2} - \frac{m-n+1}{(m+1)^2},
$$

as claimed. □

REMARK 1. Whether the maximum eigenvalue is λ_{m-n-1} or λ_{m-n} will depend on the value of m and n. For example when $\varphi = z^3\bar{z}^1$, the maximum eigenvalue will be at $\lambda_{m-n-1} = \lambda_1$, but when $\varphi = z^{11}\bar{z}^8$, the maximum eigenvalue will be at $\lambda_{m-n} = \lambda_3$. Also since $[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]$ is still a diagonal operator when $n \geqslant m$, the same argument shows that in that case

$$\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\| = \frac{(2(n-m)+1)(n-m+1)}{(2n-m+1)^2} - \frac{n-m+1}{(n+1)^2}.$$

We have the following immediate corollary.

COROLLARY 2. *Let $\varphi(z) = z^m\bar{z}^n$ with $m > n$. Then*

$$\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\| \leqslant \frac{1}{2}.$$

Proof. By Theorem 7, we have that

$$\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\| = \max\left\{\frac{2(m-n)^2}{(2m-n)^2}, \frac{(2(m-n)+1)(m-n+1)}{(2m-n+1)^2} - \frac{m-n+1}{(m+1)^2}\right\}.$$

Suppose first that the norm is at $\frac{2(m-n)^2}{(2m-n)^2}$. But then since $(2m-n)^2 = (m+m-n)^2 = m^2 + 2m(m-n) + (m-n)^2$ is clearly greater than $4(m-n)^2$, we have that the norm $\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\|$ is bounded above by

$$\frac{2(m-n)^2}{4(m-n)^2} = \frac{1}{2}.$$

Suppose, on the other hand, that $\|[T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]\| = \frac{(2(m-n)+1)(m-n+1)}{(2m-n+1)^2} - \frac{m-n+1}{(m+1)^2}$. This is strictly less than

$$\frac{2(m-n)(m-n+1)}{(2m-n+1)^2}.$$

Now, $(2m-n+1)^2 = (m-n+1)^2 + 2m(m-n+1) + m^2$. Clearly, $2m(m-n+1) \geqslant 2(m-n)(m-n+1)$. If we can then show that

$$(m-n+1)^2 + m^2 \geqslant 2(m-n)(m-n+1),$$

the claim will follow. But of course by the arithmetic-geometric mean inequality

$$2(m-n)(m-n+1) \leqslant (m-n)^2 + (m-n+1)^2 \leqslant m^2 + (m-n+1)^2,$$

since $n \geqslant 0$. It follows then that

$$\frac{2(m-n)(m-n+1)}{(2m-n+1)^2} \leqslant \frac{2(m-n)(m-n+1)}{4(m-n)(m-n+1)} = \frac{1}{2}. \qquad \square$$

By Putnam's inequality, and the spectral mapping theorem, we know that when φ is analytic we have that

$$\|[T_\varphi^*, T_\varphi]\| \leqslant \frac{\operatorname{Area}(\varphi(\mathbb{D}))}{\pi}.$$

However, in [11], it was shown that if φ is also univalent in $\mathbb{D}$, that

$$\|[T_\varphi^*, T_\varphi]\| \leqslant \frac{\operatorname{Area}(\varphi(\mathbb{D}))}{2\pi},$$

and it is a standing conjecture that the univalent condition can be dropped. Evidence for this conjecture was given in [7] where it was showed that

$$\|[T_{z^k}^*, T_{z^k}]\| = \frac{1}{2}.$$

In light of Corollary 2, and given the results of [7] and [11], we are led to conjecture:

If $\varphi \in L^\infty(\mathbb{D})$, and if T_φ is hyponormal, then $\|[T_\varphi^*, T_\varphi]\| \leqslant \frac{\operatorname{Area}(\varphi(\mathbb{D}))}{2\pi}$.

3.2. Non-harmonic polynomials

We now turn to an examination of two term non-harmonic polynomials.

THEOREM 8. *Suppose $f = a_{m,n} z^m \bar{z}^n$ and $g = a_{i,j} z^i \bar{z}^j$, with $m > n$, $i > j$ and $m - n > i - j$. Then T_{f+g} is hyponormal if for each $k \geqslant 0$ the term*

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m-n+k+1}{(m+k+1)^2} + \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i-j+k+1}{(i+k+1)^2}$$

is sufficiently large (as is defined in the following remark).

REMARK 2. Here sufficiently large means that, under the assumption $m - n > i - j$, we have the following four conditions:

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m+k-n+1}{(m+k+1)^2} + \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i+k-j+1}{(i+k+1)^2} \geqslant C_k + D_k$$

for $k \leqslant i - j - 1$, and

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m+k-n+1}{(m+k+1)^2} + \left|\frac{a_{i,j}}{a_{m,n}}\right| \left(\frac{i+k-j+1}{(i+k+1)^2} - \frac{j+k-i+1}{(j+k+1)^2}\right) \geqslant C_k + D_k$$

for $i - j \leqslant k \leqslant m - n - 1$, and

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \left(\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2}\right) + \left|\frac{a_{i,j}}{a_{m,n}}\right| \left(\frac{i+k-j+1}{(i+k+1)^2} - \frac{j+k-i+1}{(j+k+1)^2}\right) \geqslant C_k + D_k$$

for $m-n \leqslant k \leqslant m-n+i-j-1$, and

$$\left|\frac{a_{m,n}}{a_{i,j}}\right|\left(\frac{m+k-n+1}{(m+k+1)^2}-\frac{n+k-m+1}{(n+k+1)^2}\right)+\left|\frac{a_{i,j}}{a_{m,n}}\right|\left(\frac{i+k-j+1}{(i+k+1)^2}-\frac{j+k-i+1}{(j+k+1)^2}\right) \geqslant C_k+D_k$$

where

$$C_k := \begin{cases} \frac{m-n+k+1}{(m+k+1)(m-n+j+k+1)}, & \text{for } 0 \leqslant k \leqslant i-j-1, \\ \frac{m-n+k+1}{(m+k+1)(m-n+j+k+1)} - \frac{j-i+k+1}{(j+k+1)(j-i+m+k+1)}, & \text{for } k \geqslant i-j. \end{cases} \tag{4}$$

and

$$D_k := \begin{cases} 0, & \text{for } 0 \leqslant k \leqslant m-n+j-i-1, \\ \frac{i-j+k+1}{(i-j+n+k+1)(i+k+1)}, & \text{for } m-n+j-i \leqslant k \leqslant m-n-1, \\ \frac{i-j+k+1}{(i-j+n+k+1)(i+k+1)} - \frac{n-m+k+1}{(n-m+k+1)(n+k+1)}., & \text{for } k \geqslant m-n. \end{cases} \tag{5}$$

