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A SURJECTIVITY PROBLEM FOR 3 BY 3 MATRICES

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Abstract. Let P be a complex polynomial. We prove that the associated polynomial matrix-valued function $\tilde{P}$ is surjective if and only if for each $\lambda \in \mathbb{C}$ the polynomial $P-\lambda$ has at least a simple zero.

1. Natural powers for matrices of order three

For any integer number n we denote by $\mathscr{M}(n,\mathbb{C})$ the set of all complex matrices of order n. Let $A \in \mathscr{M}(3,\mathbb{C})$ be given by

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}. \tag{1.1}$$

Denote by $x,y,z \in \mathbb{C}$ its eigenvalues and by $P_A(\lambda)$ its characteristic polynomial. Recall that $P_A(\lambda) = \lambda^3 - s_1(A)\lambda^2 + s_2(A)\lambda - s_3(A)$, where $s_i(A)$ are given by

$$s_1(A) = a_1 + b_2 + c_3 \tag{1.2}$$

$$s_2(A) = det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} + \det\begin{pmatrix} a_1 & a_3 \\ c_1 & c_3 \end{pmatrix} + \det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} \tag{1.3}$$

and

$$s_3(A) = \det(A). \tag{1.4}$$

We begin this section by presenting the powers of the matrix A in a suitable form so the technicalities in the main section are minimal. In addition it enables us to obtain immediately a spectral mapping theorem. There are three cases to consider, namely $(x-y)(x-z)(y-z) \neq 0$, $x=y=z$, and finally $P_A(\lambda) = (\lambda-x)^2(\lambda-y)$ with $x \neq y$.

1. Suppose $(x-y)(x-z)(y-z) \neq 0$. Then $P_A(\lambda) = (\lambda-x)(\lambda-y)(\lambda-z)$ and from the Hamilton-Cayley Theorem we have $P_A(A) = (A-xI_3)(A-yI_3)(A-zI_3) = 0_3$; the null matrix of order three.

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PROPOSITION 1.1. *With the above notations, suppose that* $(x-y)(x-z)(y-z) \neq 0$. *Then for every nonnegative integer* n, *one has*

$$A^n = x^n B + y^n C + z^n D \tag{1.5}$$

where

$$B = \frac{(A-yI_3)(A-zI_3)}{(x-y)(x-z)}, \quad C = \frac{(A-xI_3)(A-zI_3)}{(y-x)(y-z)} \tag{1.6}$$

and

$$D = \frac{(A-xI_3)(A-yI_3)}{(z-x)(z-y)}. \tag{1.7}$$

Proof. The proof is an easy mathematical induction argument and is left to the reader. □

2. Suppose the eigenvalues of the matrix $A \in \mathscr{M}(3,\mathbb{C})$ satisfy the condition $x = y = z$. Then $P_A(\lambda) = (\lambda - x)^3$ and the Hamilton-Cayley Theorem asserts that $(A - xI)^3 = 0_3$.

PROPOSITION 1.2. *Suppose* $x = y = z \neq 0$. *Then there exists matrices* B *and* C *in* $\mathscr{M}(3,\mathbb{C})$ *such that*

$$A^n = x^n (n^2 B + nC + I_3) \textit{ for all } n \in \mathbb{Z}_+. \tag{1.8}$$

In addition, the matrices B *and* C *satisfy the matrix system*

$$\begin{cases} x(B+C+I_3) & = A \\ x^2(4B+2C+I_3) & = A^2. \end{cases} \tag{1.9}$$

Proof. Note the system (1.9) arises, for example, by taking the particular values $n = 1$ and $n = 2$ in (1.8). The solution of the system (1.9) is

$$(B,C) = \left(\frac{1}{2x^2}(A-xI_3)^2, \quad -\frac{1}{2x^2}(A-xI_3)(A-3xI_3) \right). \tag{1.10}$$

