Operators and Matrices
Volume 13, Number 1 (2019), 121–153
doi:10.7153/oam-2019-13-08

TOWARDS LOCALIZATION IN LONG–RANGE CONTINUOUS INTERACTIVE ANDERSON MODELS

VICTOR CHULAEVSKY

(*Communicated by M. Zinchenko*)

Abstract. This paper is a follow-up of [8]. The main novelty is the proof of spectral and dynamical localization for a class of interactive Anderson models in Euclidean spaces with realistic, infinite-range inter-particle and media-particle potentials featuring a power-law decay at infinity. Specifically, we prove that in an energy interval near the bottom of the spectrum, the spectral measure is pure point with probability one, and the decay rate of the averaged eigenfunction correlators in this energy interval admits a summable power-law bound, the exponent of which grows along with the growth of the decay exponents of the potentials. The localized eigenfunctions admit a fractional-exponential bound on their decay rate. Earlier rigorous works on interactive Anderson models assumed the media-particle potential to be compactly supported.

1. Introduction

1.1. The model

We study an N-particle Anderson model in $\mathbb{R}^d$ with an external random potential of the so-called alloy type. Its Hamiltonian is a random Schrödinger operator

$$\mathbf{H}(\omega) = \mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + \mathbf{U}(\mathbf{x}) + \mathbf{V}(\mathbf{x},\omega), \tag{1}$$

acting on the functions on the N-particle configuration space $(\mathbb{R}^d)^N$. The points $\mathbf{x} = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N$ are the positions of N distinguishable quantum particles in $\mathbb{R}^d$. $\boldsymbol{\Delta}$ stands for the Laplacian in $(\mathbb{R}^d)^N \cong \mathbb{R}^{Nd}$. The interaction energy $\mathbf{U}(\mathbf{x})$ is generated by a 2-body interaction potential $U^{(2)}$ (cf. Assumption (U)). $\mathbf{V}(\mathbf{x},\omega)$ is the operator of multiplication by a *random* function

$$\mathbf{x} = (x_1,\dots,x_N) \mapsto V(x_1,\omega) + \cdots + V(x_N,\omega); \tag{2}$$

here $V(x,\omega)$ is a random field on $\mathbb{R}^d$ of the so-called alloy-type:

$$V(x,\omega) = \sum_{\mathrm{a}\in\mathcal{Z}} \omega_{\mathrm{a}}\, \varphi(|x-\mathrm{a}|), \tag{3}$$

Mathematics subject classification (2010): 60H25, 37K55.
Keywords and phrases: Multi-particle Anderson localization, infinite-range interactions, eigenvalue concentration estimates.

Paper OaM-13-08

where $\mathcal{Z} \subset \mathbb{R}^d$ is a periodic lattice with d linearly independent generators (we usually assume $\mathcal{Z} = \mathbb{Z}^d$), $\{\omega_a,\, a \in \mathcal{Z}\}$ are IID (independent and identically distributed) random variables on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and $\varphi : [0, +\infty) \ni r \mapsto \mathbb{R}$ is usually referred to as a scatterer (or site) potential. Algebraically, one has

$$\mathbf{V}(\mathbf{x}, \omega) = \sum_{j=1}^{N} \left(\bigotimes_{i=1}^{j-1} \mathbf{1} \right) \otimes V(x_j, \omega) \otimes \left(\bigotimes_{k=j+1}^{N} \mathbf{1} \right). \tag{4}$$

1.2. The motivation and comparison with the existing results

An infinite range of the particle-media interaction φ results in stochastic correlations of $V(x, \omega)$ at arbitrary distances. Recall that shortly after the reformulation in [14] of the Multi-Scale Analysis (MSA) from [22, 21], a correlated model was considered in [15]. However, the stochastic correlation of the scatterers and the infinite range of site potentials are two different mechanisms which are in general non-equivalent, especially in continuous spaces. A continuous model with an inverse polynomial decay of the site potential was considered in [27] where the MSA has been adapted to the new context. In particular, the main analytic and probabilistic MSA estimates originating from [22, 21, 14] were replaced by their "stable" analogs. We follow this general path, but deviate from the technology of [27] in several ways: the presence of the inter-particle interaction brings its own lot of technical challenges, as the reader will see.

In the last three decades, a great wealth of mathematically rigorous results has been accumulated in the spectral theory of disordered media; an historical survey along with a rich bibliography can be found, e.g., in the monograph [32]. It has been understood that the phenomenon of Anderson localization, originally discovered by a physicist Philip W. Anderson [1], may have various manifestations. Quite often, one can prove that in a certain interval (or intervals) of energies $I \subset \mathbb{R}$ the spectrum of a given random operator $H(\omega)$ is pure point with probability one (spectral localization), and all the eigenfunctions with eigenvalues in I decay exponentially at infinity (exponential spectral localization). One can often prove also that the averaged eigenfunction correlators feature a fast, summable decay at infinity (strong dynamical localization). The strongest results establish exponential decay of these correlators, in which case the strong dynamical localization is qualified as exponential. However, in certain models various technical difficulties result in a slower rate of decay of either eigenfunctions or of their averaged correlators that one manages to prove. This is particularly true for the N-particle Anderson models in a continuous, *Euclidean* configuration space. For example, Fauser and Warzel [20] proved exponential strong dynamical localization and exponential decay (with respect to the so-called Hausdorff pseudo-metric) of localized eigenfunctions in the N-particle continuous Anderson model for compactly supported scatterer potentials and exponentially decaying interaction potentials $U^{(2)}$; however, the decay rate of the eigenfunction correlators (and consequently, that of the eigenfunctions) was proved in [20] to be at least sub-exponential (not necessarily exponential) under the assumption of a slower decay of the interaction potential.

In a prior work [8] we proved spectral localization (namely, the exponential one) and strong dynamical localization (with a fractional-exponential decay of eigenfunction

correlators) in a continuous N-particle model with a rather artificial site potential φ. Such a choice was made to circumvent a hard technical problem with an analog of Wegner's estimate [34] for pairs of cubes (in the N-particle space) not separated in the Hausdorff pseudo-metric, used implicitly in [11] and explicitly in [2]. Fauser and Warzel [20] faced a similar problem, but, unfortunately, it is yet to be solved in the context of the Fractional Moments Method (FMM). A recent work by Klein *et al.* [30], building on [24], gave a characterization of the metal-insulator transition in a 2-particle Anderson model. The choice $N = 2$ was motivated, at least partially, by the fact that the Hausdorff pseudo-metric is equivalent to the symmetrized norm-distance only for $N = 2$.

Note in particular that for $N > 2$, decay in the Hausdorff metric does not imply decay in the symmetrized norm-distance. The latter is physically more natural as it becomes the norm-distance in a properly constructed configuration space of physically indistinguishable (bosonic or fermionic) quantum particles.

On the other hand, it was shown in the work [30] that its results can be extended to any $N \geqslant 3$, under the conditions proposed in our papers [8, 12].

Notice that an important progress has been achieved recently in the area of the 1D lattice fermionic systems with short-range interactions; cf. [18, 16, 17, 4].

One could have expected that the infinite range of the site (or scatterer, particle-media) potential φ in (3) would result in a greater complexity of the localization analysis, but, surprisingly, the effect of a physically more realistic potential φ is quite the opposite.

The main novelty of the present work is two-fold:

♦ We prove a uniform decay of eigenfunction correlators at low energies, with respect to the symmetrized norm-distance in a domain $\mathbf{B} \subseteq (\mathbb{R}^d)^N$, assuming that:

(i) the site potential φ features a sufficiently fast power-law decay (cf. (8)), and

(ii) the inter-particle potential $U^{(2)}$ also admits a power-law *upper* bound (7).

♦ The eigenvalue concentration (EVC) estimate used in this work does not rely on the well-known Wegner's argument going back to [34] and based on *local* random fluctuations of the random potential $V(x;\omega)$ (see (3)) in a finite volume where the EVC is to be assessed. Instead, we make use of 'almost flat' background fluctuations produced by *remote* amplitudes ω_{a}. In view of a recent preprint [9], this opens a way for the localization analysis of random media with arbitrarily singular disorder.

We also make a step towards a proof of N-particle localization in an energy interval $I_* = [0, E_*]$ near the bottom of spectrum independent of the number of particles $N \geqslant 1$. In the first work [10] on continuous N-particle models, as well as in more recent papers [20, 8], the N-dependence of the localization interval $I_*(N)$ has been a thorny problem. Now we prove the initial length scale estimate for all $N > 1$ in the same energy interval as for $N = 1$. Alas, some intrinsic limitations of the existing MSA techniques do not allow us at this moment to prove similar bounds on all larger scales for $N \geqslant 4$.

In the present paper we focus from the beginning on the analysis of the decay of the eigenfunction correlators, and so the bulk of technical work serves this purpose.

We prove that the averaged correlators decay (at worst) at an inverse polynomial rate $r \mapsto r^{-\mathfrak{b}}$ where $\mathfrak{b} > 0$ can be made arbitrarily large, provided the scatterer potential (cf. (3)) is $\varphi(r) = r^{-A}$ with $A > 0$ large enough.

As to the decay of the localized eigenfunctions, we enhance the power-law decay bounds on the resolvents, established in the course of the main scale induction (cf. Section 3), making use of a convenient particularity of the Multi-Scale Analysis (MSA): a first scale induction can often be complemented by one or more additional stages resulting in stronger final decay estimates. Specifically, in subsection 5.3 we start with the power-law decay bounds for the finite-cube resolvents from Section 3, and prove that the localized eigenfunctions decay at a fractional-exponential rate (see assertion **(B)** of the main Theorem 1).

The 1-particle model treated long ago by Kirsch *et al.* [27] suggests that even under a polynomial decay of the scatterer potential one may achieve exponential decay of the localized eigenfunctions. However, this would require a different variant of the N-particle multiscale analysis, and so we do not discuss this issue in the present work.

The pivot of the present paper is Theorem 4 providing an eigenvalue *comparison* bound for the pairs of Hamiltonians in distant cubes; a significant part of the paper is devoted to its proof (see Section 4). Without such a result, the analysis from Section 3 combined with an old argument due to Martinelli and Scoppola [31] would imply only absence of absolutely continuous spectrum at low energies.

While our techniques can be adapted so as to prove N-particle localization under faster-decaying interactions (e.g., $r \mapsto r^{-\ln^a r}$ with $a > 1$), it seems important to focus first on the most difficult case (viz. $r \mapsto r^{-A}$), single out the problematic points of the N-particle MSA, and give at least a partial solution to these problems in a situation ($N = 3$) which cannot be treated with the existing techniques of a powerful and valuable alternative method, the multi-particle variant of the FMM developed in [2, 20].

One last remark concerns the domains where uniform upper bounds on the eigenfunction correlators are established. In physics of disordered systems, a sample of disordered media is always of a finite size, but the first mathematically rigorous results on N-particle Anderson localization [11, 2] operated with an infinitely extended configuration space ($\mathbb{R}^d$ or $\mathbb{Z}^d$). A statement on pure point spectrum in a bounded domain would be trivial: if a self-adjoint operator has a compact resolvent, then its spectrum is actually discrete and not just pure point. In the 1-particle systems essentially the same amount of efforts is required for the analysis of (arbitrarily) large but finite, and of infinite domains, but surprisingly enough, this changes radically when we turn to the N-particle systems, starting with $N = 3$. In fact, the first rigorous papers on the N-particle localization [11, 2] proved localization bounds in the entire lattice $\mathbb{Z}^d$, but the techniques used were insufficient for proving similar bounds in arbitrarily large bounded domains. This is an artefact of the Hausdorff pseudo-metric used in [11, 2], and, unfortunately, it still remains an unresolved issue in the N-particle FMM, although it has been solved in the framework of the N-particle MSA (cf. [8, 12]). For this reason we emphasize in Theorem 1 that the general bound (9) holds both in the entire configuration space and in arbitrarily large bounded sub-domains thereof.

1.3. Notation, assumptions and the main result

The method of induction on the number of particles, initially developed in [11], when applicable, enables one to prove localization bounds for the n-particle systems with $1 \leqslant n \leqslant N$, where $N \geqslant 2$ is an integer to be fixed before the induction starts; many important parameters depend upon N. In the present paper, the main results are proved for $N \leqslant 3$, but some important ingredients of the multi-particle MSA (including Theorems 2 and 4) are established for arbitrary $N \geqslant 2$.

We work in Euclidean spaces $(\mathbb{R}^d)^N$, $d \geqslant 1$, $1 \leqslant N \leqslant 3$. A configuration of $N \geqslant 1$ distinguishable particles in $\mathbb{R}^d$ is identified with a vector $\mathbf{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$. In general, boldface symbols are reserved for "multi-particle" objects (Hamiltonians, resolvents, etc.). All Euclidean spaces will be endowed with the max-norm denoted by $|\cdot|$: $|\mathbf{x}| = \max_i |x_i|$, $\mathbf{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$. The symmetrized max-norm distance $\mathrm{d_S}(\cdot,\cdot)$ in $(\mathbb{R}^d)^N$ is defined in terms of permutations $\pi : \{1, \dots, N\} \leftrightarrow$: for $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N) \in (\mathbb{R}^d)^N$ one has

$$\mathrm{d_S}(\mathbf{x}, \mathbf{y}) = \min_\pi |\mathbf{x} - \pi\mathbf{y}|, \quad \text{where } \pi\mathbf{y} = (y_{\pi(1)}, \dots, y_{\pi(N)}). \tag{5}$$

The bulk of technical work concerns the operators related to the cubes $\mathbf{B}_L(\mathbf{u}) = \mathbf{B}_L^{(N)}(\mathbf{u}) = \left\{\mathbf{x} \in (\mathbb{R}^d)^N : |\mathbf{x} - \mathbf{u}| < L + \frac{1}{2}\right\}$ usually centered at lattice points $\mathbf{u} \in (\mathbb{Z}^d)^N$ (some arguments allow for any $\mathbf{u} \in (\mathbb{R}^d)^N$); here $\mathbb{Z}^d$ is unrelated to the scatterers' lattice $\mathcal{Z}$: in some arguments it is convenient to have a covering of the configuration space with a countable collection of cubes.

For notational brevity, we introduce the unit cells $\mathbf{C}_\mathbf{x} := \mathbf{B}_{1/2}(\mathbf{x})$.

The spectral projection of a given self-adjoint operator $\mathbf{H}$ onto an interval $I \subset \mathbb{R}$ is denoted $\mathbf{P}_I(\mathbf{H})$. The distances and diameters are relative to the max-norm. Integer intervals $[a, b] \cap \mathbb{Z}$ will be denoted by $[\![a, b]\!]$. Finally, we define the full projection

$$\Pi : (x_1, \dots, x_N) \mapsto \{x_1, \dots, x_N\} \in \mathbb{R}^d, \tag{6}$$

so that for $\mathbf{u} = (u_1, \dots, u_N)$ one has $\Pi\mathbf{B}_L(\mathbf{u}) = \cup_{j=1}^N \mathrm{B}_L(u_j)$.

For notational simplicity, we often use the standard notation $f(s) \lesssim g(s)$, meaning that for some $C \in (0, +\infty)$ one has $f(s) \leqslant Cg(s)$.

• **Interaction potential.** We assume the following:

(U) $\mathbf{U}$ is generated by a 2-body potential $U^{(2)} : \mathbb{R}_+ \to \mathbb{R}_+$ obeying

$$0 \leqslant U^{(2)}(r) \leqslant \mathrm{Const}\, r^{-A^\#}, \quad A^\# > d \tag{7}$$

(later we may require $A^\#$ and A in (8) to be even larger). At the moment, our technique requires the exponents $A^\# > A$, as their roles in the multi-particle MSA are quite different. Recall, however, that in prior rigorous works in this direction one never assumed φ and $U^{(2)}$ to be identical. Moreover, in some works (e.g., [20, 8, 12]) $U^{(2)}$ was assumed to have an infinite range while φ had a compact support. Admittedly, further progress is to be expected here.

• **Random external potential.** The following condition will be assumed.