Proof. Recall that for $f,g \in L^\infty(\mathbb{D})$, and $u \in A^2$, we have

$$\begin{aligned} &\left\langle [T_{f+g}^*, T_{f+g}]u, u \right\rangle \\ &= \left\|T_f u\right\|^2 - \left\|T_f^* u\right\|^2 + \left\|T_g u\right\|^2 - \left\|T_g^* u\right\|^2 + 2\mathrm{Re}\left[\left\langle T_f u, T_g u\right\rangle - \left\langle T_f^* u, T_g^* u\right\rangle\right]. \end{aligned} \tag{6}$$

We begin to calculate the cross-term $2\mathrm{Re}\left[\left\langle T_f u, T_g u\right\rangle - \left\langle T_f^* u, T_g^* u\right\rangle\right]$. Without loss of generality, we may assume that $m-n > i-j$. Under this assumption, we find

$$\begin{aligned} &2\mathrm{Re}\left[\left\langle T_f u, T_g u\right\rangle - \left\langle T_f^* u, T_g^* u\right\rangle\right] \\ &= 2\mathrm{Re}\left(a_{m,n}\overline{a_{i,j}}\right)\left[\left\langle \sum_{k=0}^{\infty} \frac{m+k-n+1}{m+k+1} u_k z^{m+k-n}, \sum_{k=0}^{\infty} \frac{i+k-j+1}{i+k+1} u_k z^{i+k-j} \right\rangle \right. \\ &\qquad \left. - \left\langle \sum_{k=m-n}^{\infty} \frac{n+k-m+1}{n+k+1} u_k z^{n+k-m}, \sum_{k=i-j}^{\infty} \frac{j+k-i+1}{j+k+1} u_k z^{j+k-i} \right\rangle \right] \\ &= 2\sum_{k=0}^{\infty} C_k \mathrm{Re}\left(a_{m,n}\overline{a_{i,j}} u_k \overline{u_{k+m-n+i-j}}\right), \end{aligned}$$

where, for the purposes of slightly less daunting expressions, we used C_k as defined by (4) in the above remark. We will also, for reasons that will soon be clear, use D_k as defined by (5).

Unfortunately, as we have seen, we cannot control the sign of these cross terms. Therefore, we will assume that we must always subtract them. Further, by the inequality $2\mathrm{Re}\left(a\bar{b}\right) \leqslant |a|^2+|b|^2$, we have

$$2\mathrm{Re}\left(a_{m,n}\overline{a_{i,j}} u_k \overline{u_{k+m-n+i-j}}\right) \leqslant \left|a_{m,n}a_{i,j}\right|\left(|u_k|^2 + \left|u_{k+m-n+i-j}\right|^2\right).$$

We combine equation (6) with the calculations performed in the proof of Theorem 7 to evaluate

$$\left\|T_f u\right\|^2-\left\|T_f^* u\right\|^2+\left\|T_g u\right\|^2-\left\|T_g^* u\right\|^2$$

applied to our given f and g. Thereby we may conclude that T_φ will be hyponormal if

$$\begin{aligned}
&|a_{m,n}|^2\left(\sum_{k=0}^{m-n-1}\frac{m+k-n+1}{(m+k+1)^2}|u_k|^2+\sum_{k=m-n}^{\infty}\left(\frac{m+k-n+1}{(m+k+1)^2}-\frac{n+k-m+1}{(n+k+1)^2}\right)|u_k|^2\right)\\
&+\left|a_{i,j}\right|^2\left(\sum_{k=0}^{i-j-1}\frac{i+k-j+1}{(i+k+1)^2}|u_k|^2+\sum_{k=i-j}^{\infty}\left(\frac{i+k-j+1}{(i+k+1)^2}-\frac{j+k-i+1}{(j+k+1)^2}\right)|u_k|^2\right)\\
&\geqslant\left|a_{m,n}a_{i,j}\right|\sum_{k=0}^{\infty}C_k\left(|u_k|^2+\left|u_{k+m-n+j-i}\right|^2\right)\\
&=\sum_{k=0}^{\infty}C_k|u_k|^2+\sum_{k=m-n+i-j}^{\infty}D_k|u_k|^2.
\end{aligned}$$

Thus, an appropriate term by term comparison of the coefficients of $|u_k|^2$ will show that operator T_{f+g} is hyponormal, if the bounds given in the above remark hold.

In particular, we obtain the stronger estimate

$$\left\langle[T_{f+g}^*,T_{f+g}]u,u\right\rangle\geqslant\sum_{k=0}^{\infty}A_k|u_k|^2,$$

where A_k is non-negative for all k. $\square$

The next theorem examines the case when one of the terms in our binomial is the symbol of a cohyponormal operator (i.e. an operator whose adjoint is hyponormal). This is in contrast to Theorem 8 where each term individually yielded a hyponormal operator.

THEOREM 9. *Suppose* $f=a_{m,n}z^m\bar{z}^n$ *and* $g=a_{i,j}\bar{z}^i z^j$, *with* $m>n$ *and* $i>j$. *Then* T_{f+g} *is hyponormal if for each* $k\geqslant 0$

$$\left|\frac{a_{m,n}}{a_{i,j}}\right|\frac{m-n+k+1}{(m+k+1)^2}-\left|\frac{a_{i,j}}{a_{m,n}}\right|\frac{i-j+k+1}{(i+k+1)^2}$$

is sufficiently large (as is defined in the following remark).