With these values of B and C the proof of (1.8) is immediate. □

COROLLARY 1.1. *Let* $P(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ *be a polynomial with complex coefficients, and let* x, a, b *and* c *be given complex numbers. For*

$$A_1 = A_1(x,a,b,c) := \begin{pmatrix} x & a & c \\ 0 & x & b \\ 0 & 0 & x \end{pmatrix}, \tag{1.11}$$

$\tilde{P}(A_1) := a_n A_1^n + a_{n-1} A_1^{n-1} + \cdots a_1 A_1 + a_0 I_3$, *is given by*

$$\begin{pmatrix} P(x) & aP'(x) & \frac{1}{2!}cP''(x) \\ 0 & P(x) & bP'(x) \\ 0 & 0 & P(x) \end{pmatrix}. \tag{1.12}$$

Proof. Is enough to see that (1.8) and (1.10) yield

$$A_1^n = \begin{pmatrix} x^n & anx^{n-1} & \frac{1}{2!}n(n-1)cx^{n-2} \\ 0 & x^n & bnx^{n-1} \\ 0 & 0 & x^n \end{pmatrix}. \tag{1.13}$$

The details are omitted. $\square$

3. Suppose $P_A(\lambda) = (\lambda - x)^2(\lambda - y)$ with $x \neq y$. Then the Hamilton-Cayley Theorem asserts that $(A - xI_3)^2(A - yI_3) = 0_3$.

PROPOSITION 1.3. *If the matrix A has the eigenvalues x,x,y, with $x \neq y$ and $x \neq 0$ then its natural powers are given by*

$$A^n = x^n(nB + C) + y^n D, \quad n \in \mathbb{Z}_+, \tag{1.14}$$

where (B,C,D) is the solution of the matrix system

$$\begin{cases} C + D & = I_3 \\ xB + xC + yD & = A \\ 2x^2B + x^2C + y^2D & = A^2. \end{cases} \tag{1.15}$$

Proof. Note the system (1.15) and one has

$$B = \frac{1}{x(x-y)}(A - xI_3)(A - yI_3), \tag{1.16}$$

$$C = -\frac{1}{(x-y)^2}[A - (2x-y)I_3](A - yI_3), \tag{1.17}$$

$$D = \frac{1}{(x-y)^2}(A - xI_3)^2. \quad \square \tag{1.18}$$

COROLLARY 1.2. *Let $P(z)$ be a polynomial as in Corollary 1.1 and let x,y and a be given complex numbers. For*

$$A_2 = A_2(x,y,a) := \begin{pmatrix} x & a & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \tag{1.19}$$

$\tilde{P}(A_2)$ is given by

$$\begin{pmatrix} P(x) & aP'(x) & 0 \\ 0 & P(x) & 0 \\ 0 & 0 & P(y) \end{pmatrix}. \tag{1.20}$$

Proof. Is enough to see that (1.14), (1.16), (1.17) and (1.18) yield

$$A_2^n = \begin{pmatrix} x^n & anx^{n-1} & 0 \\ 0 & x^n & 0 \\ 0 & 0 & y^n \end{pmatrix}. \tag{1.21}$$

The details are omitted. □

Let $A \in \mathscr{M}(n,\mathbb{C})$. A monic polynomial of least degree (denoted by m_A) having the property that $m_A(A) = 0_n$ is called the minimal polynomial of A. The characteristic polynomial and the minimal polynomial of a matrix A above must have the same zeros but the multiplicity could be different. The next Theorem in its general form (i.e. for matrices n by n) is called the *Jordan canonical form Theorem* in honor of the French mathematician Camille Jordan (1833–1922) who first published a proof of it. Next we present the case $n = 3$ which is more convenient to write. The proof of the general case can be found for example in [4] on page 65.