(V) The random potential $V(x;\omega)$ is as in (3), where ω_a, $a \in \mathcal{Z}$, are IID random variables. We assume a summable inverse-polynomial decay of the scatterer function:

$$\varphi(r) = \begin{cases} r^{-A}, \text{ if } r \geqslant r_0; A > d, \\ 1, \quad \text{if } 0 \leqslant r < r_0. \end{cases} \tag{8}$$

REMARK 1. The above assumption on φ can certainly be relaxed. As was said, we intentionally explore the class of relatively slowly decaying interactions which present a certain number of technical challenges. The explicit power-law form is quite convenient, but what we need is the following:
(1) Reasonable lower (and not just upper) bounds on $\varphi(r)$. The latter is not a mere nuisance at large distances, but also an important mechanism providing eigenvalue concentration estimates difficult to establish for compactly supported potentials in the continuous multi-particle Anderson models.
(2) φ has to be a "slowly varying" function in the following sense: for large $r > 0$, $\varphi'(r)$ must be much smaller than $\varphi(r)$ itself, so that at large distances a restriction of the random potential $x \mapsto V(x,\omega)$ to a large cube could be approximated (with a controllable accuracy) by a random constant ("almost flat" potential). There certainly is a room for generalizations here, but also some constraints. To avoid unproven statements, we do not describe here more specific extensions which may appear artificial and yet leave an impression that a larger class of scatterer potentials can be treated. In fact, the staple of the present paper, Section 4, is already overly technical and lengthy.

We also assume ω_a to have a Lipschitz continuous, compactly supported probability measure μ with $0 \in \operatorname{supp}\mu \subset \mathbb{R}_+$. (These conditions can be relaxed.)

We call a connected domain $\mathbf{D} \subseteq (\mathbb{R}^d)^N$ *regular* if $\mathbf{D}$ is a union of 'cells'– *closed* cubes $\mathbf{C_u}$ of diameter 1 centered at the points $\mathbf{u} \in (\mathbb{Z}^d)^N$. If $\mathbf{D}$ is regular and bounded, then $\mathbf{H}_\mathbf{D}^{(N)}$ with Dirichlet's boundary conditions is self-adjoint in $L^2(\mathbf{D})$, and the resolvent $(\mathbf{H}_\mathbf{D}^{(N)} - \lambda\mathbf{1})^{-1}$ is a compact operator for λ outside the spectrum of $\mathbf{H}_\mathbf{D}^{(N)}$. The location of the almost sure spectrum $\Sigma_{\text{a.s.}}(\mathbf{H}^{(N)}(\omega))$ can be determined with the help of Weyl's criterion (cf. [29, Proposition A.1]).

PROPOSITION 1. *Under assumptions* (V)–(U), $\Sigma_{\text{a.s.}}(\mathbf{H}^{(N)}(\omega)) = [0, +\infty)$.

The main result of the paper is the following

THEOREM 1. *Under the assumptions* (U)–(V) *there exists an interval* $I^* = [0, E^*]$, $E^* > 0$, *such that for any* $\mathfrak{b} > 0$ *and some* $\hat{A}_\mathfrak{b}$, $\hat{A}^\#_\mathfrak{b}$, $\hat{Q}_\mathfrak{b} > 0$, *if* $A > \hat{A}_\mathfrak{b}$, $A^\# \geqslant \hat{A}^\#_\mathfrak{b}$ *and* $L_0 \geqslant \hat{Q}_\mathfrak{b}$, *then the following holds for any* $N \in [\![1,3]\!]$*:*
(A) *For all* $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ *with* $R_{\mathbf{x},\mathbf{y}} := \mathrm{d}_S(\mathbf{x},\mathbf{y}) \geqslant 1$, *and for any regular domain* $\mathbf{D} \subseteq (\mathbb{R}^d)^N$ *(bounded or not) such that* $\mathbf{D} \supset \mathbf{B}_{\frac{1}{2}R_{\mathbf{x},\mathbf{y}}}(\mathbf{x}) \cup \mathbf{B}_{\frac{1}{2}R_{\mathbf{x},\mathbf{y}}}(\mathbf{y})$

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}} \big\| \mathbf{1}_{\mathbf{C}_\mathbf{y}} \mathbf{P}_{I^*}\big(\mathbf{H}_\mathbf{D}^{(N)}\big)\, \mathrm{e}^{-\mathrm{i}t\mathbf{H}_\mathbf{D}^{(N)}}\, \mathbf{1}_{\mathbf{C}_\mathbf{x}} \big\| \Big] \lesssim R_{\mathbf{x},\mathbf{y}}^{-\mathfrak{b}}. \tag{9}$$

(B) *With probability one, the spectrum of* $\mathbf{H}^{(N)}(\omega) \equiv \mathbf{H}^{(N)}_{(\mathbb{R}^d)^N}(\omega)$ *in* I^* *is pure point. Moreover, there exists some non-random* $\zeta > 0$ *such that for each normalized eigenfunction* $\boldsymbol{\psi} = \boldsymbol{\psi}(\omega)$ *with eigenvalue in* I^* *and some* $C_{\boldsymbol{\psi}}(\omega) > 0$

$$\forall \mathbf{x} \in (\mathbb{R}^d)^N \qquad \|\mathbf{1}_{\mathbf{C}_\mathbf{x}} \boldsymbol{\psi}\|_2 \leqslant C_{\boldsymbol{\psi}}(\omega) \mathrm{e}^{-|\mathbf{x}|^\zeta}. \tag{10}$$

The norm figuring in the LHS of (10) is the vector norm in the Hilbert space $\mathrm{L}^2\big((\mathbb{R}^d)^N\big)$, and the norm in the LHS of (9) is the operator norm relative to $\mathrm{L}^2(\mathbf{D})$.

For the readers unfamiliar with standard results of the rigorous Anderson localization theory, we note that actually $\zeta \in (0,1)$. The techniques of the MSA do not allow one to achieve $\zeta = 1$, and the available, accurate analysis of the particular case of 1-particle random Hamiltonians in $\mathbb{R}^1$ suggests that one should not expect $\zeta > 1$.

2. Eigenvalue concentration bound

The main result of this section is Theorem 2. It suffices for the fixed-energy MSA (cf. Section 3), while for the proof of localization we need Theorem 4 relying on a rather tedious construction, which is the pivot of the present work, so we postpone it until Section 4.

A reader familiar with the proofs of Wegner-type estimates, starting with the seminal paper [34], can notice that the proof of Theorem 2 does not exploit the usual mechanisms ensuring regularity of the eigenvalue distribution (this is why the claim is insufficient for proving the existence of the so-called density of states), and relies instead on the infinite range and tempered decay of the scatterer potential.

In this section we work with a fixed interval $I^* = [0, E^*]$ where $E^* > 0$, and with $1 \leqslant N \leqslant 3$. Let $\Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{u})})$ be the spectrum of $\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}$ (with Dirichlet boundary conditions) and denote $\Sigma^{I^*}_{\mathbf{u},L}(\omega) = \Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}(\omega)) \cap I^*$.

Given a subset $\Lambda \subsetneq \mathcal{Z}$, we denote by ω_Λ the sub-samples of random amplitudes $(\omega_\mathrm{a}, \mathrm{a} \in \Lambda)$, by $\omega^\perp_\Lambda$ the complementary sub-samples $(\omega_\mathrm{a}, \mathrm{a} \in \mathcal{Z} \setminus \Lambda)$. We identify a sub-sample ω_Λ, considered as a function $\Lambda \ni \mathrm{a} \mapsto \omega_\mathrm{a} \in \mathbb{R}$, with its zero-extension $\mathcal{Z} \ni \mathrm{a} \mapsto \omega_\mathrm{a} \mathbf{1}_\Lambda(\mathrm{a})$ to the entire lattice (the same goes for its complement $\omega^\perp_\Lambda$), and often use the decomposition $\omega = \omega_\Lambda + \omega^\perp_\Lambda$. It is convenient to introduce an empty sub-sample $\varnothing^\perp_\Lambda$, so that it makes sense to write the zero-extension of ω_Λ as the sum $\omega_\Lambda + \varnothing^\perp_\Lambda$. The symbol $\mathfrak{F}^\perp_\Lambda$ denotes the sigma-algebra generated by random variables ω_a with $\mathrm{a} \notin \Lambda$. Recall that the support of the common probability measure of ω_a is compact, so the functions $\mathrm{a} \mapsto \omega_\mathrm{a}$ are uniformly bounded.

THEOREM 2. *Let be given an interval* $I^* = [0, E^*]$, *an integer* $N \in [\![1,3]\!]$ *and* $\tau > 0$. *There exists a positive integer* L_* *and* $\mathrm{q}_* > 0$ *large enough such that for* $L \geqslant L_*$ *and* $\mathrm{q} \geqslant \mathrm{q}_*$ *the following holds true for all* $\mathbf{u} \in (\mathbb{R}^d)^N$. *Let* $\mathbf{B} = \mathbf{B}_L(\mathbf{u})$, $\bar{\mathbf{B}} = \mathbf{B}_{\mathrm{q}L^{1+2\tau}}(\mathbf{u})$, $\bar{\Lambda} := \Pi\bar{\mathbf{B}} \cap \mathcal{Z}$ *(cf. (6)). Then for any fixed* $\omega^\perp_{\bar{\Lambda}}$ *and* $\varepsilon > 0$

$$\text{(A)} \qquad \forall E \in I^* \quad \mathbb{P}\Big\{\omega_{\bar{\Lambda}} : \operatorname{dist}\big[\Sigma^{I^*}_{\mathbf{u},L}(\omega_{\bar{\Lambda}} + \omega^\perp_{\bar{\Lambda}}), E\big] \leqslant \varepsilon\Big\} \lesssim L^{Nd+A}\varepsilon. \tag{11}$$

(B) *Moreover, for all* $\varepsilon \geqslant L^{-(1+2\tau)(A-d)}$ *and all* $E \in I^*$ *one has*

$$\mathbb{P}\Big\{\omega_{\bar\Lambda}:\ \inf_{\omega^{\perp}_{\bar\Lambda}} \operatorname{dist}\big[\Sigma^{I^*}_{\mathbf{u},L}(\omega_{\bar\Lambda},\omega^{\perp}_{\bar\Lambda}),E\big]\leqslant\varepsilon\Big\}\lesssim L^{Nd+A}\varepsilon\,. \tag{12}$$

Proof. (A) Fix an N-particle cube $\mathbf{B}\equiv\mathbf{B}_L(\mathbf{u})=\mathrm{B}_L(u_1)\times\cdots\times\mathrm{B}_L(u_N)$ and consider its full projection $\Pi\mathbf{B}_L(\mathbf{u})=\cup_{1\leqslant j\leqslant N}\mathrm{B}_L(u_j)$ (cf. (6)). By positivity of $\mathbf{V}(\mathbf{x},\omega)$, it follows from (4) that for any $\hat u\in\Pi\mathbf{B}$ we have an inequality $\mathbf{V}_{\mathbf{B}}(\omega)\geqslant V(\hat u,\omega)\cdot\mathbf{1}_{\mathbf{B}}$ for the restriction $\mathbf{V}_{\mathbf{B}}(\omega)=\mathbf{1}_{\mathbf{B}}\mathbf{V}(\omega)\mathbf{1}_{\mathbf{B}}$, in the sense of quadratic forms. For example,

$$\mathbf{1}_{\mathbf{B}}\mathbf{V}_{\mathbf{B}}(\omega)\mathbf{1}_{\mathbf{B}}\geqslant\varphi(L)\omega_{\hat u}\mathbf{1}_{\mathbf{B}}\geqslant L^{-A}\omega_{\hat u}\mathbf{1}_{\mathbf{B}}\,,\ \ \hat u=u_1\,. \tag{13}$$

For every $\mathrm{a}\in\Pi\mathbf{B}$ the random amplitude ω_{a} appears in the decomposition of $\mathbf{V}(\cdot,\omega)$ at least once with a factor given by the potential $\varphi(\cdot-\mathrm{a})$, so one has

$$\mathbf{V}_{\mathbf{B}}=\widetilde{\mathbf{V}}_{\mathbf{B}}(\omega^{\perp}_{\hat u})+\omega_{\hat u}\mathbf{\Phi}_{\hat u}\,,\quad \mathbf{\Phi}_{\hat u}\gtrsim L^{-A}\omega_{\hat u}\mathbf{1}_{\mathbf{B}}\,, \tag{14}$$

where $\widetilde{\mathbf{V}}_{\mathbf{B}}(\omega^{\perp}_{\hat u})$ is independent of $\omega_{\hat u}$. To establish (11) it suffices to prove a similar bound conditional on the σ-algebra $\mathfrak{F}^{\perp}_{\hat u}$ generated by $\{\omega_{\mathrm{a}},\,\mathrm{a}\in\Pi\bar\Lambda\setminus\hat u\}$. Until the end of the argument, we split the sample ω into the sum $\omega_{\hat u}+\omega^{\perp}_{\hat u}$ and keep the component $\omega^{\perp}_{\hat u}=(\omega_{\mathrm{a}},\mathrm{a}\neq\hat u)$ fixed. Decompose $\mathbf{H}_{\mathbf{B}}(\omega)$ as follows:

$$\mathbf{H}_{\mathbf{B}}(\omega)=\mathbf{A}_{\mathbf{B}}(\omega^{\perp}_{\hat u})+\omega_{\hat u}\mathbf{\Phi}_{\hat u}\,, \tag{15}$$

with $\mathbf{A}=\mathbf{A}_{\mathbf{B}}(\omega^{\perp}_{\hat u}):=\Big(-\mathbf{\Delta}_{\mathbf{B}}+\mathbf{U}_{\mathbf{B}}+\widetilde{\mathbf{V}}(\omega^{\perp}_{\hat u})\Big)$. Conditional on $\mathfrak{F}^{\perp}_{\hat u}$, the operator $\mathbf{A}$ is non-random. Next, consider an analytic family of self-adjoint operators (with compact resolvent) $\mathbf{K}(s)=\mathbf{A}+s\mathbf{\Phi}_{\hat u}\,,\ \ s\in\mathbb{R}$, Let $\{\lambda_j(s),j\geqslant1\}$ be the corresponding eigenvalues numbered in ascending order, counting multiplicity; their analytic s-dependence follows from standard results of functional analysis (cf., e.g., [33, Chapter XII]). Let $\boldsymbol{\psi}_j(s)$ the corresponding eigenfunctions; then on account of (14), we have by the Hellmann–Feynman formula

$$\frac{\mathrm{d}\lambda_j(s)}{\mathrm{d}s}=\left\langle\boldsymbol{\psi}_j(s),\frac{\mathrm{d}\mathbf{K}(s)}{\mathrm{d}s}\,\boldsymbol{\psi}_j(s)\right\rangle=\langle\boldsymbol{\psi}_j(s),\mathbf{\Phi}_{\hat u}\,\boldsymbol{\psi}_j(s)\rangle\gtrsim L^{-A}>0,$$

hence identifying the random eigenvalues $E_j(\omega_{\hat u})$ with $\lambda_j(s)\big|_{s=\omega_{\hat u}}$ we see that the probability measure of any $E_j(\omega_{\hat u})$ is the image of the probability measure of $\omega_{\hat u}$ by a strictly monotone mapping $\lambda_j:\mathbb{R}\to\mathbb{R}$ with the derivative lower-bounded by $cL^{-A}>0$, $c>0$.

By Weyl's estimate (cf., e.g., [5, Section VI.2.1]) for the kinetic energy operator $(-\mathbf{\Delta})$ and positivity of $\mathbf{U}$ and $\mathbf{V}$, the number of eigenvalues in $I=[0,E^*]$ is bounded by $\overline{C}L^{Nd}$ with $\overline{C}=\overline{C}(E_*)\in(0,\infty)$. This proves the asserted estimate (11).

(B) Decompose the N-particle random potential as follows: $\mathbf{V}(\omega)=\mathbf{V}'(\omega)+\mathbf{V}''(\omega)$, where $\mathbf{V}'(\omega)$ is generated by the finite sample of random potentials

$$V'(x,\omega)=\sum_{\mathrm{a}\in\mathcal{Z}\cap\bar\Lambda}\omega_{\mathrm{a}}\mathfrak{u}(\mathrm{a}-x)$$

and $\mathbf{V}''(x,\omega)$ is generated by $\omega_{\mathrm{a}}\mathrm{u}(\mathrm{a}-x)$ with $\mathrm{a}\in\mathcal{Z}\setminus\overline{\Lambda}$, hence independent of $\mathbf{V}'(\cdot,\omega)$. Since $\operatorname{dist}\big[\Pi\mathbf{B}_L(\mathbf{u}),\,\mathcal{Z}\setminus\overline{\Lambda}\big]\gtrsim \mathrm{q}L^{1+2\tau}$, we get by a straightforward calculation a norm-bound on the potential $\mathbf{V}''$ induced by remote scatterers,

$$\sup_{\omega}\|\mathbf{V}''(\cdot,\omega)\mathbf{1}_{\mathbf{B}_L(\mathbf{u})}(\cdot)\|\lesssim \mathrm{q}^{-A}L^{-(A-d)(1+2\tau)}\leqslant\frac{1}{2}L^{-(A-d)(1+2\tau)},$$

valid for $\mathrm{q}>1$ large enough. Thus by the min-max principle, we have for any fixed sub-sample $\widetilde{\omega}_{\overline{\Lambda}}$ and any complementary sub-sample $\omega_{\overline{\Lambda}}^{\perp}$

$$\operatorname{dist}\big[\Sigma^{I^*}_{\mathbf{u},L}(\widetilde{\omega}_{\overline{\Lambda}},\omega_{\overline{\Lambda}}^{\perp}),E\big]\geqslant\operatorname{dist}\big[\Sigma^{I^*}_{\mathbf{u},L}(\widetilde{\omega}_{\overline{\Lambda}},\varnothing),E\big]-\frac{1}{2}L^{-(A-d)(1+2\tau)}. \tag{16}$$

Taking into account assertion (A), this proves assertion (B). □

In Section 3 we will introduce positive numbers τ_N playing the role of τ from Theorem 2, and the length scales $L_{N,k}$, $N\in[\![1,3]\!]$, $k\geqslant 0$ (cf. (24)).