REMARK 3. Here, as in Theorem 8, we can specify what sufficiently large means. To do so, we abbreviate

$$\widetilde{A}_k:=\left|\frac{a_{m,n}}{a_{i,j}}\right|\left(\frac{m+k-n+1}{(m+k+1)^2}-\frac{n+k-m+1}{(n+k+1)^2}\right),\text{ and}\tag{7}$$

$$\widetilde{B}_k:=\left|\frac{a_{i,j}}{a_{m,n}}\right|\left(\frac{i+k-j+1}{(i+k+1)^2}-\frac{j+k-i+1}{(j+k+1)^2}\right),\tag{8}$$

as well as

$$\widetilde{C}_k := \frac{m-n+k+1}{(m+k+1)(m-n+i+k+1)} - \frac{i-j+k+1}{(i+k+1)(i-j+m+k+1)}, \text{ and} \tag{9}$$

$$\widetilde{D}_k := \frac{k+j-i+1}{(n+j-i+k+1)(j+k+1)} - \frac{k+n-m+1}{(m-n+j+k+1)(n+k+1)}. \tag{10}$$

Now sufficiently large means that the following four conditions are satisfied:

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m+k-n+1}{(m+k+1)^2} - \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i+k-j+1}{(i+k+1)^2} \geqslant \widetilde{C}_k$$

for $k \leqslant \min\{m-n, i-j\} - 1$, and

$$\widetilde{C}_k \leqslant \begin{cases} \left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m+k-n+1}{(m+k+1)^2} - \widetilde{B}_k & \text{when } m-n > i-j \\ \\ \widetilde{A}_k - \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i+k-j+1}{(i+k+1)^2} & \text{when } m-n < i-j \end{cases}$$

for $\min\{m-n, i-j\} \leqslant k \leqslant \max\{m-n, i-j\} - 1$, and

$$\widetilde{A}_k - \widetilde{B}_k \geqslant \widetilde{C}_k$$

for $\max\{m-n, i-j\} \leqslant k \leqslant m-n+i-j-1$, and

$$\widetilde{A}_k - \widetilde{B}_k \geqslant \widetilde{C}_k + \widetilde{D}_k$$

for $k \geqslant m-n+i-j$.

Proof. Recall that for $f, g \in L^\infty(\mathbb{D})$, and $u \in A^2(\mathbb{D})$, we have

$$\begin{aligned} &\left\langle [T_{f+g}^*, T_{f+g}]u, u \right\rangle \\ &= \|T_f u\|^2 - \|T_f^* u\|^2 + \|T_g u\|^2 - \|T_g^* u\|^2 + 2\mathrm{Re}\left[\langle T_f u, T_g u\rangle - \langle T_f^* u, T_g^* u\rangle\right]. \end{aligned}$$

Again, the calculations performed in the proof of Theorem 7 applied to the current f and g show

$$\begin{aligned} &\|T_f u\|^2 - \|T_f^* u\|^2 + \|T_g u\|^2 - \|T_g^* u\|^2 \\ &= |a_{m,n}|^2 \sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 + \sum_{k=m-n}^{\infty} \widetilde{A}_k |u_k|^2 \qquad (11) \\ &\quad - |a_{i,j}|^2 \sum_{k=0}^{i-j-1} \frac{i+k-j+1}{(i+k+1)^2} |u_k|^2 - \sum_{k=i-j}^{\infty} \widetilde{B}_k |u_k|^2, \qquad (12) \end{aligned}$$

where we used $\widetilde{A}_k$ and $\widetilde{B}_k$ as defined in (7) and (8).

This proof differs from that of Theorem 8 in the calculation for the cross-terms. Under the assumption $i > j$, we have

$$\begin{aligned}
&2\mathrm{Re}\left[\left\langle T_f u, T_g u\right\rangle - \left\langle T_f^* u, T_g^* u\right\rangle\right] \\
&= 2\mathrm{Re}\left(a_{m,n}\overline{a_{i,j}}\right)\left[\left\langle \sum_{k=0}^{\infty} \frac{m+k-n+1}{m+k+1} u_k z^{m+k-n}, \sum_{k=i-j}^{\infty} \frac{j+k-i+1}{j+k+1} u_k z^{j+k-i} \right\rangle\right. \\
&\qquad \left. - \left\langle \sum_{k=m-n}^{\infty} \frac{n+k-m+1}{n+k+1} u_k z^{n+k-m}, \sum_{k=i-j}^{\infty} \frac{i+k-j+1}{i+k+1} u_k z^{i+k-j} \right\rangle\right] \\
&= 2\mathrm{Re}\left(a_{m,n}\overline{a_{i,j}}\right) \sum_{k=0}^{\infty} \widetilde{C}_k u_k \overline{u_{m-n+i-j+k}}. \qquad (13)
\end{aligned}$$

via direct calculation and with $\widetilde{C}_k$ from (9).

The argument now follows *mutatis mutandis* as in Theorem 8. In particular, with $\widetilde{D}_k$ from (10) and once again taking advantage of the inequality $2\mathrm{Re}\left(a\overline{b}\right) \leqslant |a|^2 + |b|^2$, we have that if the conditions given in Remark 3 hold, then operator T_{f+g} will be hyponormal. $\square$

Both of the above theorems are rather cumbersome to apply directly. Further, it is not immediately clear *a priori* that the relevant bounds are ever actually attainable. In the following example we look at a symbol which shows that the bounds in Theorem 9 can be attained. This shows that while a seemingly "nice" symbol like $T_{z-3|z|^2}$ might fail to be hyponormal even though it is the sum of a sub-normal operator and a self-adjoint operator, the sum of a hyponormal and co-hyponormal operator might still produce an operator which is hyponormal.