THEOREM 1.1. *Let $A \in \mathscr{M}(3,\mathbb{C})$ be a matrix with the characteristic polynomial P_A and the minimal polynomial m_A.*

1. *If $P_A(\lambda) = m_A(\lambda) = (\lambda - x)(\lambda - y)(\lambda - z)$ with x,y,z mutually different then there exists an invertible complex matrix T_1 such that*

$$T_1^{-1}AT_1 = J_1(A) = diag(x,y,z) := \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}, \tag{1.22}$$

2. *If $P_A(\lambda) = m_A(\lambda) = (\lambda - x)^2(\lambda - y)$ with $x \neq y$ then there exists an invertible complex matrix T_2 such that*

$$T_2^{-1}AT_2 = J_2(A) := \begin{pmatrix} x & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}. \tag{1.23}$$

3. *If $P_A(\lambda) = (\lambda - x)^2(\lambda - y)$ and $m_A(\lambda) = (\lambda - x)(\lambda - y)$ with $x \neq y$ then there exists an invertible complex matrix T_3 such that*

$$T_2^{-1}AT_3 = J_3(A) := \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}. \tag{1.24}$$

4. *If $P_A(\lambda) = m_A(\lambda) = (\lambda - x)^3$ then there exists an invertible complex matrix T_4 such that*

$$T_4^{-1}AT_4 = J_4(A) := \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}. \tag{1.25}$$

5. *If* $P_A(\lambda) = (\lambda - x)^3$ *and* $m_A(\lambda) = (\lambda - x)^2$ *then there exists an invertible complex matrix* T_5 *such that*

$$T_5^{-1}AT_5 = J_5(A) := \begin{pmatrix} x & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}. \tag{1.26}$$

6. *If* $P_A(\lambda) = (\lambda - x)^3$ *and* $m_A(\lambda) = (\lambda - x)$ *then* $A = J_6(A) := xI_3$.

Recall that the spectrum of a matrix A, denoted by $\sigma(A)$, is the set of all its eigenvalues and that the resolvent set of A is $\rho(A) := \mathbb{C} \setminus \sigma(A)$, i.e. the set of all complex numbers z for which the matrix $zI_3 - A$, is invertible.

REMARK 1.1. **(1).** Note (see below)

$$\sigma(\tilde{P}(A)) = P(\sigma(A)). \tag{1.27}$$

(2). If $z \mapsto f(z) := \sum_{k=0}^{\infty} a_k z^k$ is an integer function (i.e. it is holomorphic on $\mathbb{C}$) then for each matrix $A \in \mathscr{X}$, the matrix

$$\tilde{f}(A) := \sum_{k=0}^{\infty} a_k A^k \tag{1.28}$$

is well defined. The convergence in (1.28) is considered with respect to the operator norm of matrices. Thus one has

$$\sigma(\tilde{f}(A)) = f(\sigma(A)). \tag{1.29}$$

In particular,

$$e^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}, \text{ and } \sigma(e^{tA}) = e^{t\sigma(A)}, t \in \mathbb{R}. \tag{1.30}$$

Proof. Let $A \in \mathscr{M}(3, \mathbb{C})$, $k \in \{1,2,3,4,5,6\}$ and T_k be an invertible matrix such that $T_k^{-1}AT_k = J_k(A)$. Then

$$\sigma(\tilde{f}(A)) = \sigma(T_k^{-1}\tilde{f}(A)T_k) = \sigma(\tilde{f}(T_k^{-1}AT_k)) = \sigma(\tilde{f}(J_k(A))) = f(\sigma(A)). \tag{1.31}$$

□

For matrices and operators we refer the reader to [2], [3], [4] and [5].

2. Global problems in the space of matrices

LEMMA 2.1. *If the polynomial* $P \in \mathbb{C}[z]$ *has no simple zeros then the matrix equation*

$$\tilde{P}(X) = Y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

has no solutions in $\mathscr{M}(3, \mathbb{C})$.