DEFINITION 2.1. A cube $\mathbf{B}^{(N)}_{L_{N,k}}(\mathbf{u})$ is called strongly interactive (SI) if

$$\operatorname{diam}\Pi\mathbf{u}\equiv\max_{i\neq j}|u_i-u_j|\leqslant 3\,\mathrm{q}NL_k^{1+2\tau_N}, \tag{17}$$

and weakly interactive (WI), otherwise.

Any WI cube $\mathbf{B}^{(N)}_L(\mathbf{u})$ admits a factorization $\mathbf{B}^{(N)}_L(\mathbf{u})=\mathbf{B}^{(n')}_L(\mathbf{u}')\times\mathbf{B}^{(n'')}_L(\mathbf{u}'')$ where

$$\operatorname{dist}\Big(\Pi\mathbf{B}^{(n')}_L(\mathbf{u}'),\Pi\mathbf{B}^{(n'')}_L(\mathbf{u}'')\Big)>\frac{3N\mathrm{q}}{N-1}L^{1+2\tau_N}-2L>\mathrm{q}L^{1+2\tau_N} \tag{18}$$

(we omit the proof which is quite simple; cf. a similar statement in [2]). Such a factorization may not be unique, but we will assume that one factorization satisfying (18) is fixed for each WI cube; it will be referenced to as the canonical one. Note that for any cube $\mathbf{B}_L(\mathbf{u})$ one has

$$\max_{\mathbf{x}\in\mathbf{B}_L(\mathbf{u})}\operatorname{dist}(\Pi\mathbf{x},\Pi\mathbf{u})\leqslant L, \tag{19}$$

and for an SI cube $\mathbf{B}_L(\mathbf{u})$ one has in addition, for L large enough,

$$\operatorname{diam}\Pi\mathbf{B}_L(\mathbf{u})\leqslant\operatorname{diam}\Pi\mathbf{u}+2L\leqslant 3N\mathrm{q}L^{1+2\tau_N}+2L<4N\mathrm{q}L^{1+2\tau_N}. \tag{20}$$

COROLLARY 1. *Fix $k\in\mathbb{N}$ and an arbitrary collection of strongly interactive cubes $\mathbf{B}_{L_k}(\mathbf{u}_i)\subset\mathbf{B}=\mathbf{B}_{L_{k+1}}(\mathbf{y})$, $1\leqslant i\leqslant M$, $M\geqslant 2$, with $\mathrm{d}_S(\mathbf{u}^{(i)},\mathbf{u}^{(j)})>8N\mathrm{q}L_k^{1+2\tau_N}$ for $i\neq j$. Let $\overline{\Lambda}^i=\Pi\mathbf{B}_{L_k^{1+2\tau_N}}(\mathbf{u}^{(i)})\cap\mathcal{Z}$. Then the events*

$$\Big\{\omega_{\overline{\Lambda}^i}:\inf_{\omega^{\perp}_{\overline{\Lambda}^i}}\operatorname{dist}\Big[\Sigma^{I^*}_{\mathbf{u}^{(i)},L_k}\big(\omega_{\overline{\Lambda}^i},\omega^{\perp}_{\overline{\Lambda}^i}\big),E\Big]\leqslant\varepsilon\Big\},\quad 1\leqslant i\leqslant M, \tag{21}$$

are independent. Consequently, for any $E\in I^*$ and $\varepsilon\geqslant L_k^{-(1+2\tau_N)(A-d)}$ one has

$$\mathbb{P}\Big\{\omega_{\overline{\Lambda}^{(i)}}:\min_{1\leqslant i\leqslant M}\inf_{\omega^{\perp}_{\overline{\Lambda}^i}}\operatorname{dist}\Big[\Sigma^{I^*}_{\mathbf{u}^{(i)},L_k}\big(\omega_{\overline{\Lambda}^i},\omega^{\perp}_{\overline{\Lambda}^i}\big),E\Big]\leqslant\varepsilon\Big\}\lesssim\Big(L_k^{A+Nd}\,\varepsilon\Big)^M. \tag{22}$$

Proof. Check the independence of the events (21), so the claim would follow from assertion (B) of Theorem 2. Since all cubes $\mathbf{B}_L(\mathbf{u}^{(i)})$ are strongly interactive, we have $\max_l |u_1^{(i)} - u_l^{(i)}| \leqslant 3qL_k^{1+2\tau_N}$, so by (19) we get $\Pi\mathbf{B}_{L_k}(\mathbf{u}^{(i)}) \subset \overline{\Lambda}^i$, and (20) implies

$$\operatorname{dist}\left(\overline{\Lambda}^i, \overline{\Lambda}^j\right) > \operatorname{dist}\left(u_1^{(i)}, u_1^{(j)}\right) - 2 \cdot 4Nq L_k^{1+2\tau_N} > 0. \tag{23}$$

Thus $\overline{\Lambda}^i \cap \overline{\Lambda}^j = \varnothing$ for $i \neq j$, whence the independence of the events (21). □

3. Decay of the Green functions at a fixed energy

We shall prove localization bounds for the N-particle Green functions of the Hamiltonians, for every energy E in an interval $I^* = [0, E^*]$ with $E^* > 0$ determined by the parameters of the model. With $N = 2, 3$, we carry out a scale induction for the finite-volume Green functions, with recursively defined length scales $L_{N,k}$:

$$L_{N,k+1} = \left\lfloor L_{N,k}^{\alpha_N} \right\rfloor, \; L_0 = L_{0,N} > 1, \;\; \alpha_N = (1+\sigma)(1+2\tau_N), \; \sigma \in (0, \tfrac{1}{4}). \tag{24}$$

For brevity, we often write L_k instead of $L_{N,k}$. The goal of the scale induction is to derive analytic and probabilistic bounds on the decay of the Green functions of the N-particle systems from their counterparts for n-particle systems, $1 \leqslant n \leqslant N-1$. The base of induction is of course the analysis of single-particle Hamiltonians. As in prior works (cf., e.g., [8, 12]), the case $N = 1$, which could be treated separately, is merely a simpler variant of the general situation: there are no non-trivial subsystems, hence no "weakly interactive" cubes (cf. Definition 2.1).

The principal assumptions on α_N and other key parameters of the scaling scheme are summarized in (40)–(42). The two most important quantities are b_N giving the guaranteed decay rate $L_0 \mapsto L_0^{-b_N}$ of the N-particle Green functions at the initial scale L_0, and the exponent s_N figuring in the probabilities of various unwanted events relative to that scale, e.g., $L_0^{-s_N}$. Both b_N and s_N are determined by the key parameters of the model; L_0 is to be sufficiently large. The N-dependence of various quantities is often suppressed in notation. Given a cube $\mathbf{B}_{L_k}(\mathbf{x}) \equiv \mathbf{B}_{L_k}^{(N)}(\mathbf{x})$, we introduce a convenient shortcut

$$\left\|\mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{x})}(E)\right\|^{\curlywedge} := \left\|\mathbf{1}_{\partial\mathbf{B}_{L_k}(\mathbf{x})}\, \mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{x})}(E)\, \mathbf{1}_{\mathbf{B}_{L_k/3}(\mathbf{x})}\right\| \tag{25}$$

($\|\cdot\|$ is the operator norm), where

$$\partial\mathbf{B}_L(\mathbf{x}) := \mathbf{B}_L(\mathbf{x}) \setminus \mathbf{B}_{L-2}(\mathbf{x}), \; L \geqslant 4.$$

Here $\curlywedge$ symbolizes the decay from the center to the boundary of the cube.

DEFINITION 3.1. Let be given real numbers $\delta, \varepsilon > 0$, E, and integers $k > 0$, $N \geqslant 1$. A sample $\omega : \mathcal{Z} \to \mathbb{R}$ of site amplitudes ω_{a}, $\mathrm{a} \in \mathcal{Z}$ (cf. (2)), is called :

(i) (E, δ)-non-singular in a cube $\mathbf{B}_{L_k}^{(N)}(\mathbf{u})$ ((E, δ)-NS, in short) iff

$$(3L_k)^{Nd} \left\|\mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{x})}(E)\right\|^{\curlywedge} \leqslant \delta; \tag{26}$$

(ii) (E,ε)-non-resonant in $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{u})$ (denoted (E,ε)-NR) iff

$$\operatorname{dist}\bigl(\Sigma_{\mathbf{u},L_{k+1}},E\bigr)\geqslant 2\varepsilon\,; \tag{27}$$

(iii) (E,ε)-completely non-resonant in the cube $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{u})$ (denoted (E,ε)-CNR) iff for all $\ell\in[\![L_k,L_{k+1}-L_k-2]\!]$

$$\operatorname{dist}\bigl(\Sigma_{\mathbf{u},\ell},E\bigr)\geqslant 2\varepsilon. \tag{28}$$

It is customary to associate the "singularity" or "resonance" properties with a given cube, once ω is fixed. So, we often say, e.g., that a cube $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ is (E,δ)-NS (for a sample ω). We shall need stronger variants of the above three properties $\mathscr{P}$, viz. (E,δ)-NS, (E,ε)-NR and (E,ε)-CNR. Given a cube $\mathbf{B}=\mathbf{B}_{L_k}(\mathbf{u})$, we denote $\bar{\mathbf{B}}=\mathbf{B}_{L_k^{1+\tau}}(\mathbf{u})$ and say that a sub-sample $\omega_{\Pi\bar{\mathbf{B}}}$ has a strong (or stable) property $\mathscr{P}$ in $\mathbf{B}$ iff for any complementary sub-sample $\omega^{\perp}_{\Pi\bar{\mathbf{B}}}$, the full sample $\omega=(\omega_{\Pi\bar{\mathbf{B}}},\omega^{\perp}_{\Pi\bar{\mathbf{B}}})$ has the property $\mathscr{P}$ in $\mathbf{B}$. The respective abbreviations become SNS, SNR and SCNR.

Specifically, the (E,δ)-SNS property of a cube $\mathbf{B}=\mathbf{B}_L(\mathbf{u})$ with $\Pi\mathbf{B}=\cup_{j=1}^N \mathrm{B}_L(u_i)$ $=:\Lambda$ reads as follows:

$$\sup_{\omega_\Lambda}(3L_k)^{Nd}\bigl\|\mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{x})}(E,\omega_{\bar{\mathbf{B}}}+\omega^{\perp}_{\bar{\mathbf{B}}})\bigr\|^{\lambda}\leqslant\delta, \tag{29}$$

and for the (E,ε)-SNR property we have

$$\inf_{\omega_\Lambda}\operatorname{dist}\bigl(\Sigma_{\mathbf{u},L_{k+1}}(\omega_{\bar{\mathbf{B}}}+\omega^{\perp}_{\bar{\mathbf{B}}}),E\bigr)\geqslant 2\varepsilon. \tag{30}$$

The SCNR property enhances CNR in the same way as the item **(iii)** enhances **(ii)** in Definition 3.1.

The norm $\bigl\|\mathbf{G}_{\mathbf{B}_{L_k}\mathbf{x}}(E)\bigr\|^{\lambda}$ is usually assessed with the help of the Simon–Lieb inequality (a.k.a. Geometric Resolvent Inequality); cf., e.g., [21, 14, 23, 29]. This technique is well-known by now, so we skip below some technical details in the proofs.

Induction hypothesis $\mathsf{S}(N,k)$

Given the integer sequence $\{L_{j,n}\}$ *(cf.* (24)*) and positive numbers* b_n, s_n, $1\leqslant n\leqslant 3$, *the following property is fulfilled for all* $1\leqslant n\leqslant N$ *and* $0\leqslant k'\leqslant k$*:*

$$\forall\, E\in I^*=[0,E^*]\quad \mathbb{P}\Bigl\{\mathbf{B}^{(n)}_{L_{k,n}}(\mathbf{x})\ \textit{is}\ \Bigl(E,L_{k,n}^{-b_n}\Bigr)\textit{-SNS}\Bigr\}\geqslant 1-L_{k,n}^{-s_n}. \tag{31}$$

Naturally, in the course of induction on $k\geqslant 0$ with N fixed, we are entitled to use $\mathsf{S}(n,k)$ with $n\leqslant N-1$ and any $k\geqslant 0$.

3.1. Initial length scale (ILS) estimate

As observed in [10], the N-particle version of the ILS estimate (cf., e.g., [32, Theorems 2.2.3, 3.3.3]) is proved essentially in the same way as for $N=1$, provided the interaction potential is non-negative. The reason is that the Lifshitz tails phenomenon, the main mechanism used in [32, Theorems 2.2.3, 3.3.3], is actually *enhanced* by a

nontrivial non-negative interaction. The next statement is an adaptation of a result proved in [32]; in the form given below it was used in [8]. Specifically, the estimate (32) appears in the proof of [32, Theorem 2.2.3], but a fractional-exponential (in L) bound was ultimately replaced there by a weaker, power-law upper bound in order to suit the polynomial scaling employed in [32].

Let $\mathbf{B}_L = \mathbf{B}(\mathbf{u}) = \bigtimes_{j=1}^{N} \mathrm{B}_L(u_j)$, $\Lambda = \Pi\mathbf{B}_L \cap \mathcal{Z} = \cup_{j=1}^{N} \mathrm{B}_L(u_j) \cap \mathcal{Z}$. We identify the finite sample $\omega_\Lambda = (\omega_\mathrm{a})_{\mathrm{a}\in\Lambda}$ with its zero-extension $\mathcal{Z} \ni \mathrm{a} \mapsto \omega_\mathrm{a}\mathbf{1}_\Lambda(\mathrm{a})$. This identification makes it possible to represent the entire sample $\omega = \big(\omega_\mathrm{a}\big)_{\mathrm{a}\in\mathcal{Z}}$ as a sum $\omega_\Lambda + \omega_\Lambda^\perp$, where $\omega_\Lambda^\perp = (\omega_\mathrm{a})_{\mathrm{a}\in\mathcal{Z}\setminus\Lambda}$ is also identified with $\mathcal{Z} \ni \mathrm{a} \mapsto \omega_\mathrm{a}\mathbf{1}_{\mathcal{Z}\setminus\Lambda}(\mathrm{a})$.

Next, let $H^{[i]}(\omega)$, $i = 1,\dots,N$, be the 1-particle operators, each acting on its respective variable x_i (but with the same ω for all $i \in [\![1,N]\!]$), and denote

$$\mathbf{H}^{[i]}(\omega) = \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{i-1 \text{ factors}} \otimes H^{[i]}_{\mathrm{B}_L(u_i)}(\omega) \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{N-i \text{ factors}},$$

so $\mathbf{H}^{\mathrm{ni}}(\omega) := \sum_{i=1}^{N} \mathbf{H}^{[i]}(\omega) = \mathbf{H}^{(N)}(\omega) - \mathbf{U}(\mathbf{x})$ (here the superscript "ni" stands for "non-interactive"). By positivity of $\mathbf{U}$ one has $\mathbf{H}_{\mathbf{B}_L}(\omega) \geqslant \mathbf{H}^{\mathrm{ni}}_{\mathbf{B}_L}(\omega_\Lambda)$ pointwise in ω. At the same time, $\mathbf{H}_{\mathbf{B}_L}(\omega_\Lambda)$ is measurable with respect to the σ-algebra $\mathfrak{F}_\Lambda$ generated by the finite sample $(\omega_\mathrm{a})_{\mathrm{a}\in\Lambda}$.

These notations will be used in the formulation and the proof of the next statement.