EXAMPLE 2. Consider $\varphi(z) = z^2\overline{z} + \frac{1}{7}\,\overline{z}^4 z^3$. We can plug this into the relevant calculations from Theorem 9 to test for hyponormality. In particular, we find that

$$\left|a_{m,n}a_{i,j}\right| \widetilde{A}_k = \frac{3k+8}{(k+3)^2(k+2)^2},$$

and that

$$\begin{aligned}
&\left|a_{m,n}a_{i,j}\right| \left(\widetilde{B}_k + \widetilde{C}_k + \widetilde{D}_k\right) \\
&= \frac{1}{7}\left(\frac{7k+32}{7(k+5)^2(k+4)^2} + \frac{3k^3+21k^2+46k+8}{(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)}\right). \qquad (14)
\end{aligned}$$

Thus, we find that T_φ will be hyponormal if

$$\begin{aligned}
&\left|a_{m,n}a_{i,j}\right| \left(\widetilde{A}_k - \widetilde{B}_k - \widetilde{C}_k + \widetilde{D}_k\right) \\
&= \frac{3k+8}{(k+3)^2(k+2)^2} - \frac{1}{7}\left(\frac{7k+32}{7(k+5)^2(k+4)^2} + \frac{3k^3+21k^2+46k+8}{(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)}\right) \\
&= \frac{119k^7+3475k^6+41785k^5+267977k^4+985764k^3+2061168k^2+2228760k+927168}{49(k+6)(k+5)^2(k+4)^2(k+3)^2(k+2)^2(k+1)} > 0
\end{aligned}$$

for all $k \geqslant 2$, since the checks for $k = 0, 1$ show the desired inequalities hold.

However in fact, it is clear from observation that this rational function is positive for all $k > 0$, and in particular for $k \geqslant 2$. Thus T_φ is hyponormal. This example will be explored more in depth in Theorem 10.

Note however that for our choice of φ, since we have that the expression in (14) is less than $\frac{3k+8}{(k+3)^2(k+2)^2}$ for all $k > 0$, only one check was actually necessary to show that T_φ is hyponormal. Indeed the construction of this example was based on ensuring a sufficiently quick decay of the expression in (14) while also ensuring that for small values of k the required inequalities would still hold. In the following theorem, we generalize the idea of this construction to find a general construction for hyponormal operators whose symbol is of the form in the hypothesis of Theorem 9.

THEOREM 10. *Fix $\delta \in \mathbb{N}$. For every integer $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$, such that T_φ with symbol $\varphi(z) = z^{n+\delta}\bar{z}^n + \frac{1}{2j+\delta}\bar{z}^{j+\delta}z^j$ is hyponormal.*

Proof. The idea of the proof lies in constructing the symbol in such a way that the bounds found in Theorem 9 are satisfied.

We let $m = n+\delta$ and $i = j+\delta$. Since $m-n = i-j = \delta$, the formulas from Theorem 9 become somewhat simplified. Recall that $\widetilde{A}_k = \left|\frac{a_{m,n}}{a_{i,j}}\right| \left(\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2}\right)$, as well as $\widetilde{B}_k = \left|\frac{a_{i,j}}{a_{m,n}}\right| \left(\frac{i+k-j+1}{(i+k+1)^2} - \frac{j+k-i+1}{(j+k+1)^2}\right)$. In particular, for $k \geqslant \delta$ and with $a_{m,n} = 1$ and $a_{i,j} = \frac{1}{i+j}$, we obtain

$$\widetilde{A}_k = \frac{(i+j)(m+n)\delta k + (i+j)(\delta+1)(n+1)^2 + (i+j)(\delta-1)(m+1)^2}{(k+m+1)^2(k+n+1)^2},$$

and

$$\widetilde{B}_k = \frac{(i+j)^2\delta k + (i+j)(\delta+1)(j+1)^2 + (i+j)(\delta-1)(j+1)^2}{(i+j)^2(k+i+1)^2(k+j+1)^2}.$$

Finally, we have

$$\widetilde{C}_k = \frac{\delta(i-m)(k+\delta+1)}{(k+m+1)(k+m+\delta+1)(k+i+1)(k+i+\delta+1)}, \text{ and}$$
$$\widetilde{D}_k = \frac{\delta(i-m)(k-\delta+1)}{(k+m+1)(k+n-\delta+1)(k+j+1)(k+j-\delta+1)}.$$

Recall that our aim is now to prove that for $k \geqslant 2\delta$ we have

$$\widetilde{A}_k \geqslant \widetilde{B}_k + \widetilde{C}_k + \widetilde{D}_k. \tag{15}$$

This is a direct application of the bounds given in Theorem 9.

Our goal will be to prove that the numerator of $\widetilde{A}_k$ is larger than the sums of the numerators of $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{D}_k$, while ensuring that the denominator of $\widetilde{A}_k$ is smaller than each of the denominators of $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{D}_k$. If we can show this we will have shown that (15) holds for all $k \geqslant 2\delta$, and in fact, the other required bounds of Theorem 9 will also necessarily follow immediately, guaranteeing the hyponormality of T_φ.

Looking first at the numerators then, we first wish to show

$$(i+j)(m+n)\delta k \geqslant (2i-2m+1)k, \tag{16}$$

for all $k \geqslant 2\delta$. Yet since clearly $(i+j)(m+n)\delta > (2i-2m+1)$, we have that (16) holds for all $k \geqslant 0$. Looking at the constant terms of the numerators, and multiplying through by $(i+j)$ so we may use the first term of the denominator of $\widetilde{B}_k$ for cancellation and an easier comparison, it is clear that

$$\begin{aligned}(i+j)^2&\left[(\delta+1)(n+1)^2+(\delta-1)(m+1)^2\right]\\ &\geqslant (\delta+1)(j+1)^2+(\delta-1)(j+1)^2+2\delta(i-m)(i+j),\end{aligned} \tag{17}$$

since the inequality

$$(i+j)^2\left[(\delta+1)+(\delta-1)\right] \geqslant (\delta+1)(j+1)^2+(\delta-1)(j+1)^2,$$

and the inequality

$$(i+j)^2\left[(\delta+1)(n^2+2n)+(\delta-1)(m^2+2m)\right] \geqslant 2\delta(i-m)(i+j)$$

both hold by inspection. So we have that the numerator of $\widetilde{A}_k$ is larger than the sums of the numerators of $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{D}_k$, as desired.