Proof. We argue by contradiction. Suppose that there exists $A \in \mathscr{M}(3,\mathbb{C})$ such that $\tilde{P}(A) = Y$. Thus in view of (1.27), $P(\sigma(A)) = \sigma(\tilde{P}(A)) = \{0\}$, i.e. the eigenvalues of A are zeros of the polynomial P. Now if $x,y,z \in \sigma(A)$ then $P(x) = P'(x) = P(y) = P(z) = 0$ and there exists a complex invertible matrix T_k such that $T_k^{-1}AT_k = J_k(A)$. Thus $\tilde{P}(X) = T_k\tilde{P}(J_k(A))T_k^{-1} = 0_3$ (the null matrix of order 3), for $k = 1,2,3,4,5$, so we have a contradiction. We have a similar contraction for $k = 6$ (we omit the details). $\square$

PROPOSITION 2.1. *Let $P \in \mathbb{C}[z]$ be a polynomial having the property that there exists a $m \in \mathbb{C}$ such that $Q := P - m$ has no simple zeros. Then the map $X \mapsto \tilde{P}(X)$: $\mathscr{M}(3,\mathbb{C}) \to \mathscr{M}(3,\mathbb{C})$ is not surjective.*

Proof. In view of Lemma 2.1, the equation

$$\tilde{P}(X) = mI_3 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

has no solutions in $\mathscr{M}(3,\mathbb{C})$. $\square$

DEFINITION 2.1. We say that a polynomial $P \in \mathbb{C}[z]$ has the simple zero property (**SZP**) if for every $m \in \mathbb{C}$ the polynomial $Q := P - m$ has at least a simple zero.

For example, the polynomial $P_1(z) = (z-1)(z-2)(z-3)$ has **SZP** while $P_2(z) = z^3$ does not. Clearly every polynomial of degree 1 has **SZP** but polynomials of degree 2 have not the property.

THEOREM 2.1. *Let $P \in \mathbb{C}[z]$ be a polynomial satisfying* **SZP**. *Then the map*

$$X \mapsto \tilde{P}(X) : \mathscr{M}(3,\mathbb{C}) \to \mathscr{M}(3,\mathbb{C}) \tag{2.3}$$

is surjective.

Proof. Let $Y \in \mathscr{M}(3,\mathbb{C})$ be given. The argument is broken into several cases.

1. The spectrum of Y consists of three mutually different complex numbers u, v and w. Thus, there exists an invertible matrix T such that $T^{-1}YT =$ diag (u,v,w). Set $X := T$ diag $(x,y,z))T^{-1}$, where x,y and z are zeros of $P-u$, $P-v$ and $P-w$, respectively. Then $\tilde{P}(X) = T\tilde{P}($ diag $(x,y,z))T^{-1} = Y$.

2. The spectrum of Y consists of u and v, with u being a zero of P_Y of multiplicity 2.

2.1. When $m_Y(\lambda) = (\lambda-u)^2(\lambda-v)$ then there exists an invertible matrix T_2 such that

$$T_2^{-1}YT_2 = J_2(Y) := \begin{pmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix}. \tag{2.4}$$

Let x be a simple zero of $P-u$ and y as above. Thus $P'(x) \neq 0$. Set

$$X = T_2 \begin{pmatrix} x & \frac{1}{P'(x)} & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} T_2^{-1}. \tag{2.5}$$

In view of Corollary 1.2, one has

$$\tilde{P}(X) = T_2 \tilde{P}\left(\begin{pmatrix} x & \frac{1}{P'(x)} & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \right) T_2^{-1} = T_2 J_2(Y) T_2^{-1} = Y. \tag{2.6}$$

2.2. When $m_Y(\lambda) = (\lambda - u)(\lambda - v)$ then there exists an invertible matrix T_3 such that

$$T_3^{-1} Y T_3 = J_3(Y) := \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix}. \tag{2.7}$$

Let x be a simple zero of $P-u$ and y as above. Set

$$X = T_3 \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} T_3^{-1}. \tag{2.8}$$

In view of Corollary 1.2, one has

$$\tilde{P}(X) = T_3 \tilde{P}\left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \right) T_3^{-1} = T_3 J_3(Y) T_3^{-1} = Y. \tag{2.9}$$