PROPOSITION 2. (Cf. [8, Sect. 3.1]) *Consider the N-particle Anderson Hamiltonian (1) with the single-particle random potential of the form (2), where $\{\omega_\mathrm{a}, \mathrm{a} \in \mathcal{Z}\}$ are IID random variables with the probability measure μ. Assume that $0 \in \operatorname{supp}\mu \subset [0,+\infty)$ and $\operatorname{supp}\mu \neq \{0\}$. Then there exists $E_* > 0$ with the following properties.*

For spatial dimension $d \geqslant 1$, any $\beta \in (0,1)$ and all sufficiently small $\varepsilon > 0$ there exist $L_*(\beta,\varepsilon) \in \mathbb{N}$ and $c, \nu > 0$ such that for all $L \geqslant L_*$, all $E \in [0,E_*]$, any $N \geqslant 1$ and any cube $\mathbf{B}_L(\mathbf{u}) = \mathbf{B}_L^{(N)}(\mathbf{u})$, denoting $\Lambda := \Pi\mathbf{B}_L(\mathbf{u}) \cap \mathcal{Z}$, one has

$$\mathbb{P}\left\{ \sup_{\omega_\Lambda^\perp} \|\mathbf{1}_{\partial\mathbf{B}_L(\mathbf{u})}\mathbf{G}_{\mathbf{B}_L(\mathbf{u})}(E,\omega_\Lambda+\omega_\Lambda^\perp)\mathbf{1}_{\mathbf{C}_\mathbf{u}}\| > \mathrm{e}^{-cL^\beta} \right\} \leqslant \mathrm{e}^{-\nu L^{\frac{1-\beta}{2}-\varepsilon}}, \tag{32}$$

where $\mathbf{G}_{\mathbf{B}_L(\mathbf{u})}(E,\omega) = (\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}(\omega) - E)^{-1}$.

Observe that the event in the LHS of (32) is $\mathfrak{F}_\Lambda$-measurable.

Proof. First, let us show that the required bound can be essentially reduced to its counterpart for 1-particle systems, and taking $N > 1$ actually enhances the Lifshitz tails phenomenon responsible for it. Ultimately it also enhances the decay bound on the Green functions resulting from the Combes–Thomas estimate [13, 3].

By separation of variables in $\mathbf{H}^{\mathrm{ni}}(\omega)$ (where the interaction is switched off) and on account of $\mathbf{H}_{\mathbf{B}_L}(\omega) \geqslant \mathbf{H}^{\mathrm{ni}}(\omega_\Lambda)$, one has, for any $\lambda \in \mathbb{R}$ and pointwise in ω,

$$E_0\left(\mathbf{H}^N(\omega)\right) \geqslant E_0\left(\mathbf{H}^{\mathrm{ni}}(\omega_\Lambda)\right) = \sum_{i=1}^{N} E_0\left(H^{[i]}_{\mathrm{B}_L(u_i)}(\omega_\Lambda)\right) \geqslant E_0\left(H^{[1]}_{\mathrm{B}_L(u_1)}(\omega_\Lambda)\right),$$

thus for any $\lambda \in \mathbb{R}$

$$\mathbb{P}\left\{\inf_{\omega_\Lambda^+} E_0\left(\mathbf{H}^{(N)}(\omega_\Lambda + \omega_\Lambda^+)\right) \leqslant \lambda\right\} \leqslant \mathbb{P}\left\{E_0\left(H^{[1]}_{\mathrm{B}_L(u_1)}(\omega_\Lambda)\right) \leqslant \lambda\right\}.$$

Now the problem of estimation of the lower edge of the N-particle spectrum is reduced to that of the 1-particle operator $H^{[1]}_{\mathrm{B}_L(u_1)}(\omega)$. Next, let $\ell = L^\theta$ with $\theta \in (0,1)$ to be fixed later, decompose $\mathrm{B} := \mathrm{B}_L(u_1)$ into a union of $M := L^d/\ell^d = \ell^{(\theta^{-1}-1)d}$ adjacent ℓ-cubes $\mathrm{B}_\ell(v_i)$, and consider a new 1-particle operator $\widetilde{H}_\mathrm{B}$ with Neumann boundary conditions on all the boundaries of these ℓ-cubes. By the Neumann decoupling,

$$\widetilde{H}(\omega) = \bigoplus_{i=1}^{M} H^{(1,\mathrm{N})}_{\mathrm{B}_\ell(v_i)}(\omega),$$

where the upright "N" in $H^{(1,\mathrm{N})}_{\mathrm{B}_\ell(v_i)}$ stands for "Neumann boundary conditions", and "1" refers to the single-particle nature of the operator. As is well-known (cf., e.g., [33, Section XIII.15]), Neumann boundaries lower the spectrum, so

$$E_0\big(H^{[1]}(\omega)\big) \geqslant E_0\big(\widetilde{H}(\omega)\big) = \min_{1\leqslant i\leqslant M} E_0\left(H^{(1,\mathrm{N})}_{\mathrm{B}_\ell(v_i)}(\omega)\right).$$

A probabilistic argument based on the large deviations estimates, used in the proof of [32, Theorem 2.2.3], shows that there exist $s, \nu > 0$ and $\ell_* \in \mathbb{N}$ such that for any $\ell \geqslant \ell_*$

$$\forall\, i \in [\![1, M]\!] \quad \mathbb{P}\left\{E_0\left(H^{(1,\mathrm{N})}_{\mathrm{B}_\ell(u_i)}(\omega)\right) \leqslant s\ell^{-2}\right\} \leqslant \mathrm{e}^{-\nu\ell^d} \tag{33}$$

whence, for $\ell = L^\theta$ large enough and with $M = \ell^{(\theta^{-1}-1)d}$, $s\ell^{-2} = sL^{-2\theta}$,

$$\mathbb{P}\left\{\min_{1\leqslant i\leqslant M} E_0\left(H^{(1,\mathrm{N})}_{\mathrm{B}_\ell(u_i)}(\omega)\right) \leqslant sL^{-2\theta}\right\} \leqslant M\mathrm{e}^{-\nu\ell^d} \leqslant \mathrm{e}^{-\nu L^{d\theta-\varepsilon}}, \tag{34}$$

where one can take $\varepsilon \downarrow 0$ as $L \uparrow +\infty$. The proof of (34) in [32] relies upon a so-called complete covering condition for the alloy type random potential. In our notations, it reads as follows:

$$\sum_{\mathrm{a}\in\mathcal{Z}} \varphi(|x-\mathrm{a}|) \geqslant C > 0. \tag{35}$$

This is indeed a restrictive condition in the case where φ (assumed to be non-negative) is compactly supported: the supports of $\varphi(\cdot - \mathrm{a})$ have to cover the entier space. For our strictly positive site potentials $x \mapsto \varphi(|x|) > 0$ the complete covering condition is trivially fulfilled. Indeed, the lattice $\mathcal{Z} \hookrightarrow \mathbb{R}^d$ is assumed to have d linearly independent periods, so there is one site "a" per a unit cell of $\mathcal{Z}$. Let R be the common diameter of these cells, then by definition of φ and (8), $\sum_{\mathrm{a}\in\mathcal{Z}} \varphi(|x-\mathrm{a}|) \geqslant R^{-A} > 0$.

Now we can draw functional analytic conclusions from the lower bound on the ground state energy E_0 which holds with sub-exponential probability in a sufficiently large cube $\mathbf{B}_L(\mathbf{u})$. This is usually done with the help of the well-known estimate due to

Combes and Thomas [13], later streamlined by Barbaroux *et al.* [3]. It holds for a large class of Schrödinger operators; one mainly needs a uniform ellipticity condition (see the details in [32, Section 2.4]). For the reader's convenience, we quote its encapsulation given in [32], better adapted to our notations. Below $\rho(H_\Lambda)$ stands for the resolvent set of the corresponding self-adjoint operator H_Λ. One can choose the Dirichlet, Neumann or periodic boundary conditions (see [32]).

LEMMA 1. (Cf. [32, Theorem 2.4.1]) *Let* $\lambda > 0$. *There exist* $c_1(\lambda), c_2(\lambda) > 0$ *with the following properties. Let* $\Lambda \subset \mathbb{R}^D$, $D \geqslant 1$, *be an open cube,* $X, Y \subset \Lambda$ *measurable subsets satisfying* $\operatorname{dist}(X,Y) > R > 0$ *and* $Y \supset \{x \in \Lambda : \operatorname{dist}(x, \partial\Lambda) \leqslant 1\}$. *Further, consider the Schrödinger operator* $H_\Lambda = -\Delta + W(x)$ *in* Λ *and a non-empty open interval* $(\lambda_-, \lambda_+) \subset \rho(H_\Lambda) \cap (-\lambda, \lambda)$, *and let* $\eta := \operatorname{dist}\big(E, \mathbb{R} \setminus (\lambda_-, \lambda_+)\big) > 0$. *Then*

$$\|\mathbf{1}_X G_\Lambda(E,\omega)\mathbf{1}_Y\| \leqslant c_1 \eta^{-1} \mathrm{e}^{-c_2 R\sqrt{(\lambda_+ - \lambda_-)\eta}} \equiv \mathrm{e}^{-c_2 R\sqrt{(\lambda_+ - \lambda_-)\eta} + \ln(c_1/\eta)}. \tag{36}$$

In our case, $D = Nd$ and $\mathbb{R}^D$ is replaced by $(\mathbb{R}^d)^N$. As evidences the statement, the structure of the potential is irrelevant, once we have the above spectral gap condition $E \in (\lambda_-, \lambda_+)$ with $\eta = \operatorname{dist}\big(E, \mathbb{R} \setminus (\lambda_-, \lambda_+)\big) > 0$. Now the condition

$$\|\mathbf{1}_{\partial \mathbf{B}_L(x)} \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(E,\omega)\mathbf{1}_{\mathbf{C}_\mathbf{x}}\| \leqslant \mathrm{e}^{-cL^\beta} \tag{37}$$

follows from Lemma 1. Indeed, we have $R = L(1 - \mathrm{o}(1))$, $\lambda_- = 0$, $\lambda_+ = qL^{-2\theta}$, $\eta = \frac{1}{2}\lambda_+$, so $R\sqrt{(\lambda_+ - \lambda_-)\eta} > \frac{1}{2} s L^\beta$, $\beta = 1 - 2\theta > 0$. Clearly, any value $\beta \in (0,1)$ can be achieved by taking $\theta = (1-\beta)/2$. Since we have $R\sqrt{(\lambda_+ - \lambda_-)\eta} \gg 1$ for $L \gg 1$, the logarithmic term in the last equation in (36) can be absorbed in the exponent by replacing c_2 with $c = c_2/2$.

Collecting (37) and with the probability estimate (34) for this property to hold, we recover the claim, replacing θ in (34) with the value $(1-\beta)/2$ suitable for any dimension d (here $d = 1$ is the worst case scenario). □

PROPOSITION 3. *Under the assumptions (U)–(V), there exists a nontrivial interval* $I^* = [0, E^*]$ *such that for any* $b_N, s_N > 0$, $N \geqslant 1$, *and some integers* $\widetilde{L}^{(N)}$ *the hypothesis* $\widetilde{S}(N,0)$ *holds for all* $N \geqslant 1$ *and* $L_{N,0} \geqslant \widetilde{L}^{(N)}$.

A notable improvement provided by Proposition 3, as compared to [10, Lemma 3.1], is that the estimates for the systems of $n = 1, \ldots, N$ particles are now established in the same energy interval, while [10], as well as [28, 29, 20, 12, 8], in the course of the induction in the number of particles ($n = 1, \ldots, N$) one had to operate in the energy intervals $I^{(n)}$ of size $|I^{(n)}| \leqslant \mathrm{e}^{-Cn}$, $C > 0$. This was necessary for the proof of *exponential* decay of the localized eigenfunctions; now we consciously settle for a power-law decay, which is of course a notable concession. On the bright side, this is precisely what enables us to render n-independent the low-energy band where the localization can be proved, albeit in a weaker form. From the energy band perspective, the key point is to establish the initial length scale estimates, for then the MSA induction kicks in and carries the localization estimates to arbitrarily large scales.

We emphasize that while the core of the multi-particle MSA used in the present work remains an induction in the number of particles ($n = 1, \dots, N$), the ILS estimate from Proposition 3 is proved for all N at once, or better to say, individually for any given $N \geqslant 1$ without referring to $n < N$.

Proof. Let $\mathbf{B} = \mathbf{B}_{L_{0,n}}(\mathbf{u})$, $\bar{\mathbf{B}} = \mathbf{B}_{qL_{0,n}^{1+2\tau}}(\mathbf{u})$, $\tau > 0$, $\bar{\Lambda} := \Pi\bar{\mathbf{B}} \cap \mathcal{Z}$. According to the definition (29) of the SNS property and on account of (25), we have to show that

$$\mathbb{P}\left\{ (3L_k)^{Nd} \sup_{\omega_{\bar{\Lambda}}^{\perp}} \|\mathbf{1}_{\partial\mathbf{B}_L(x)} \mathbf{G}_{\mathbf{B}_{L_{0,N}}(\mathbf{x})}(E, \omega_{\bar{\Lambda}} + \omega_{\bar{\Lambda}}^{\perp}) \mathbf{1}_{\mathbf{C}_\mathbf{x}}\| > L_{0,N}^{-b_N} \right\} \leqslant L_{0,N}^{-s_N}. \tag{38}$$

By Proposition 2, we have for $L_{0,N}$ (replacing L in Proposition 2) large enough

$$\mathbb{P}\left\{ \sup_{\omega_{\bar{\Lambda}}^{\perp}} \|\mathbf{1}_{\partial\mathbf{B}_L(x)} \mathbf{G}_{\mathbf{B}_{L_{0,N}}(\mathbf{x})}(E, \omega_{\bar{\Lambda}} + \omega_{\bar{\Lambda}}^{\perp}) \mathbf{1}_{\mathbf{C}_\mathbf{x}}\| > \mathrm{e}^{-cL^{\beta}} \right\} \leqslant \mathrm{e}^{-\nu L^{\frac{1-\beta}{2}-\varepsilon}}. \tag{39}$$

where $\varepsilon > 0$ can be made arbitrarily small for $L_{0,N}$ large enough. It is to be stressed that the estimate (39), which is much stronger than (38) for large $L_{0,N}$, holds in a fixed energy interval (being derived from the properties of just one, single-particle component of the N-particle Hamiltonian), and it actually gets stronger with the number of particles $N \uparrow +\infty$, although we do not exploit this enhancement mechanism. Therefore, without changing the energy interval we can infer from it weaker, power-law estimates with different (depending upon N in whatever way) and arbitrarily large exponents b_N and s_N figuring in the claim (38) to be proved. Specifically, for any positive numbers $c, \beta, \theta', \nu, b_N, s_N > 0$ and $L_{*,N}$ large enough, for all $L_{0,N} \geqslant L_{*,N}$

$$L_{0,n}^{-b_N} \geqslant \mathrm{e}^{-cL_{0,N}^{\beta}}, \quad L_{0,N}^{-s_N} \geqslant \mathrm{e}^{-\nu L_{0,N}^{\theta'}}, \quad \theta' = \theta - \varepsilon = \frac{1-\beta}{2} - \varepsilon > 0,$$

which shows that the initial length scale non-singularity condition for N-particle Hamiltonians holds with a satisfactory probability $L_{0,N}^{-s_N}$, provided $L_{0,N}$ is large enough. □

Perhaps it is worth noticing also that this argument, as is well-known, can be used solely in the initial scale, because in the subsequent scaling steps the relevant energies can be inside the spectrum and not strictly below it with a satisfactory spectral gap. The above mentioned enhancement with $N \uparrow +\infty$, alas, is a rather exceptional particularity of the initial length scale analysis, but in many other aspects growing N is a source of various hard technical problems.

Notice also that we use in the claim an augmented cube $\bar{\mathbf{B}} = \mathbf{B}_{qL_{0,N}^{1+2\tau}}(\mathbf{u})$, as per the scale induction requirements, but as was made clear in the proof, we actually make use only of the scatterers $\mathrm{a} \in \Pi\mathbf{B}_L(\mathbf{u})$, and all exterior scatterer amplitudes ω_a can only enhance the resolvent decay and probability estimates, even if $\mathrm{a} \in \bar{\mathbf{B}}_L(\mathbf{u}) \setminus \mathbf{B}_L(\mathbf{u})$.