It remains to show our desired inequalities for the denominators. It is clear by inspection that if $j > m$, then we have that

$$(k+m+1)^2(k+n+1)^2 \leqslant (k+i+1)^2(k+j+1)^2$$

and

$$(k+m+1)^2(k+n+1)^2 \leqslant (k+m+1)(k+m+\delta+1)(k+i+1)(k+i+\delta+1).$$

We take a moment to show that it is possible to choose j large enough so that

$$(k+m+1)^2(k+n+1)^2 < (k+m+1)(k+n-\delta+1)(k+j+1)(k+j-\delta+1) \tag{18}$$

for all $k \geqslant 2\delta$. Since we have already assumed that $j > m$, we have that $j-\delta > n$, and thus (18) follows so long as

$$(k+m+1)(k+n+1) < (k+n-\delta+1)(k+j+1).$$

Or equivalently, since $k \geqslant 2\delta$, inequality (18) follows so long as

$$j > \frac{(k+m+1)(k+n+1)}{k+n-\delta+1} - k - 1 = \frac{k(m+\delta)+mn+m+\delta}{k+n-\delta+1} =: q(k).$$

Since the rational function $q(k)$ remains bounded for $k \in [2\delta, \infty)$, it is possible to choose an appropriate $j \in \mathbb{N}$. Thus (15) holds for all $k \geqslant 2\delta$.

The same argument will show that $\widetilde{A}_k \geqslant \widetilde{B}_k + \widetilde{C}_k$ holds for $\delta \leqslant k \leqslant 2\delta$. The required bounds for $k < \delta$ hold trivially.

Thus, by Theorem 9, operator T_φ is hyponormal. $\square$

4. Polynomials of fixed relative degree

We now turn to operators whose symbol is a polynomial of the form

$$\varphi(z) = a_1 z^{m_1}\,\bar{z}^{n_1} + \ldots + a_k z^{m_k}\,\bar{z}^{n_k}, \qquad \text{with } m_1 - n_1 = \ldots = m_k - n_k = \delta \geqslant 0.$$

We shall call these *polynomials of fixed relative degree.* Though working with non-harmonic symbols can be difficult, some results are known in these special cases. One which we will be interested in for this paper is due to Y. Liu and C. Lu in [9, Theorem 3.1]. There they make use of the *Mellin transform* of φ.

DEFINITION 1. Suppose $\varphi \in L^1([0,1], rdr)$. For $\operatorname{Re} z \geqslant 2$, the *Mellin transform* of φ, is given by

$$\widehat{\varphi}(z) := \int_0^1 \varphi(x) x^{z-1}\, dx.$$

For $\varphi(re^{i\theta}) = e^{ik\theta}\varphi_0(r)$, with $k \in \mathbb{Z}$ and φ_0 radial, we can compute the action of T_φ on z^n. Specifically,

$$T_\varphi z^n = \begin{cases} 2(n+k+1)\,\widehat{\varphi}_0(2n+k+2) z^{n+k} & n+k \geqslant 0 \\ 0 & n+k < 0, \end{cases}$$

and

$$T_{\bar{\varphi}} z^n = \begin{cases} 2(n-k+1)\,\widehat{\varphi}_0(2n-k+2) z^{n-k} & n-k \geqslant 0 \\ 0 & n-k < 0. \end{cases}$$

Using this, Y. Liu and C. Lu proved the following theorem in [9, Theorem 3.1].

THEOREM 11. *Let $\varphi(re^{i\theta}) = e^{i\delta\theta}\varphi_0(r) \in L^\infty(\mathbb{D})$, where $\delta \in \mathbb{Z}$ and φ_0 is radial. Then T_φ is hyponormal if and only if one of the following conditions holds:*

1) $\delta = 0$ *and* $\varphi_0 \equiv 0$*;*

2) $\delta = 0$*;*

3) $\delta > 0$ *and for each* $\alpha \geqslant \delta$*,*

$$|\widehat{\varphi}_0(2\alpha + \delta + 2)| \geqslant c_{\alpha,\delta}\,|\widehat{\varphi}_0(2\alpha - \delta + 2)|,$$

where we abbreviate

$$c_{\alpha,\delta} := \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}}.$$

The first situation immediately implies that if $\varphi(z)$ is a polynomial in z and $\bar{z}$ where the degree of $\bar{z}$ is larger than the degree of z in each term, then T_φ cannot be hyponormal. The second situation is a consequence of the fact that whenever φ is real valued in $\mathbb{D}$, then T_φ is actually self-adjoint and thus trivially hyponormal. The final situation, when $\delta > 0$, will be of interest to us.

REMARK 4. One can prove Theorem 7 by applying Theorem 11, however the proof is non-trivial.

The following is a corollary of Theorem 13. However, a direct proof is simple enough that we showcase it here for the convenience of the reader.

COROLLARY 3. *Let* $\varphi(z) = a_1 z^{m_1}\bar{z}^{n_1} + \ldots + a_k z^{m_k}\bar{z}^{n_k}$*, with* $m_1 - n_1 = \ldots = m_k - n_k = \delta \geqslant 0$*, and* a_i *all lying along the same ray for* $1 \leqslant i \leqslant k$ *(i.e.* $\arg(a_1) = \ldots = \arg(a_k)$*), then* T_φ *is hyponormal.*

Proof. Write $\varphi = \varphi_1 + \ldots + \varphi_k$, where $\varphi_i = a_i e^{i\delta\theta}\varphi_{0,i}(r)$. Recall that $c_{\alpha,\delta} = \sqrt{\frac{\alpha-\delta+1}{\alpha+\delta+1}}$. By Theorem 11 and Theorem 7, we have that for each $\alpha \geqslant \delta$

$$|a_i\widehat{\varphi}_{0,i}(2\alpha+\delta+2)| \geqslant c_{\alpha,\delta}\,|a_i\widehat{\varphi}_{0,i}(2\alpha-\delta+2)|.$$

Since the $a_i's$ all lie along the same ray, we have that for each $n \geqslant \delta$

$$\left|\sum_{i=1}^{k} a_i\widehat{\varphi}_{0,i}(2\alpha+\delta+2)\right| = \sum_{i=1}^{k}|a_i|\,|\widehat{\varphi}_{0,i}(2\alpha+\delta+2)| \geqslant \sum_{i=1}^{k} c_{\alpha,\delta}\,|a_i|\,|\widehat{\varphi}_{0,i}(2\alpha-\delta+2)|.$$

The claim now follows by Theorem 11. □

One is tempted to conjecture that the argument of these coefficients should not matter. However the following example shows that this is not the case.