3. The spectrum of Y consists of u, being a zero of P_Y of multiplicity 3. We divide the proof into three steps.

3.1. When $m_Y(\lambda) = (\lambda - u)^3$ then there exists an invertible matrix T_4 such that

$$T_4^{-1} Y T_4 = J_4(Y) := \begin{pmatrix} u & 1 & 0 \\ 0 & u & 1 \\ 0 & 0 & u \end{pmatrix}. \tag{2.10}$$

Let x be a simple zero of $P-u$. Note $P'(x) \neq 0$. Set

$$X = T_4 \begin{pmatrix} x & \frac{1}{P'(x)} & 0 \\ 0 & x & \frac{1}{P'(x)} \\ 0 & 0 & x \end{pmatrix} T_4^{-1}. \tag{2.11}$$

In view of Corollary 1.1 one has

$$\tilde{P}(X) = T_4 \tilde{P}\left(\begin{pmatrix} x & \frac{1}{P'(x)} & 0 \\ 0 & x & \frac{1}{P'(x)} \\ 0 & 0 & x \end{pmatrix} \right) T_4^{-1} = T_4 J_{31}(Y) T_4^{-1} = Y. \tag{2.12}$$

3.2. When $m_Y(\lambda) = (\lambda - u)^2$ then there exists an invertible matrix T_5 such that

$$T_5^{-1} Y T_5 = J_5(Y) := \begin{pmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \tag{2.13}$$

and in view of Corollary 1.1 one has

$$\tilde{P}(X) = T_5 \tilde{P}\left(\begin{pmatrix} x & \frac{1}{P'(x)} & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}\right) T_5^{-1} = T_5 J_5(Y) T_5^{-1} = Y. \tag{2.14}$$

3.3. When $m_Y(\lambda) = (\lambda - u)$ then $Y = uI_3$. Let x a zero of $P - u$ and set $X = xI_3$. Then $\tilde{P}(X) = Y$. $\square$

THEOREM 2.2. *Let P be a polynomial. The map*

$$X \mapsto \tilde{P}(X) : \mathscr{M}(3, \mathbb{C}) \to \mathscr{M}(3, \mathbb{C}) \tag{2.15}$$

is surjective if and only if P has the simple zero property.

The proof of Theorem 2.2 follows by combining Proposition 2.1 with Theorem 2.1.

Finally, as an immediate consequence, we present the following $\mathbb{C}^9$ version of the Ax-Grothendieck's Theorem; see [1].

Let n be a positive integer and denote (ad-hoc) by $\mathscr{P}_n$ the set of all polynomial functions $p : \mathbb{C}^n \to \mathbb{C}^n$ (that is, all components of p are scalar valued polynomials of n complex variables). As is well-known (Ax-Grothendieck's Theorem) if $p \in \mathscr{P}_n$ is injective then it is surjective as well. In what follows we refer to the particular case $n = 9$.

Denote by $\mathscr{A}_9$ the set of all polynomials $p \in \mathscr{P}_9$ having the property that there exist a scalar polynomial P (of one complex variable) and a linear transformation $T : \mathscr{M}(3, \mathbb{C}) \to \mathbb{C}^9$ such that

$$p = T\tilde{P}T^{-1}. \tag{2.16}$$

COROLLARY 2.1. *If $p \in \mathscr{A}_9$ is injective then it is surjective as well. Moreover, in this case, the inverse of p is also a polynomial.*

Proof. From (2.16) we have

$$\tilde{P} = T^{-1} p T. \tag{2.17}$$

Now, the assumption on p yields the injectivity of $\tilde{P}$ and thus (as is very easy to see), the polynomial P has degree equal to 1. In particular P has **SZP**, so $\tilde{P}$ is surjective. Now, from (2.16), p is surjective. Moreover, since P has degree equal to 1, (2.16) yields that p has the degree equal to 1 and thus its inverse is a polynomial. $\square$

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