3.2. Assumptions on the key parameters

In the course of the scale induction for N-particle systems we work with lengths $L_k = L_{N,k}$ defined by $L_{N,k+1} = \lfloor L_{N,k}^{\alpha_N} \rfloor$, where $\alpha_1 = \alpha^3$ and $\alpha_N = \alpha = \varrho^2$ for $2 \leqslant N \leqslant 3$, and $\varrho > 0$ will be made large enough to have $\mathfrak{b} > 0$ in Theorem 1 arbitrarily large. Further, $\tau_N > 0$ are given by $\alpha_N = (1+\sigma)(1+2\tau_N)$, $0 < \sigma < \frac{1}{4}$ (cf. (24)).

We assume $A > 8 \cdot 3 \cdot d = 24d$ (or larger, if necessary); here "3" is the maximal value of N in our main results. The exponents s_N, b_N in main analytic and probabilistic estimates for N-particle systems, are defined by

$$s_N = \tfrac{1}{2}\, \varrho^{8-3N}\, A\,, \quad b_N = 4\alpha_N s_N\,, \tag{40}$$

with $\varrho \geqslant 4$ (or as large as needed), so $s_{N-1} = \varrho\, \alpha s_N$, $b_{N-1} \geqslant 4 b_N$. The main requirement for the exponent $A^{\#}$ (cf. (7)) is as follows:

$$A^{\#} > \frac{2b_N}{1+2\tau_N}\,. \tag{41}$$

Since $\sigma < \frac{1}{4}$, it suffices to have $A^{\#} \geqslant 10 s_N$. The latter implies in particular

$$A^{\#}(1+2\tau_N) - d > 2\alpha_N s_N + d. \tag{42}$$

Other conditions on the key parameters used in the proofs are summarized and checked in Appendix A.

In the course of the scale induction we shall use the small parameters

$$\delta_k = L_k^{-b_N}\,, \qquad \varepsilon_{k+1} = L_{k+1}^{-2s_N} \equiv L_k^{-2s_N\alpha_N}. \tag{43}$$

3.3. Bad and singular cubes. Deterministic analysis

DEFINITION 3.2. Given $K \in \mathbb{N}$, a cube $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{x})$ is called (E,δ,K)-bad iff
(i) either it contains a weakly interactive (E,δ)-S cube of radius L_k,
(ii) or it contains K, (E,δ)-S, SI cubes $\mathbf{B}_{L_k}(\mathbf{u}_i)$ with $\mathrm{d_S}(\mathbf{u}_i,\mathbf{u}_j) > 8\mathrm{q}NL_k^{1+2\tau_N}$ for $i \neq j$.
Otherwise, it is called (E,δ,K)-good.

The next statement is a standard result of the Multi-Scale Analysis, essentially going back to the work [14], so its proof will be omitted; it is deterministic, does not rely on a specific structure of the potential and thus applies to single- and multi-particle Hamiltonians.

LEMMA 2. (Conditions for non-singularity) *Fix some $k \in \mathbb{N}$ and consider a cube $\mathbf{B} = \mathbf{B}_{L_{k+1}}(\mathbf{u})$. Suppose that*
(i) $\mathbf{B}$ *is (E,ε)-NR with $\varepsilon \geqslant \delta^{1-c}$, for some $\varepsilon,\delta,c \in (0,1)$;*
(ii) $\mathbf{B}$ *is (E,δ,K)-good, with $K \geqslant 0$ such that*

$$\mathrm{M} := \lfloor L_{k+1}/L_k \rfloor - 200\,\mathrm{q}NK\lfloor L_k^{1+2\tau_N} \rfloor \geqslant 1. \tag{44}$$

If L_0 is large enough, then $\mathbf{B}$ is $(\delta^{\mathrm{M}+c}, E)$-NS.

Due to unboundedness of $\operatorname{supp}\varphi$, we need a "stable" variant of Lemma 2:

LEMMA 3. (Conditions for strong non-singularity) *Consider a cube* $\mathbf{B}=\mathbf{B}_{L_{k+1}}(u)$, $k\geqslant 0$, *and suppose that*

(i) $\mathbf{B}$ *is* (E,ε_{k+1})-SNR*;*

(ii) $\mathbf{B}$ *is* (E,δ_k,τ_N,K)-S-good, *with* $K\geqslant 0$ *such that* (44) *holds with* $\mathrm{M}>2\alpha_N$.

If L_0 *is large enough, then* $\mathbf{B}$ *is* (E,δ_{k+1})-SNS.

Proof. Let $\bar{\mathbf{B}}=\mathbf{B}^{1+2\tau_N}_{L_{k+1}}(\mathbf{u})$, $\bar{\Lambda}:=\Pi\bar{\mathbf{B}}\cap\mathcal{Z}$. One has to show that, with a fixed sample $\omega^{\perp}_{\bar{\mathbf{B}}}$ satisfying the hypotheses **(i)–(ii)**, the cube $\mathbf{B}$ is (E,δ_{k+1})-NS for the sample $(\omega_{\bar{\Lambda}},\omega^{\perp}_{\bar{\Lambda}})$ regardless of the complementary sample $\omega^{\perp}_{\bar{\Lambda}}$.

First, notice that the condition **(i)** is already stable with respect to $\omega^{\perp}_{\bar{\Lambda}}$.

Next, by **(ii)** there exist at most $K-1$ cubes $\mathbf{B}_{17\mathrm{q}NL_k^{1+2\tau_N}}(\mathbf{x}_i)$ such that any cube $\mathbf{B}_{L_k}(\mathbf{x})$ with $\mathbf{x}\notin\mathscr{S}:=\cup_{i=1}^{K-1}\mathbf{B}_{17L_k^{1+2\tau_N}}(\mathbf{x}_i)$ is (E,δ_k)-SNS. The support of $\omega^{\perp}_{\bar{\Lambda}}$ lies outside all the cubes $\mathbf{B}_{17\mathrm{q}NL_k^{1+2\tau_N}}(\mathbf{x}_i)$, hence the distant sample $\omega^{\perp}_{\bar{\Lambda}}$ does not affect the strong non-singularity property of the cubes $\mathbf{B}_{L_k}(\mathbf{x})$ with $\mathbf{x}\notin\mathscr{S}$. Thus by Lemma 2, the cube $\mathbf{B}$ is $(E,\tilde{\delta}_{k+1})$-NS with

$$\tilde{\delta}_{k+1}\leqslant L_k^{-b_N(\mathrm{M}+c)}\leqslant(3L_{k+1})^{-Nd}\cdot L_{k+1}^{-b_N\left(\frac{\mathrm{M}}{\alpha_N}-\frac{Nd}{b_N}\right)}\leqslant(3L_{k+1})^{-Nd}\cdot L_{k+1}^{-b_N} \tag{45}$$

(recall $c>0$) provided L_0 is large enough and $\frac{\mathrm{M}}{\alpha_N}-\frac{Nd}{b_N}>1$; the latter holds true, since $\frac{Nd}{b_N}<1$ by (41) and $\mathrm{M}>2\alpha_N$ by hypothesis. Therefore, $\mathbf{B}$ is (E,δ_{k+1})-SNS. □

3.4. Non-singularity of weakly interactive cubes

The next lemma adapts earlier results [10, Lemma 4.1], [12, Lemma 4.5], [8, Lemma 6] and similar results from [28, 29]. Recall that if $N=1$, there are no WI cubes (decomposable systems), and then subsections 3.4-3.5 become unnecessary.

LEMMA 4. *Consider a WI cube* $\mathbf{B}=\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$ *with a canonical factorization* $\mathbf{B}=\mathbf{B}'\times\mathbf{B}''$ *(cf.* (18)*), and the respective sub-system Hamiltonians* $\mathbf{H}'=\mathbf{H}^{(n')}_{\mathbf{B}'}$ *and* $\mathbf{H}''=\mathbf{H}^{(n'')}_{\mathbf{B}''}$. *Assume that* $\mathbf{B}$ *is* $(E,2L_k^{-2\alpha_{N^{S_N}}})$*-SNR. Suppose further that*

- $\forall\lambda'\in\Sigma(\mathbf{H}')$ *the cube* $\mathbf{B}''$ *is* $(E-\lambda',\delta_k)$*-SNS, and*
- $\forall\lambda''\in\Sigma(\mathbf{H}'')$ *the cube* $\mathbf{B}'$ *is* $(E-\lambda'',\delta_k)$*-SNS.*

Then $\mathbf{B}$ *is* (E,δ_k)*-SNS.*

Proof. In order to fully exploit the WI property of the cube $\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$, we perform a cut-off of the interaction potentials at distances $r>L_k^{1+2\tau_N}$, thus introducing the operator $\mathbf{H}^{\mathrm{ni}}$ and its resolvent $\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)$ (here "ni" stands for "non-interactive"), assess

the localization properties of the latter, and finally make use of the second resolvent equation which implies the bound

$$\|\mathbf{G}_{\mathbf{B}}^{(N)}(E) - \mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\| \leqslant \|\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\|\,\|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\|\,\|\mathbf{G}_{\mathbf{B}}^{(N)}(E)\|\,. \tag{46}$$

By definition of a WI cube, $\operatorname{dist}[\Pi\mathbf{B}', \Pi\mathbf{B}''] \geqslant \mathrm{q}L_k^{1+2\tau_N}$, so by the hypothesis (U), with $\mathrm{q} \gg 1$, $\|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \leqslant L_k^{-A^{\#}(1+2\tau_N)+d}$ The hypothesis $\operatorname{dist}\bigl[\Sigma(\mathbf{H}^{(N)}, E\bigr] \geqslant 2L_k^{-2\alpha_N s_N}$, implies, by the min-max principle, that for L_0 large enough

$$\operatorname{dist}\Bigl[\Sigma(\mathbf{H}^{\mathrm{ni}}, E\Bigr] \geqslant 2L_k^{-2\alpha_N s_N} - \|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \geqslant 2L_k^{-2\alpha_N s_N} - L_k^{-A^{\#}(1+2\tau_N)+d} \geqslant L_k^{-2\alpha_N s_N}\,, \tag{47}$$

because $2\alpha_N s_N < A^{\#}(1+2\tau_N) - d$ (cf. (42)). Therefore, we have in (46)

$$\|\mathbf{G}_{\mathbf{B}}^{(N)}(E)\| \leqslant \frac{1}{2}L_k^{2\alpha_N s_N}, \quad \|\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\| \leqslant L_k^{2\alpha_N s_N}. \tag{48}$$

It remains to assess the decay of the resolvent $\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)$. The operators $\mathbf{H}'$ and $\mathbf{H}''$ have orthonormal bases $\{\mathbf{\Psi}'_a\}$, $\{\mathbf{\Psi}''_b\}$. Let $\mathbf{P}'_a = |\mathbf{\Psi}'_a\rangle\langle\mathbf{\Psi}'_a|$, $\mathbf{P}''_b = |\mathbf{\Psi}''_b\rangle\langle\mathbf{\Psi}''_b|$ be the respective projections (we use Dirac's bra-ket notation). It follows from the identity $\mathbf{H}^{\mathrm{ni}} = \mathbf{H}'\otimes\mathbf{1}'' + \mathbf{1}'\otimes\mathbf{H}''$ that $\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E) = \sum_a \mathbf{P}'_a \otimes \mathbf{G}_{\mathbf{B}''}(E - E'_a)$, where $\|\mathbf{P}'_a\| = 1$, and so with $\boldsymbol{\chi}_{\mathbf{u}} = \boldsymbol{\chi}_{\mathbf{u}'}\otimes\boldsymbol{\chi}_{\mathbf{u}''}$, $\boldsymbol{\chi}_{\mathbf{v}} = \boldsymbol{\chi}_{\mathbf{v}'}\otimes\boldsymbol{\chi}_{\mathbf{v}''}$ we get

$$\bigl\|\boldsymbol{\chi}_{\mathbf{u}}\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\boldsymbol{\chi}_{\mathbf{v}}\bigr\| \leqslant \left(\sum_{1\leqslant a\leqslant \mathcal{K}_1} + \sum_{a>\mathcal{K}_1}\right)\bigl\|\boldsymbol{\chi}_{\mathbf{u}''}\mathbf{G}_{\mathbf{B}''}(E - E'_a)\,\boldsymbol{\chi}_{\mathbf{v}''}\bigr\|\,,$$

where $\mathcal{K}_1 := \max\,[a \geqslant 1 : E^* - E'_a \geqslant -1] \lesssim \bigl|\mathbf{B}_{L_{k+1}}^{(n')}\bigr| \lesssim L_k^{n'd}$ by Weyl's estimate for the Laplacian (cf. [5, Section VI.2.1]) and positivity of interactions. Recall $E \in [0, E^*]$, so by the Combes–Thomas estimate [13] the sum over $a > \mathcal{K}_1$ is bounded by $C'\mathrm{e}^{-c'L_k}$, $c' > 0$. Next, estimate the finite sum over $1 \leqslant a \leqslant \mathcal{K}_1 \lesssim L_k^{Nd}$, making use of the induction hypothesis for the n-particle systems with $n < N$ (if any):

$$\sum_{a=1}^{\mathcal{K}_1}\bigl\|\boldsymbol{\chi}_{\mathbf{u}''}\mathbf{G}_{\mathbf{B}''}(E - E'_a)\boldsymbol{\chi}_{\mathbf{v}''}\bigr\| \leqslant \mathcal{K}_1\,L_k^{-b_{n'}} \lesssim L_k^{Nd}\,L_k^{-4b_N}\,, \tag{49}$$

since $b_{n'} \geqslant b_{N-1} \geqslant 4b_N$ by (40). From $b_N = 4\alpha_N s_N > Nd$ (cf. (81)) we thus get

$$\|\boldsymbol{\chi}_{\mathbf{u}}\mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\boldsymbol{\chi}_{\mathbf{v}}\| \lesssim L_k^{Nd}\left(L_k^{-4b_N} + \mathrm{e}^{-c'L_k}\right) \leqslant \frac{1}{2}L_k^{-b_N}\,. \tag{50}$$

From (46), (48), $A^{\#}(1+2\tau_N) > 2b_N$ (cf. (41)), and $b_N = 4\alpha_N s_N$, we infer

$$\|\mathbf{G}_{\mathbf{B}}^{(N)}(E) - \mathbf{G}_{\mathbf{B}}^{\mathrm{ni}}(E)\| \lesssim L_k^{4\alpha_N s_N}L_k^{-A^{\#}(1+2\tau_N)} < \frac{1}{2}L_k^{-2b_N+4\alpha_N s_N} \leqslant \frac{1}{2}L_k^{-b_N}\,. \tag{51}$$

Collecting (50) and (51), the claim follows. □

3.5. Probabilistic analysis of weakly interactive cubes

LEMMA 5. (Cf. [8, Lemma 6]) *Assume the property* $\mathsf{S}(N-1,k)$. *If* L_0 *is large enough, then for any weakly interactive cube* $\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$ *one has*

$$\mathbb{P}\left\{\mathbf{B}^{(N)}_{L_k}(\mathbf{u}) \textit{ is } (E,\delta_k)\textit{-SNS}\right\} \geqslant 1 - L_k^{-\frac{3}{2}s_N}. \tag{52}$$

Consequently, for any cube $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{x})$

$$\mathbb{P}\left\{\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{x}) \textit{ contains a WI } L_k\textit{-cube which is not } (E,\delta_k)\textit{-SNS}\right\} \leqslant \frac{1}{4}L_{k+1}^{-s_N}. \tag{53}$$

Proof. By Lemma 4, if $\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$ is not (E,δ_k,m_N)-SNS, then
(i) either for some $\lambda' \in \Sigma(\mathbf{H}')$ the cube $\mathbf{B}''$ is not $(E-\lambda',\delta_k)$-SNS,
(ii) or for some $\lambda'' \in \Sigma(\mathbf{H}'')$ the cube $\mathbf{B}'$ is not $(E-\lambda'',\delta_k)$-SNS,
(iii) or $\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$ is not $(E,2\varepsilon_{k+1})$-SNR, with $\varepsilon_{k+1} = L_{k+1}^{-2s_N}$.