EXAMPLE 3. Let $\varphi(z) = z^2\bar{z} - z^3\bar{z}^2$. Then $\widehat{\varphi}_0(k) = \frac{1}{k+3} - \frac{1}{k+5}$, and we find that

$$\frac{1}{2\alpha+6} - \frac{1}{2\alpha+8} < \sqrt{\frac{\alpha}{\alpha+2}}\left(\frac{1}{2\alpha+4} - \frac{1}{2\alpha+6}\right),$$

whenever $\alpha \geqslant 2$. This violates the conditions of Theorem 11, and so T_φ cannot be hyponormal.

So, can we find sufficient conditions, beyond all coefficients lying along the same ray, to guarantee that such functions yield hyponormal operators? The answer is yes, and depends somewhat on the number of terms, as well as the relative position of the coefficients, as the following two theorems demonstrate.

THEOREM 12. *Let* $\varphi(z) = a_1 z^m\bar{z}^n + a_2 z^i\bar{z}^j$*, with* $m - n = i - j = \delta \geqslant 0$*. Then* T_φ *is hyponormal if* a_1 *and* a_2 *lie in the same quarter-plane (i.e.* $|\arg(a_1) - \arg(a_2)| \leqslant \frac{\pi}{2}$*). Further, under the additional condition that*

$$0 \leqslant \frac{|a_1|}{\alpha+m+1} - \frac{|a_2|}{\alpha+i+1} < c_{\alpha,\delta}^2\left(\frac{|a_1|}{\alpha+n+1} - \frac{|a_2|}{\alpha+j+1}\right) \qquad \text{for all } \alpha, \tag{19}$$

the requirement that $|\arg(a_1) - \arg(a_2)| \leqslant \frac{\pi}{2}$ *is also necessary for the hyponormality of* T_φ.

Proof. We begin with some general observations. Without loss of generality, we may assume that a_1 is a positive real number and that $a_2 = r_2 e^{i\theta}$ with $\frac{-\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. We have $\widehat{\varphi}_0(k) = \frac{a_1}{m+n+k} + \frac{a_2}{i+j+k}$. Recall that $c_{\alpha,\delta} = \sqrt{\frac{\alpha-\delta+1}{\alpha+\delta+1}}$. By Theorem 11, T_φ will be hyponormal if and only if

$$|\widehat{\varphi}_0(2\alpha+\delta+2)|^2 \geqslant c_{\alpha,\delta}^2 \, |\widehat{\varphi}_0(2\alpha-\delta+2)|^2,$$

which is equivalent to

$$\left|\frac{a_1}{\alpha+m+1} + \frac{a_2}{\alpha+i+1}\right|^2 \geqslant c_{\alpha,\delta}^2 \left|\frac{a_1}{\alpha+n+1} + \frac{a_2}{\alpha+j+1}\right|^2,$$

as well as to

$$\begin{aligned}
&\left(\frac{a_1}{\alpha+m+1} + \frac{r_2\cos\theta}{\alpha+i+1}\right)^2 + \frac{r_2^2\sin^2\theta}{(\alpha+i+1)^2} \\
&\geqslant c_{\alpha,\delta}^2 \left[\left(\frac{a_1}{\alpha+n+1} + \frac{r_2\cos\theta}{\alpha+j+1}\right)^2 + \frac{r_2^2\sin^2\theta}{(\alpha+j+1)^2}\right],
\end{aligned} \tag{20}$$

for all $\alpha \geqslant \delta$.

Let us focus on proving the first statement. By the hypothesis that $i = \delta + j$, we can verify

$$\frac{r_2^2\sin^2\theta}{(\alpha+i+1)^2} \geqslant c_{\alpha,\delta}^2 \frac{r_2^2\sin^2\theta}{(\alpha+j+1)^2}$$

for all $\alpha \geqslant \delta$. Similarly,

$$\left(\frac{a_1}{\alpha+m+1} + \frac{r_2\cos\theta}{\alpha+i+1}\right)^2 \geqslant c_{\alpha,\delta}^2 \left(\frac{a_1}{\alpha+n+1} + \frac{r_2\cos\theta}{\alpha+j+1}\right)^2$$

so long as $\cos\theta \geqslant 0$. That is, when a_2 is in the closed right half-plane. Thus, it follows that when $|\arg(a_1) - \arg(a_2)| \leqslant \frac{\pi}{2}$, then the estimate in equation (20) holds for all $\alpha \geqslant \delta$. And so T_φ is hyponormal by Theorem 11.

To show the converse, we assume the extra condition (19). We will show that if $\frac{\pi}{2} < \theta \leqslant \pi$, then there exists an α for which (20) fails, and consequently T_φ must fail to be hyponormal by Theorem 11.

First, fix $\alpha \geqslant \delta$. We construct two circles.

$$C_1 := \left\{ z : \left| z - \frac{a_1}{\alpha+m+1} \right| = \frac{r_2}{\alpha+i+1} \right\},$$

centered at $\frac{a_1}{\alpha+m+1}$ with radius $\frac{r_2}{\alpha+i+1}$, and

$$C_2 := \left\{ z : \left| z - c_{\alpha,\delta}^2 \frac{a_1}{\alpha+n+1} \right| = c_{\alpha,\delta}^2 \frac{r_2}{\alpha+j+1} \right\},$$

centered at $c_{\alpha,\delta}^2 \frac{a_1}{\alpha+n+1}$ with radius $c_{\alpha,\delta}^2 \frac{r_2}{\alpha+j+1}$. Without loss of generality, we may always assume that both of these circles lie in the right half-plane.

So long as the difference of their centers is bounded by the difference of their radii, i.e.

$$\frac{a_1}{\alpha+m+1} - c_{\alpha,\delta}^2 \frac{a_1}{\alpha+n+1} < \frac{r_2}{\alpha+i+1} - c_{\alpha,\delta}^2 \frac{r_2}{\alpha+j+1},$$

we have that C_2 lies completely in the region bounded by C_1. Such a scenario is illustrated in Figure 1 for one value of $\alpha = 6$.

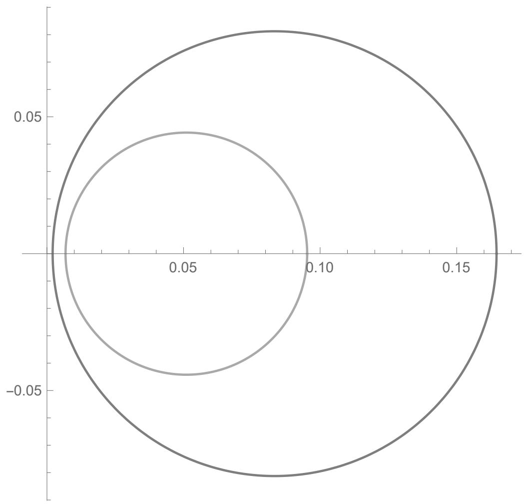

Figure 1: *The situation when* $\alpha = 6$, $m = 5$, $i = 9$, *and* $\delta = 4$.

In this case, it is clear that there exists a $\frac{\pi}{2} < \theta < \pi$ such that

$$\left(\frac{a_1}{\alpha+m+1} + \frac{r_2\cos\theta}{\alpha+i+1}\right)^2 - c_{\alpha,\delta}^2 \left(\frac{a_1}{\alpha+n+1} + \frac{r_2\cos\theta}{\alpha+j+1}\right)^2 = 0. \tag{21}$$

Then if $\theta \to \pi$, the left hand side of (21) will converge to a negative real number by condition (19). At the same time, since

$$\lim_{\theta\to\pi}\left(\frac{r_2^2\sin^2\theta}{(\alpha+i+1)^2} - \frac{r_2^2\sin^2\theta}{(\alpha+j+1)^2}\right) = 0,$$

there exists some θ for which (20) fails. Define

$$\theta_\alpha := \inf\{\theta : \text{ equation (20) fails}\}.$$

We will now show that $\theta_\alpha \to \frac{\pi}{2}$ as $\alpha \to \infty$. Define

$$F_\alpha(\theta) := \left(\frac{a_1}{\alpha+m+1} + \frac{r_2\cos\theta}{\alpha+i+1}\right)^2 + \frac{r_2^2\sin^2\theta}{(\alpha+i+1)^2} - c_{\alpha,\delta}^2\left[\left(\frac{a_1}{\alpha+n+1} + \frac{r_2\cos\theta}{\alpha+j+1}\right)^2 + \frac{r_2^2\sin^2\theta}{(\alpha+j+1)^2}\right].$$

As shown above, there exists a θ such that $F_\alpha(\theta) = 0$. It must be the case that $\theta = \theta_\alpha$ is a root, since $F_\alpha(\theta) > 0$ for $\theta < \theta_\alpha$, and since $F_\alpha(\theta) < 0$ for $\theta > \theta_\alpha$. Solving for this θ_α, we find that

$$\begin{aligned}\theta_\alpha &= \arccos\left(\frac{c_{\alpha,\delta}^2\left[\frac{1}{(\alpha+n+1)^2} + \frac{r_2^2}{(\alpha+j+1)^2}\right] - \left[\frac{1}{(\alpha+m+1)^2} + \frac{r_2^2}{(\alpha+i+1)^2}\right]}{2r_2\left(\frac{1}{(\alpha+m+1)(\alpha+i+1)} - \frac{c_{\alpha,\delta}^2}{(\alpha+n+1)(\alpha+j+1)}\right)}\right)\\ &= \arccos\left(\frac{\mathcal{O}\left(\alpha^5\right)}{2r_2\left(1+r_2^2\right)\alpha^7}\right).\end{aligned}$$

Since

$$\frac{\mathcal{O}\left(\alpha^5\right)}{2r_2\left(1+r_2^2\right)\alpha^7} \to 0 \qquad \text{as} \qquad \alpha \to \infty,$$

this means that $\theta_\alpha \to \frac{\pi}{2}$ when $\alpha \to \infty$, as claimed. In particular, this shows that for all $\frac{\pi}{2} < \theta \leqslant \pi$, there exists an α for which $F_\alpha(\theta) < 0$. For such θ then, the Toeplitz operator with the symbol $a_1 z^m \bar{z}^n + r_2 e^{i\theta} z^i \bar{z}^j$ is not hyponormal. $\square$

The next example will demonstrate that the extra conditions we used for necessity in Theorem 12 cannot be completely dropped.

EXAMPLE 4. Let $\varphi_\theta(z) = z^2\bar{z} + \frac{1}{10}e^{i\theta} z^3 \bar{z}^2$. Here again, thinking in terms of two circles as in the proof of Theorem 12, we see that in this case the interiors of the two circles are disjoint for small α as shown in Figure 2.

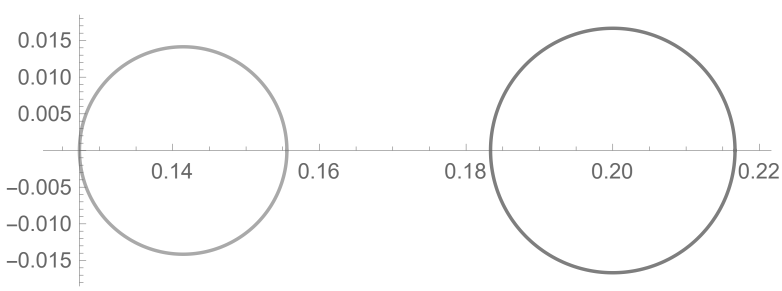

Figure 2: *The situation for* $\varphi_\theta(z) = z^2\bar{z} + \frac{1}{10}e^{i\theta} z^3 \bar{z}^2$ *with* $\alpha = 2$.