By $\mathsf{S}(N-1,k)$, the events **(i)–(ii)** have probability bounded by

$$|\mathbf{B}^{(N-1)}_{L_k}(\cdot)|\, L_k^{-s_{N-1}} \lesssim L_k^{-s_{N-1}+(N-1)d} \leqslant \frac{1}{4}L_{k+1}^{-\frac{3}{4}\frac{s_{N-1}}{\alpha_N}} \leqslant \frac{1}{4}L_{k+1}^{-3s_N} \tag{54}$$

(recall $s_{N-1} > 4Nd$, $s_{N-1} = \varrho\,\alpha_N s_N$, $\varrho \geqslant 4$, cf. (81), (82)). By (80) we have $L_{k+1}^{-2s_N} \geqslant L_{k+1}^{-(A-d)(1+2\tau_N)}$, thus Theorem 2 applies and gives, with $s_N > \frac{2A+2Nd}{\alpha_N}$ (cf. (80)),

$$\mathbb{P}\left\{\operatorname{dist}\left[\Sigma(\mathbf{H}^{(N)}_{\mathbf{B}_{L_k}(\mathbf{u})}), E\right] < 2L_{k+1}^{-2s_N}\right\} \lesssim L_{k+1}^{\frac{A+Nd}{\alpha_N}} L_{k+1}^{-2s_N} < \frac{1}{4}L_{k+1}^{-\frac{3}{2}s_N}. \tag{55}$$

Therefore, $\mathbf{B}_{L_k}(\mathbf{u})$ is $(E,L_k^{-b_N})$-SNS with probability $1 - L_{k+1}^{-\frac{3}{2}s_N}$; this proves (52).

Since $s_N > 4Nd$ (cf. (81)), the second assertion (53) follows by counting the number of L_k-cubes (or their centers) inside $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{x})$. □

3.6. Conclusion of the fixed-energy MSA

LEMMA 6. *Assume the relations* (24). *If* L_0 *is large enough, then*

$$\forall\, k \geqslant 0 \quad \mathbb{P}\left\{\mathbf{B}_{L_{k+1}}(\mathbf{u}) \text{ is not } (E,L_{k+1}^{-2s_N})\text{-SCNR}\right\} \leqslant \frac{1}{4}L_{k+1}^{-s_N}. \tag{56}$$

Proof. By definition of an SCNR cube (28), if $\mathrm{B}_{L_k}(\mathbf{u})$ is not $(E,2L_{k+1}^{-2s_N})$-SCNR, then for some $\ell \in [\![L_k, L_{k+1}-L_k-2]\!]$ the cube $\mathbf{B}_\ell(\mathbf{u})$ is not $(E,2L_{k+1}^{-2s_N})$-SNR. By Wegner estimate (11), the probability of this event is bounded by $C L_{k+1}^{-2s_N+Nd+1}$. Since $s_N > Nd+1$ (cf. (81)), this implies (56) for L_0 large enough. □

THEOREM 3. *Assume* (24) *and let* $p_{N,k} := \mathbb{P}\left\{\mathbf{B}_{L_k}(u) \text{ is not } (E, L_k^{-b_N})\text{-SNS}\right\}$. *If* $p_{N,0} \leqslant L_{N,0}^{-s_N}$ *and* $L_{0,N}$ *is large enough, then for all* $k \geqslant 1$ *one has* $p_{N,k} \leqslant L_k^{-s_N}$.

Proof. It suffices to infer from $p_{N,k} \leqslant L_k^{-s_N}$ its counterpart for $k+1$. By Lemma 3, if $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ is not (E, δ_k)-SNS, then it is either (E, δ_k, K)-bad or not $(E, L_{k+1}^{-s_N})$-SCNR. Recalling Definition 3.2 of bad cubes, we see that there are three options:

(i) $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ is not $(E, L_{k+1}^{-s_N})$-SCNR, with probability $\leqslant \frac{1}{4} L_{k+1}^{-s_N}$, by Lemma 6;

(ii) $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ contains a WI L_k-cube which is not $(E, L_{k+1}^{-b_N})$-SNS; the probability of this event is bounded by $\frac{1}{4} L_{k+1}^{-s_N}$ (cf. (53));

(iii) $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ contains K SI cubes $\mathbf{B}_{L_k}(\mathbf{u}_i)$, with $|\mathbf{u}_i - \mathbf{u}_j| > 8N\mathrm{q}L_k^{1+2\tau_N}$ for $i \neq j$, which are not (E, δ_k)-SNS.

Recalling that $s_N > \alpha_N d$ (cf. (81)) and $K > \frac{2\alpha_N s_N}{s_N - \alpha_N d}$, we conclude that the probability of the latter event does not exceed

$$\left(L_k^{-s_N} L_{k+1}^d\right)^K \leqslant L_{k+1}^{-\frac{K}{\alpha_N}(s_N - \alpha_N d)} \leqslant L_{k+1}^{-2s_N} \leqslant \frac{1}{4} L_{k+1}^{-s_N}, \tag{57}$$

since the events $\left\{\mathbf{B}_{L_k}(\mathbf{u}_i) \text{ is not } (E, \delta_k)\text{-SNS}\right\}$, $1 \leqslant i \leqslant K$, in the cubes $\mathbf{B}_{L_k}(\mathbf{u}_i)$ with pairwise $8N\mathrm{q}L_k^{(1+2\tau_N)}$-distant centers are independent. Putting together the probability bounds for the options (i)–(iii), the claim follows. □

4. Eigenvalue comparison estimate

The fixed-energy MSA estimates established in some energy interval $I \subset \mathbb{R}$ do not imply directly spectral localization (a.s. pure point spectrum) in I, although an old argument by Martinelli and Scoppola [31] rules out any a.c. spectrum in I. One needs additional arguments; in the present paper we employ a variant of the method proposed in the single-particle context by Elgart *et al.* [19]. In the form used below, it was formulated in [7] and applied to multi-particle Hamiltonians in [12, 8].

For the derivation of dynamical localization from the energy-interval MSA we will follow the same path as in [7, 12, 8] and use the general approach by Germinet and Klein [23] which becomes particularly simple when applied to finite volume operators ([23] can treat operators in an infinite space). The pivot of this derivation is the eigenvalue comparison estimate for spectra of pairs of random Hamiltonians (cf. Theorem 4). Its implementation is rather technical and requires some preparation statements of geometrical nature, Lemmas 7–8, building on earlier used techniques from [6].

♦ **Cluster partitions.** Introduce the following distances to be used below, in a context where there will always be a reference length scale $L > 1$ (usually $L = L_k, k \geqslant 0$):

$$r_l = \mathrm{q}^{2Nl} L, \quad l = 1, \ldots, N+1; \tag{58}$$

$\mathrm{q} > 1$ will be chosen large enough, on the as-needed basis.

We will also need a finer scale of intermediate distances,

$$\rho_{l,s} = \mathrm{q}^s \rho_l = \mathrm{q}^{2Nl+s} L, \quad s = 0, \dots, N, \tag{59}$$

so that $r_l \equiv \rho_{l,0} \ll \rho_{l,1} \ll \cdots \ll \rho_{l,N} \ll r_{l+1}$, where $f(\mathrm{q}) \ll g(\mathrm{q})$ means, here and below, that $f(\mathrm{q}) = \mathrm{o}(g(\mathrm{q}))$ as $\mathrm{q} \to +\infty$.

Further, we need a refinement of a geometrical statement used earlier in [6, 12, 8] and based on the notion of the so-called "weakly separated" cubes.

LEMMA 7. (Sparse clustering) *For any cube* $\mathbf{B}_L^{(N)}(\mathbf{x})$ *there exists an integer* $M \in [\![1, N-1]\!]$ *and a partition* $[\![1,N]\!] = \sqcup_{j=1}^{S} I_j$, $S \geqslant 1$, *such that the respective subsets of* $\{x_1, \dots, x_N\}$ *(called below clusters)* $\Gamma_j = \cup_{i \in I_j} \{x_i\}$, $1 \leqslant j \leqslant S$, *obey for all* $j' \neq j''$

$$\begin{aligned} \operatorname{dist}\left(\Gamma_{j'}, \Gamma_{j''}\right) &> r_{M+1}, \\ \operatorname{diam} \Gamma_{j'} &\leqslant (N-1) r_M. \end{aligned} \tag{60}$$

Proof. Fix a finite sequence $r_1 < r_2 < \cdots < r_M$ satisfying (58). Decompose the points $\{x_1, \dots, x_N\}$ into clusters with the help of the following recursive algorithm:

► Start with the singletons $\{x_j\}$, merge identical singletons, and call them clusters of rank 0. A rank-0 cluster separated from its complement by a distance larger than $2r_1$ is called *complete* (*but it may eventually undergo a fusion at a later stage*).

For each $n = 0, 1, \dots$ until the algorithm terminates, repeat the following steps:

• If there is no incomplete rank-0 cluster, the algorithm terminates.

• Otherwise, taking incomplete clusters of rank n one by one, until their list is exhausted, merge the incomplete rank-n clusters into maximal clusters of rank $n+1$ defined as maximal $2r_{n+1}$-connected subsets. Specifically,

• the union of cubes $\mathrm{B}_{r_{n+1}}(x_j)$ for x_j within a given cluster of rank $(n+1)$ must be connected in $\mathbb{R}^d$;

• the maximality of a rank-$(n+1)$ cluster means that it is separated from its complement by a distance larger than r_{n+1} (*unless the complement is empty*).

• A rank-$(n+1)$ cluster separated from its complement by a distance larger than $2r_{n+2}$ is called *complete* (*but it may undergo a fusion at a later stage*).

► The algorithms stops after some number $M \leqslant N-1$ of iterations, when one obtain the final complete clusters $\Gamma_j = \cup_{i \in I_j} \{x_i\}$, $1 \leqslant j \leqslant S$, satisfying (60). □

REMARK 2. We need to decompose the points x_j into clusters in such a way that each cluster be surrounded by an annular area of width much larger than the cluster's size. For this reason, a mere R-clusterization with some large but fixed R, as in [6, 8], may not suffice: the inter-cluster distance may be comparable to the cluster diameter.

EXAMPLE. Let $d = 1$, $N = 4$, $\mathbf{x} = (0, R, 4R, 5R)$. R-clusterization gives two clusters, $(0, R)$ and $(4R, 5R)$, each of diameter R, but the distance between the two clusters ($3R > 2R$) is comparable to R.

In general, one may have a single final cluster, e.g., for $\mathbf{x} = (x, x, \dots, x)$. However, if $\operatorname{diam}(\Pi\mathbf{x}) > r_{N+1}$, there must be more than one final cluster, for otherwise one would come to a contradiction:

$$r_{N+1} < \operatorname{diam}(\Pi\mathbf{x}) = \operatorname{diam}\{x_1, \dots, x_N\} \leqslant (N-1)r_M < r_{M+1} \leqslant r_N .$$

♦ The integer M figuring in Lemma 7 will be called the *rank of the partition*.

We will also need concentric cubes of various radii covering each cluster; to this end, we pick for each cluster Γ_i its "center" $\hat{x}(\Gamma_i)$, so that

$$\forall r \geqslant \operatorname{diam}(\Gamma_i) \qquad \hat{x}(\Gamma_i) \in \Gamma_i \subset \mathbf{B}_r\big(\hat{x}(\Gamma_i)\big). \tag{61}$$

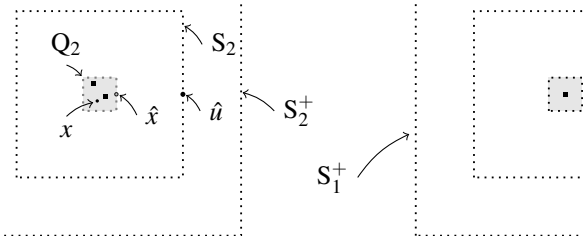

Figure 1: *Example of clustering (in the* **1-*particle*** *space!). Here* $N = 3$, $d = 2$. *Call a function* $f : A \to \mathbb{R}_+$ *"flat" if* $\max_{x\in A} f(x) = (1 + \mathrm{o}(1)) \max_{x\in A} f(x)$. *Here* $f : x \mapsto \operatorname{dist}^{-A}(\hat{u}, x)$ *is "flat" on the left gray square (denoted* Q_2*) covering the cluster* Γ_2. *For any* y *from the right gray square (covering a mono-cluster* Γ_1*), the function* $g : x \mapsto \operatorname{dist}^{-A}(y, x)$ *is also "flat" on* Q_2, *and in addition* $g = \mathrm{o}(f)$. *With a satisfactory accuracy, one can replace* f *and* g *by constants on* Q_2 *(*f *dominant,* g *negligible), which greatly simplifies the eigenvalue comparison analysis. (Notations* Q_i, S_i *and* S_i^+ *are introduced after Corollary 2).*

EXAMPLE. (*Note that it suffices for Theorem 1!*) Let $N = 3$, and suppose $\operatorname{diam}\Pi\mathbf{x} > r_{N+1}$. Start with the singletons $\{x_1\}$, $\{x_2\}$, $\{x_3\}$ and merge them into r_1-clusters.

► If for all $i \neq j$ one has $|r_i - r_j| > 2r_2$, then we obtain $S = 3$ mono-clusters $\Gamma_j = \{x_j\}$, $1 \leqslant j \leqslant 3$, pairwise separated by distances larger than $2r_2$. The construction procedure is terminated after $M = 1$ iteration, and we have indeed the clusters of diameter $0 < (N-1)r_1$ with inter-cluster distances $\operatorname{dist}(\Gamma_i, \Gamma_j) > 2r_{M+1}$.

► Since there must be at least two clusters, the only alternative to the three-cluster situation is a partition into one mono-cluster and one two-point cluster. Changing if necessary the numeration, we get the clusters $\Gamma_1 = \{x_1\}$ and $\Gamma_2 = \{x_2, x_3\}$ with

$$\operatorname{dist}(\Gamma_1, \Gamma_2) > 2r_2, \quad 0 = \operatorname{diam}\Gamma_1 < (N-1)r_1, \quad \operatorname{diam}\Gamma_2 \leqslant (N-1)r_1 .$$

♦ **Dominant clusters. Weak separation of cubes.**

DEFINITION 4.1. (Weak separation of cubes) Fix some $\mathrm{q} > 1$ and consider the sequences r_i and $\rho_{l,s}$ defined by (58)–(59), and the cluster partition of $\Pi\mathbf{x}$ relative to

the sequences $(r_\bullet)$ and $(\rho_{\bullet,\bullet})$, with the cluster index sets $\mathcal{I}_i$, $i = 1,\dots,K$. An N-particle cube $\mathbf{B}_L^{(N)}(\mathbf{x})$ is called weakly separated from $\mathbf{B}_L^{(N)}(\mathbf{y})$ iff the cluster partition of $\mathbf{x}$ relative to the sequences $(r_\bullet)$ and $(\rho_{\bullet,\bullet})$ fulfills the following property.

Define the integers

$$\mathbf{n}_{\Gamma_i}(\mathbf{x}) := |\mathcal{I}_i|, \quad \mathbf{n}_{\Gamma_i}(\mathbf{y}) := \mathrm{card}\Big\{ j : y_j \in \mathbf{B}_{\rho_{M,N}}\big(\hat{x}(\Gamma_i)\big) \Big\}, \tag{62}$$

where $M \geqslant 1$ is the maximal rank of clusters attained in the clustering procedure. Then there exists at least one cluster $\Gamma_{i_\circ}$ and a permutation $\pi \in \mathfrak{S}_N$ of the index set $[\![1,N]\!]$ such that $\mathbf{n}_{i_\circ}(\mathbf{y}) < \mathbf{n}_{i_\circ}(\mathbf{x})$.

A pair of cubes $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$ is called weakly separated iff at least one of these cubes is weakly separated from the other.

Pictorially, $\mathbf{n}_{i_\circ}(\mathbf{x})$ and $\mathbf{n}_{i_\circ}(\mathbf{y})$ are the "occupation numbers" of the zones of the $\mathbf{x}$-clusters by the particles from $\mathbf{x}$ and, respectively, from $\mathbf{y}$. Any $\mathbf{x}$-cluster $\Gamma_{i_\circ}$ figuring in Definition 4.1 will be called *dominant*; actually, it is *locally* dominant. The following simple argument, in essence going back to [6], provides a sufficient condition for the existence of a dominant cluster relative to a pair $(\mathbf{x},\mathbf{y})$.

LEMMA 8. *Let be given a pair of cubes* $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$. *Fix some* $\mathrm{q} > 1$ *and consider the cluster partition of* $\Pi\mathbf{x}$ *relative to the sequences* $(r_i),(\rho_{l,s})$ *defined by* (58)–(59). *Assume that* $\mathrm{d}_S(\mathbf{x},\mathbf{y}) > r_{N+1}$. *If* q *is large enough, then there exists at least one cluster* $\Gamma_{i_\circ}$ *such that* $0 \leqslant \mathbf{n}_{i_\circ}(\mathbf{y}) < \mathbf{n}_{i_\circ}(\mathbf{x})$.

Proof. Let $\{\Gamma_1,\dots,\Gamma_K\}$ be the cluster partition of $\Pi\mathbf{x}$ and $\{\mathcal{I}_1,\dots,\mathcal{I}_K\}$ the respective particle index partition of $[\![1,N]\!]$. Let M be the maximal rank of the clusters (the number of iterations of the clustering algorithm). Consider the following two cases. **(I)** $\mathrm{diam}\,\Pi\mathbf{x} > r_N \equiv \mathrm{q}^{2N^2}L$, thus there are $K \geqslant 2$ clusters. First, let us show that one cannot have $\mathbf{n}_{\Gamma_i}(\mathbf{x}) = \mathbf{n}_{\Gamma_i}(\mathbf{y})$ for all $i \in [\![1,K]\!]$. Indeed, assume otherwise; then there exists a permutation $\pi \in \mathfrak{S}_N$ such that (cf. (62))

$$\forall\, i \in [\![1,K]\!] \quad \forall\, j \in \mathcal{I}_i \qquad \begin{cases} x_j \in \Gamma_i\,, \\ y_{\pi(j)} \in \mathbf{B}_{\rho_{M,N}}\big(\hat{x}(\Gamma_i)\big) \end{cases},$$

hence $|x_j - y_{\pi(j)}| \leqslant \rho_{M,N}$, which leads to a contradiction:

$$r_{N+1} < \mathrm{d}_S(\mathbf{x},\pi(\mathbf{y})) = \max_{1\leqslant j\leqslant N} |x_j - y_{\pi(j)}| \leqslant \rho_{M,N} \leqslant \rho_{N-1,N} < r_N\,.$$

Therefore, not all integers $\mathbf{n}_{\Gamma_i}(\mathbf{x}) - \mathbf{n}_{\Gamma_i}(\mathbf{y})$ are zero, yet we have

$$\sum_i \big(\mathbf{n}_{\Gamma_i}(\mathbf{x}) - \mathbf{n}_{\Gamma_i}(\mathbf{y})\big) = \sum_i \mathbf{n}_{\Gamma_i}(\mathbf{x}) - \sum_i \mathbf{n}_{\Gamma_i}(\mathbf{y}) \geqslant N - N = 0\,, \tag{63}$$

hence at least one integer summand in the above LHS must be strictly positive, which proves the claim in the case **(I)** (see Fig. 2).

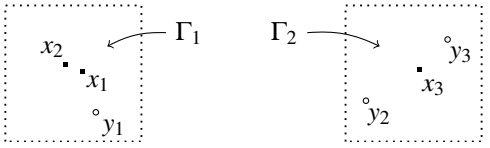

Figure 2: *Example for the proof of Lemma 8, case (**I**). Here $N=3$, $d=2$. The points y_i may dominate around the* $\mathbf{x}$*-cluster Γ_2, but the* $\mathbf{x}$*-particles dominate around Γ_1.*

(II) $\operatorname{diam}\Pi\mathbf{x} \leqslant r_N$. Since $\mathrm{d}_S(\mathbf{x},\mathbf{y}) > r_{N+1}$, there exists a permutation π such that $\max_{1\leqslant j\leqslant N} |x_j - y_{\pi(j)}| > r_{N+1}$, hence for some j' and $j'' := \pi(j')$ one has $|x_{j'} - y_{j''}| > r_{N+1}$. Since now $\operatorname{diam}\Pi\mathbf{x} \leqslant r_N$, it follows that for any $j \in [\![1,N]\!]$

$$|x_j - y_{j''}| \geqslant |x_{j'} - y_{j''}| - |x_j - x_{j'}| > r_{N+1} - \operatorname{diam}\Pi\mathbf{x} \geqslant r_{N+1} - r_N > r_N\,,$$

thus $\operatorname{dist}\big(\Pi\mathbf{x}, y_{j'}\big) \geqslant r_N > \rho_{M,N}$, so $y_{j'}$ lies outside all spheres $\partial \mathrm{B}_{\rho_{M,N}}(\hat{x}(\Gamma_i))$ around the $\mathbf{x}$-clusters $\Gamma_i(\mathbf{x})$ (recall $M \leqslant N-1$,). We thus have a stronger version of (63):

$$\sum_i \Big(\mathbf{n}_{\Gamma_i}(\mathbf{x}) - \mathbf{n}_{\Gamma_i}(\mathbf{y})\Big) = \sum_i \mathbf{n}_{\Gamma_i}(\mathbf{x}) - \sum_i \mathbf{n}_{\Gamma_i}(\mathbf{y}) \geqslant N - \operatorname{card}\big([\![1,N]\!] \setminus \{j'\}\big) = 1\,,$$

so again, there exists $i_\circ$ such that $\mathbf{n}_{i_\circ}(\mathbf{x}) - \mathbf{n}_{i_\circ}(\mathbf{y}) \geqslant 1$.

Alternatively, one can resort in the case **(II)** to a cluster partition of $\Pi\mathbf{y}$ and find an $\mathbf{y}$-cluster far from all $\mathbf{x}$-particles, thus locally dominant (the mono-cluster $\{y_3\}$ on Fig. 3). □

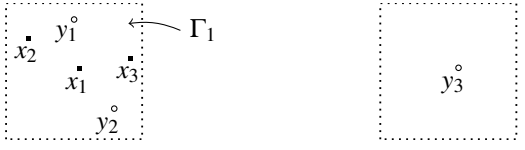

Figure 3: *Example for the proof of Lemma 8, case (**II**). Here $N=3$, $d=2$.*

REMARK 3. The case **(II)** corresponds in terminology of the works [11, 10] to a pair of "separated cubes": they are separated in the so-called Hausdorff pseudo-metric in the space of N-particle configurations, explicitly used in [2, 28, 20, 29]. Some pairs in the case **(I)** are *much* more difficult for the eigenvalue comparison analysis, both in the models with short- and long-range potentials, for there is no way to stochastically "decouple" the eigenvalues of $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$ and $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}(\omega)$: to "freeze" one of the two spectra by an appropriate conditioning and wriggle the other. This is a major difficulty encountered in the multi-particle FMM [2, 20], even for short-range interactions.

COROLLARY 2. *Under the assumptions of Lemma 8, there exists a cluster partition of some rank $M \leqslant N-1$ with at least one dominant cluster $\Gamma_{i_\circ}$, and integers $s \in [\![1,N]\!]$, $0 \leqslant l \leqslant \mathbf{n}_{i_\circ}(\mathbf{y})$, such that $\mathbf{n}_{i_\circ}(\mathbf{x}) > \mathbf{n}_{i_\circ}(\mathbf{y})$ and*

$$\begin{aligned} \forall\, j \in [\![1,l]\!] \qquad & y_{\pi(j)} \in \mathrm{B}_{\rho_{K,s}}\big(\hat{x}(\Gamma_i)\big)\big\}, \\ \forall\, j \in [\![l+1,N]\!] \qquad & y_{\pi(j)} \in \mathbb{R}^d \setminus \mathrm{B}_{\rho_{K,s+2}}\big(\hat{x}(\Gamma_i)\big)\big\}. \end{aligned} \tag{64}$$

In other words, one can find a zone in the 1-particle space such that (cf. Fig. 4)

• deep inside it, viz. inside some cube $\mathrm{B}_{\rho_{K,s}}\big(\hat{x}(\Gamma_i)\big) =: \mathrm{Q}_i$ with $K \leqslant M$ ("dominant core"), there are more particles from the configuration $\mathbf{x}$ than from $\mathbf{y}$;

• the "dominant core" is separated from the rest of the configurations $\mathbf{x}$ and $\mathbf{y}$ by a sphere $\mathrm{S}_i := \partial \mathrm{B}_{\rho_{K,s+1}}\big(\hat{x}(\Gamma_i)\big)$ ("separating shell") of much larger radius, so

$$\rho_{K,s} \ll \mathrm{dist}\,(\mathrm{Q}_i, \mathrm{S}_i)\,; \tag{65}$$

• the "separating shell" is itself isolated from the rest of the configuration $\mathbf{y}$ by a sphere $\mathrm{S}_i^+ := \partial \mathrm{B}_{\rho_{K,s+2}}\big(\hat{x}(\Gamma_i)\big)$ of an even larger radius ("isolating shell"):

$$\rho_{K,s+1} \ll \mathrm{dist}\,\Big(\mathrm{B}_{\rho_{K,s+1}}\big(\hat{x}(\Gamma_i)\big), \mathrm{S}^+_{K,s+2}\big(\hat{x}(\Gamma_i)\big)\Big)\,. \tag{66}$$

Proof. Consider any dominant cluster $\Gamma_{i_\circ}$ with $\mathbf{n}_{i_\circ}(\mathbf{x})$ particles from $\mathbf{x}$. By construction of the cluster partition, inside the separating sphere $\mathrm{S}_{\rho_{K,N}}$, there are $\mathbf{n}_{i_\circ}(\mathbf{y}) \leqslant \mathbf{n}_{i_\circ}(\mathbf{x}) - 1 \leqslant N - 1$ particles $\mathbf{y}$. These $\mathbf{n}_{i_\circ}(\mathbf{y}) \leqslant N-1$ particles cannot be present in all the N, pairwise non-overlapping annuli delimited by $(\mathrm{S}_{\rho_{K,s}}, \mathrm{S}_{\rho_{K,s+2}})$, $s = 1, \dots, N$ (see Fig. 4). □

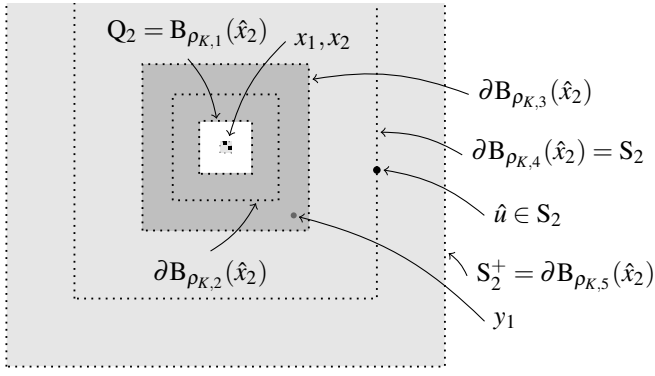

Figure 4: *Example for Corollary 2. $N = 3$, $d = 2$. We consider the zone around the dominant cluster Γ_2 with $\mathbf{n}_2(\mathbf{x}) = 2 > 1 = \mathbf{n}_2(\mathbf{y})$. The cube delimited by $\mathrm{S}_{\rho_{K,3}}$ contains the particles at x_1 and x_2 (tiny black squares) and exactly one y-particle (y_1). Since the (dark gray) annulus $\mathrm{B}_{\rho_{K,3}} \setminus \mathrm{B}_{\rho_{K,1}}$ contains y_1, we cannot take $\partial \mathrm{B}_{\rho_{K,2}}$ as the separating shell, and examine the (light gray) annulus $\mathrm{B}_{\rho_{K,5}} \setminus \mathrm{B}_{\rho_{K,3}}$. These two large gray annuli cannot both contain y-particles, since $\mathbf{n}_2(\mathbf{y}) = 1$, hence the second annulus has no y-particle. We pick any point $\hat{u} \in \mathrm{S}_2 = \partial \mathrm{B}_{\rho_{K,4}}$; it is far from all x-points and y-points. The particles outside $\mathrm{S}_{\rho_{K,5}}$ make a negligible contribution to the potential energy generated by $\mathfrak{u}(\hat{u} - \cdot)$, while on the entire cube delimited by $\mathrm{S}_{\rho_{K,3}}$ this potential is "almost flat", since $\rho_{K,3} \ll \rho_{K,4}$. The latter cube contains x_1, x_2 and y_1, so the energetic contribution from x-particles is approx. twice stronger than the one from y-particles.*

The main point of Corollary 2 is that one can pick a site potential $\omega_{\hat{u}}\, \mathfrak{u}(\hat{u} - \cdot)$ (which can actually be chosen in a number of ways) such that, with a high accuracy, its contributions to the potential energies in the cubes $\mathbf{B}_L(\mathbf{x})$ and $\mathbf{B}_L(\mathbf{y})$ are proportional,

to $\mathbf{n}_{i_\circ}(\mathbf{x})$ and, respectively, to $\mathbf{n}_{i_\circ}(\mathbf{y}) \in [0, \mathbf{n}_{i_\circ}(\mathbf{x}) - 1]$, thus these energies have different and explicitly controllable sensitivities to the random variable $\omega_{\hat{u}}$.

In contrast to Theorem 2, the eigenvalue *comparison* estimate in Theorem 4 below does not have to be stable, for it is not meant to be used in the framework of a scale induction. It is required only for the energy-interval analysis of pairs of N-particle cubes $\mathbf{B}_{L_k}(\mathbf{x})$, $\mathbf{B}_{L_k}(\mathbf{y})$ for which the inductive, fixed-energy decay bounds should already have been obtained. For this reason, there is no constraint (viz., a scale-dependent lower bound) imposed on the value of $\varepsilon \geqslant 0$ in (67).

THEOREM 4. (Eigenvalue comparison bound) *Under the assumptions (V) and (U), for any fixed N, d, E^* and the PDF F_V of the random amplitudes $\omega_\bullet$, the following bound holds for any pair of $\mathrm{C_{ws}}NL$-distant cubes $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$ with sufficiently large $\mathrm{C_{ws}} \geqslant 1$:*

$$\forall\, \varepsilon \in [0,1] \quad \mathbb{P}\left\{ \operatorname{dist}\left[\Sigma^{I^*}_{\mathbf{x},L}, \Sigma^{I^*}_{\mathbf{y},L} \right] \leqslant \varepsilon \right\} \lesssim L^{A+2Nd}\, \varepsilon. \tag{67}$$

Here $\mathrm{C_{ws}}$ is related to q used in the cluster partitions: $\mathrm{C_{ws}} \uparrow +\infty$ as $\mathrm{q} \uparrow +\infty$; the subscript "WS" stands for "weak separation", reminding of the important role of the notion used in the proof of the eigenvalue comparison bound.

Proof. The general strategy is the same as for Theorem 2, but now we have to rely on an elaborate clustering algorithm described above. Consider again the distances $r_l = \mathrm{q}^{2Nl} L$, $\rho_{l,s} = r_l \mathrm{q}^s = \mathrm{q}^{2Nl+s} L$, construct the cluster partition $\{\Gamma_i\}$ of $\Pi\mathbf{x}$, and the respective partition of the particle index set $\{\mathcal{I}_i\}$. Fix any dominant cluster $\Gamma = \Gamma_{i_\circ}$; let $\mathrm{Q} = \mathrm{B}_{\rho_{K,s}}(\hat{x})$ be its core and $\mathrm{S} = \partial \mathrm{B}_{\rho_{K,s+1}}(\hat{x})$ its separating shell, with $K \leqslant M \leqslant N-1$ (the cluster rank) and $1 \leqslant s \leqslant N$ (cf. (65)–(66)).