And indeed, as $\alpha \to \infty$, and these two circles come together, we find that

$$\left(\frac{1}{\alpha+3}+\frac{\cos\theta}{10(\alpha+4)}\right)^2+\frac{\sin^2\theta}{100\,(\alpha+4)^2}-c_{\alpha,\delta}^2\left[\left(\frac{1}{\alpha+2}+\frac{\cos\theta}{10(\alpha+3)}\right)^2+\frac{\sin^2\theta}{100\,(\alpha+3)^2}\right]>0$$

for all $\theta\in[0,\pi]$ and all $\alpha\geqslant 1$. Thus, the Toeplitz operator with symbol φ_θ is hyponormal for all choices of θ.

The next theorem improves slightly on the conditions of Corollary 3.

THEOREM 13. *Let $\varphi(z)=a_1z^{m_1}\,\overline{z}^{n_1}+\ldots+a_kz^{m_k}\,\overline{z}^{n_k}$, with $m_1-n_1=\ldots=m_k-n_k=\delta\geqslant 0$, and a_i all lying in the same quarter-plane $1\leqslant i\leqslant k$ (that is, we have $\max_{1\leqslant i,j\leqslant k}\left|\arg(a_i)-\arg(a_j)\right|\leqslant\frac{\pi}{2}$), then T_φ is hyponormal.*

Proof. The proof follows *mutatis mutandis* the proof of Theorem 12. Recall again that $c_{\alpha,\delta}=\sqrt{\frac{\alpha-\delta+1}{\alpha+\delta+1}}$.

We assume without loss of generality that a_1 is a positive real number, and we let $a_j=r_je^{i\theta_j}$ for $2\leqslant j\leqslant k$. The only other change is that instead of condition (20), we have hyponormality if and only if

$$\begin{aligned}&\left(\frac{a_1}{\alpha+m_1+1}+\sum_{i=2}^{k}\frac{r_i\cos\theta_i}{\alpha+m_i+1}\right)^2+\left(\sum_{i=2}^{k}\frac{r_i\sin\theta_i}{\alpha+m_i+1}\right)^2\\ &\geqslant c_{\alpha,\delta}^2\left(\left(\frac{a_1}{\alpha+n_1+1}+\sum_{i=2}^{k}\frac{r_i\cos\theta_i}{\alpha+n_i+1}\right)^2+\left(\sum_{i=2}^{k}\frac{r_i\sin\theta_i}{\alpha+n_i+1}\right)^2\right),\end{aligned} \tag{22}$$

for all $\alpha\geqslant\delta$.

Thus, in addition to needing all a_i in the right-half plane to guarantee

$$\left(\frac{a_1}{\alpha+m_1+1}+\sum_{i=2}^{k}\frac{r_i\cos\theta_i}{\alpha+m_i+1}\right)^2\geqslant c_{\alpha,\delta}^2\left(\frac{a_1}{\alpha+n_1+1}+\sum_{i=2}^{k}\frac{r_i\cos\theta_i}{\alpha+n_i+1}\right)^2,$$

for all $\alpha\geqslant\delta$, we also need all a_i in the upper half-plane to guarantee

$$\left(\sum_{i=2}^{k}\frac{r_i\sin\theta_i}{\alpha+m_i+1}\right)^2\geqslant c_{\alpha,\delta}^2\left(\sum_{i=2}^{k}\frac{r_i\sin\theta_i}{\alpha+n_i+1}\right)^2.$$

Thus, as long as $0\leqslant\theta_i\leqslant\frac{\pi}{2}$, for all i, T_φ is hyponormal.

By rotation, it is sufficient to have $\max_{1\leqslant i,j\leqslant k}\left|\arg(a_i)-\arg(a_j)\right|\leqslant\frac{\pi}{2}$. □

It is not known whether or not this condition is necessary. It may be possible *a priori* to construct φ in such a way that condition (22) holds, while allowing one of the a_i to be outside the given quarter-plane. We expect that the techniques of Example 4 can be modified to yield the desired outcome.

5. Final remarks

Our studies have focused on finding sufficient conditions for the hyponormality of Toeplitz operators having certain non-harmonic polynomials as symbols, with our methods invariably focusing on what can only be described as "hard" analysis. We would be interested in finding more function theoretic results akin to P. Ahern and Z. Čučković in [1], which would generate softer proofs and more qualitative results. For example, something along the lines of the following conjecture:

If T_f is hyponormal and T_g is co-hyponormal, then T_{f+g} is hyponormal implies that $|f_z| \geqslant |g_{\bar{z}}|$ in $\mathbb{D}$.

So far, all the examples we have conform to this prediction, but given the subtlety of hyponormality, this evidence is certainly not overwhelming.

We would also be interested in looking at necessary conditions, along the lines of much of the work that has been done by others studying operators with harmonic symbols such as Z. Čučković and R. Curto's recent work in [5].

Finally, in [3], Ch. Chu and D. Khavinson proved the following theorem for hyponormal Toeplitz operators acting on the Hardy space.

THEOREM 14. *If $\varphi = f + \overline{T_{\bar{h}} f}$ for $f, h \in H^\infty$, with $\|h\|_\infty \leqslant 1$ and $h(0) = 0$, that is, if T_φ is a hyponormal Toeplitz operator on the Hardy space H^2, then we have that*

$$\|[T_\varphi^*, T_\varphi]\| \geqslant \|P_+(\varphi) - \varphi(0)\|_2^2.$$

where P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto the Hardy space.

Combining this result with Putnam's inequality they arrive immediately at the following corollary:

COROLLARY 4. *If T_φ is a hyponormal Toeplitz operator acting on the Hardy space H^2, then*

$$\operatorname{Area}\left(\sigma\left(T_\varphi\right)\right) \geqslant \pi \|P_+(\varphi) - \varphi(0)\|_2^2.$$

Although a classification of hyponormal Toeplitz operators remains elusive for the Bergman space, it would be interesting to see under what conditions a similar lower bound could be obtained in the Bergman space setting. A cursory examination of the proof of Corollary 2 combined with Putnam's inequality shows that

$$\|P(z^m \bar{z}^n)\|_2^2 = \frac{(m-n+1)\pi}{(m+1)^2} \leqslant \operatorname{Area}\left(\sigma\left(T_{z^m \bar{z}^n}\right)\right).$$

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(Received October 3, 2017)

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