Pick an arbitrary shell point $\hat{u} \in \mathrm{S}$ (see Fig. 5 where $\Gamma_{i_\circ} = \Gamma_1$). We shall use the fact that the potential $x \mapsto \varphi(\hat{u}, x)$ is "almost flat" on the core Q (Q_1 on Fig. 5). Specifically, for all $j \in \mathcal{I}_{i_\circ}$ and all $x \in \mathrm{B}_L(u_j) \subset \mathrm{Q} = \mathrm{B}_{\rho_{K,s}}(\hat{x})$ we have

$$\frac{\big|\, |\hat{u}-x| - |\hat{u}-\hat{x}| \,\big|}{|\hat{u}-\hat{x}|} \leqslant \frac{|x-\hat{x}|}{|\hat{u}-\hat{x}|} \leqslant \frac{\operatorname{diam} \mathrm{Q}}{|\hat{u}-\hat{x}|} \lesssim \frac{\mathrm{q}^{2NK+s} L}{\mathrm{q}^{2NK+s+1} L} = \mathrm{O}\left(\mathrm{q}^{-1}\right), \tag{68}$$

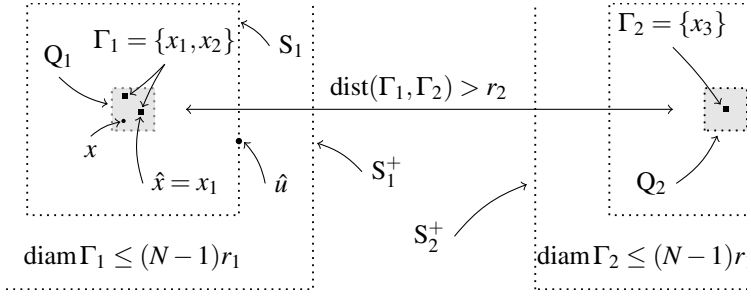

Figure 5: *Example for the proof of Theorem 4. Here $N = 3$, $d = 2$, $\Gamma_{i_\circ} \equiv \Gamma_1$.*

thus

$$|\hat{u}-x| = |\hat{u}-\hat{x}|\left(1+\frac{|\hat{u}-x|-|x-\hat{x}|}{|\hat{u}-\hat{x}|}\right) = |\hat{u}-\hat{x}|\left(1+\mathrm{O}\left(\mathrm{q}^{-1}\right)\right), \tag{69}$$

so with $\hat{a} := |\hat{u}-\hat{x}|^{-A}$, one has for all $x \in \mathrm{B}_L(u_j)$

$$\varphi(\hat{u},x)\chi_j(x) = \hat{a}\chi_j(x)\left(1+\theta_j(x)\right), \quad \|\theta_j\chi_j\|_\infty = \mathrm{O}\left(\mathrm{q}^{-1}\right). \tag{70}$$

For any non-dominant cluster Γ_i, with the core Q_i, we have

$$\min_{x\in\mathrm{Q}_i}|\hat{u}-x| \geqslant \rho_{K,s+2} = \mathrm{q}^{2NK+s+2},$$

hence a substantially smaller upper bound on the potential (for q large enough):

$$\varphi(\hat{u},x)\chi_j(x) = \mathrm{o}\left(\mathrm{q}^{-1}\right)\hat{a}\,\chi_j(x), \quad \chi_j = \mathbf{1}_{\mathrm{B}_L(u_j)}.$$

The potential energy $\mathbf{U}_{\hat{u}}(x)$ induced on $\mathbf{B}_L(\mathbf{x})$ by $\omega_{\hat{u}}\varphi(\hat{u}-\cdot)$ has therefore the form

$$\begin{aligned}\mathbf{B}_L(\mathbf{x}) \ni \mathbf{v} \mapsto \mathbf{U}_{\hat{u}}(\mathbf{v}) &= \sum_{j\in\mathcal{I}_i}\varphi(|\hat{u}-v_j|) + \sum_{j\notin\mathcal{I}_i}\mathrm{o_q}(\hat{a})\,\varphi(|\hat{u}-v_j|)\\ &= |\Gamma_i|\hat{a}\left(1+\mathrm{o_q}(1)\right)\chi_{\mathbf{B}_L(\mathbf{x})}(\mathbf{v}) + \sum_{j\in\mathcal{I}_i^{\mathrm{c}}}\mathrm{o_q}(\hat{a})\left(1+\mathrm{o_q}(1)\right)\chi_{\mathbf{B}_L(\mathbf{x})}(v)\\ &= \left(\mathbf{n}_{\Gamma_i}(\mathbf{x})+\mathrm{o_q}(1)\right)\hat{a}\,\chi_{\mathbf{B}_L(\mathbf{x})}(v), \quad \mathbf{n}_{i_\circ}(\mathbf{x}) \geqslant 1. \end{aligned} \tag{71}$$

Here and below, notation $\mathrm{o_q}(\cdot)$ refers to $\mathrm{q}\to+\infty$.

Further, introduce the decomposition $\omega = (\omega_{\hat{u}}, \omega_{\hat{u}}^{\perp})$ and fix the sub-sample $\omega_{\hat{u}}^{\perp} = \widetilde{\omega}_{\hat{u}}^{\perp}$; here $\widetilde{\omega}_{\hat{u}}^{\perp}$ is an arbitrary nonrandom collection that will be kept "frozen" until the end of the argument. Our analysis is similar to that in the proof of Theorem 2, but rather than to assess the eigenvalues of a Hamiltonian in a single cube $\mathbf{B}_L(\mathbf{x})$, we now aim to compare the random eigenvalues of the operators:

- $\mathbf{H}' = \mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ decomposed as $\mathbf{H}' = \mathbf{A}'(\widetilde{\omega}_{\hat{u}}^{\perp}) + \omega_{\hat{u}}\boldsymbol{\varphi}'_{\hat{u}}$, where $\mathbf{A}'(\omega_{\hat{u}}^{\perp})$ is now non-random, and $\boldsymbol{\varphi}'_{\hat{u}} = \boldsymbol{\varphi}\big|_{\mathcal{H}_{\mathbf{x},L}}$ is the restriction of the operator of multiplication by the function $\boldsymbol{\varphi}(\cdot) : \mathbf{x} \mapsto \sum_j \varphi(\hat{u}-x_i)$ to $\mathcal{H}_{\mathbf{x},L} = \mathrm{L}^2(\mathbf{B}_L(\mathbf{x}))$,
- similarly, $\mathbf{H}'' = \mathbf{H}_{\mathbf{B}_L(\mathbf{y})} = \mathbf{A}''(\widetilde{\omega}_{\hat{u}}^{\perp}) + \omega_{\hat{u}}\boldsymbol{\varphi}''_{\hat{u}}$, with non-random $\mathbf{A}''(\widetilde{\omega}_{\hat{u}}^{\perp})$ and $\boldsymbol{\varphi}''_{\hat{u}} = \boldsymbol{\varphi}\big|_{\mathcal{H}_{\mathbf{y},L}}$, $\mathcal{H}_{\mathbf{y},L} = \mathrm{L}^2(\mathbf{B}_L(\mathbf{y}))$. From the previous analysis, we have the representations

$$\begin{aligned}\boldsymbol{\varphi}' &= \hat{a}\,\mathbf{n}_{\Gamma_i}(\mathbf{x})\mathbf{1}_{\mathcal{H}_{\mathbf{x},L}} + \boldsymbol{\Phi}'_{\hat{u}}, \qquad \|\boldsymbol{\Phi}'_{\hat{u}}\| = \mathrm{o_q}(\hat{a}),\\ \boldsymbol{\varphi}'' &= \hat{a}\,\mathbf{n}_{\Gamma_i}(\mathbf{y})\mathbf{1}_{\mathcal{H}_{\mathbf{y},L}} + \boldsymbol{\Phi}''_{\hat{u}}, \qquad \|\boldsymbol{\Phi}''_{\hat{u}}\| = \mathrm{o_q}(\hat{a}),\ \mathbf{n}_{\Gamma_i}(\mathbf{y}) \leqslant \mathbf{n}_{\Gamma_i}(\mathbf{x}) - 1. \end{aligned} \tag{72}$$

Let $\lambda'(\omega_{\hat{u}})$ be an eigenvalue of $\mathbf{H}'(\omega_{\hat{u}})$ and, $\lambda''(\omega_{\hat{u}})$ an eigenvalue of $\mathbf{H}''(\omega_{\hat{u}})$, then

$$\begin{aligned}\lambda'(\omega_{\hat{u}}) - \lambda''(\omega_{\hat{u}}) &= \left[\widehat{\lambda}' + \hat{a}\,\mathbf{n}_{\Gamma_i}(\mathbf{x})\omega_{\hat{u}}\left(1+\mathrm{o_q}(1)\right)\right] - \left[\widehat{\lambda}'' + \hat{a}\,\mathbf{n}_{\Gamma_i}(\mathbf{y})\omega_{\hat{u}}\left(1+\mathrm{o_q}(1)\right)\right]\\ &\leqslant \left(\widehat{\lambda}' - \widehat{\lambda}''\right) + \mathbf{n}_{\mathbf{x},\mathbf{y}}\,\hat{a}\,\omega_{\hat{u}}\left(1+\mathrm{o_q}(1)\right), \end{aligned}$$

where $\mathbf{n}_{\mathbf{x},\mathbf{y}} := \mathbf{n}_{\Gamma_i}(\mathbf{x}) - \mathbf{n}_{\Gamma_i}(\mathbf{y}) \geqslant 1$, and $\widehat{\lambda}', \widehat{\lambda}''$ are $\mathfrak{F}_{\hat{u}}^{\perp}$-measurable, whence

$$\mathbb{P}\left\{|\lambda'_{\alpha'} - \lambda''_{\alpha''}| \leqslant \varepsilon \,\middle|\, \mathfrak{F}_{\hat{u}}^{\perp}\right\} \leqslant \mathbb{P}\left\{\left|\mathbf{n}_{\mathbf{x},\mathbf{y}}\hat{a}\omega_{\hat{u}}\big(1+\mathrm{o}_{\mathrm{q}}(1)\big) - (\widehat{\lambda}' - \widehat{\lambda}'')\right| \leqslant \varepsilon \,\middle|\, \mathfrak{F}_{\hat{u}}^{\perp}\right\}$$
$$\leqslant \sup_{\lambda\in\mathbb{R}} \mathbb{P}\left\{|\omega_{\hat{u}} - \lambda| \leqslant c'\mathbf{n}_{\mathbf{x},\mathbf{y}}^{-1}\hat{a}^{-1}\frac{\varepsilon}{1+\mathrm{o}_{\mathrm{q}}(1)} \,\middle|\, \mathfrak{F}_{\hat{u}}^{\perp}\right\} \lesssim \hat{a}^{-1}\varepsilon.$$

Taking the conditional expectation with respect to $\mathfrak{F}_{\hat{u}}^{\perp}$, we conclude

$$\mathbb{P}\left\{|\lambda'_{\alpha'} - \lambda''_{\alpha''}| \leqslant \varepsilon\right\} \lesssim \hat{a}^{-1}\varepsilon \lesssim L^{A}\varepsilon\,. \tag{73}$$

By Weyl's estimate for the eigenvalue distribution for the Laplacian and positivity of the potential energy, the number of eigenvalues in a bounded interval I is deterministically bounded by CL^{Nd}, so the claim (67) follows by a straightforward calculation. □

5. Spectral and dynamical localization

5.1. Energy interval estimates

Introduce the following notation: $\mathbf{F}_{\mathbf{x}}(E) = \mathbf{F}_{\mathbf{x},L}(E) := \max_{\mathbf{z}\in\partial^{-}\mathbf{B}_L(\mathbf{x})} \left\|\chi_{\mathbf{z}}\mathbf{G}_{\mathbf{B}_L(\mathbf{x})}\chi_{\mathbf{x}}\right\|$.

THEOREM 5. (Cf. [8, Theorem 6]) *Fix $L \geqslant 1$, a pair of N-particle cubes $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$, and a bounded interval $I \subset \mathbb{R}$. Assume for some $\mathfrak{a}_L, \mathfrak{r}_L > 0$ and for all $E \in I$ one has $\max_{\mathbf{z}\in\{\mathbf{x},\mathbf{y}\}} \mathbb{P}\{\mathbf{F}_{\mathbf{z}}(E) \geqslant \mathfrak{a}_L\} \leqslant \mathfrak{r}_L$. Assume also that the EVC bound of the form (67) holds for the pair $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$. Then for any $\mathfrak{c} > 0$ one has*

$$\mathbb{P}\{\exists E \in I : \min\big(\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\big) \geqslant \mathfrak{a}_L\} \leqslant 2|I|\mathfrak{c}^{-1}\mathfrak{r}_L + CL^{4Nd}b. \tag{74}$$

Hence under the assumptions (U)-(V), the bound (74) holds in the interval $I = I^*$ for any pair of $C_{\mathrm{q}}NL_k$-distant cubes of radius L_k with C_{q} large enough, by Theorem 4.

COROLLARY 3. *Under the assumptions of Theorem 1, there exists an interval $I^* = [0, E^*]$ with the following properties. For any $\mathfrak{b}, \mathfrak{s} > 0$ there exist $\hat{A}$, $\hat{A}^{\#}$, $\hat{Q} \in \mathbb{R}_+$ such that, if $A > \hat{A}$, $A^{\#} > \hat{A}^{\#}$ and $L_0 \geqslant \hat{Q}$, then for $\mathbf{x}$, $\mathbf{y}$ obeying $|\mathbf{x}-\mathbf{y}| \geqslant CL$ with C large enough*

$$\forall\ N \in [\![1,3]\!] \quad \mathbb{P}\left\{\sup_{E\in I^*} \min\left[\mathbf{F}_{\mathbf{x}}^{(N)}(E), \mathbf{F}_{\mathbf{y}}^{(N)}(E)\right] \geqslant L_k^{-\mathfrak{b}}\right\} \leqslant L_k^{-\mathfrak{s}}. \tag{75}$$

Proof. The claim follows from (74) by taking $\mathfrak{r}_L = L^{-\frac{1}{2}s_N}$ and $b = \mathfrak{r}_L^{1/2}$, provided s_N is large enough, so that $\frac{1}{2}s_N - 4Nd > \mathfrak{s}$. As was shown in Section 3, the value of s_N can be made as large as one pleases, if the the decay exponents A, $A^{\#}$ of the interaction potentials and L_0 are large enough. □

5.2. Decay of the EF correlators

The derivation of the strong dynamical localization from the energy-interval bounds given in Corollary 3 is obtained in essentially the same way as in the works by Klein and Nguyen [28, 29] (for N-particle models), with the help of the techniques developed earlier by Germinet and Klein [23, 25, 26]. The only difference is the rate of decay of the key probabilities, reflected in the decay rate of the eigenfunction correlators (power-law, in our case). See also [8], where a slightly different technical implementation of essentially the same (or very similar) general ideas was proposed, to achieve a faster decay in a model with short-range potentials.

5.3. Decay of the localized eigenfunctions

We adapt a well-known argument developed originally by von Dreifus and Klein [14] for the 1-particle disordered systems and later modified so as to suit the multi-particle models [11, 10, 28, 29]; in the cited works the decay of the resolvents and of the eigenfunctions was exponential. The specificity of the present case is that we start here with relatively weak, power-law bounds on the decay of resolvents, but ultimately derive a significantly faster, fractional-exponential decay of the localized eigenfunctions. Such a "boost" would be difficult, if at all possible, to perform in Section 3 due to multiple concurrent constraints of the scale induction (cf. Appendix A): any bound achieved on scale L_k had to be reproduced on the next scale L_{k+1}, but now we are free from this obligation.

As usual, we shall need a functional analytic statement providing a link between the decay of the resolvents and that of the (generalized) eigenfunctions. It is deterministic and applies to a large class of elliptic operators of second order (cf., e.g., [32]). In the case of operators $-\Delta + V(x)$ it suffices to assume that V is measurable and bounded. For notational consistency, we keep the boldface notations $\mathbf{H}$ and $\boldsymbol{\psi}$.

LEMMA 9. (Cf., e.g., [32, Lemma 3.3.2]) *For any bounded interval $I \subset \mathbb{R}$ there exists a constant $C \in (0, +\infty)$ such that every generalized eigenfunction $\boldsymbol{\psi}$ of $\mathbf{H}$ with associated generalized eigenvalue $E \in I$ satisfies*

$$\|\boldsymbol{\psi}\mathbf{1}_{\mathbf{C_u}}\| \leqslant C\,\|\mathbf{1}_{\partial\mathbf{B}}\mathbf{G}_{\mathbf{B}}(E)\,\mathbf{1}_{\mathbf{C_u}}\|\,\|\boldsymbol{\psi}\mathbf{1}_{\partial\mathbf{B}}\|. \tag{76}$$

Another result, closely related to (76) and usually called the Simon–Lieb inequality (or Geometric Resolvent Inequality) for the resolvents, is actually required for the proof of Lemma 2 (which was omitted for brevity and replaced with bibliographic references). Like (76), it can be found in a number of papers and books; we refer to [32] treating the 1-particle models, but its formulation below is adapted to our notations. It is purely deterministic, and the structure of the potential (distinguishing the 1-particle and multi-particle Hamiltonians) is irrelevant for the proof. Specifically, consider two embedded cubes $\mathbf{B}_\ell(\mathbf{x}) \subset \mathbf{B}_{L-3}(\mathbf{u})$; then for some $C' > 0$

$$\|\mathbf{1}_{\partial\mathbf{B}_L(\mathbf{u})}\mathbf{G}_{\mathbf{B}_L(\mathbf{u})}(E)\,\mathbf{1}_{\mathbf{C_x}}\| \leqslant C'\,\|\mathbf{1}_{\partial\mathbf{B}_L(\mathbf{u})}\,\mathbf{G}_{\mathbf{B}_L(\mathbf{u})}(E)\,\mathbf{1}_{\partial\mathbf{B}_\ell(\mathbf{x})}\|\cdot\|\mathbf{1}_{\partial\mathbf{B}_\ell(\mathbf{x})}\,\mathbf{G}_{\mathbf{B}_\ell(\mathbf{x})}\,\mathbf{1}_{\mathbf{C_x}}\|\,. \tag{77}$$

An important feature of the Simon–Lieb inequality is that it can be iterated, by surrounding a given point $\mathbf{x} \in \mathbf{B}_L(\mathbf{u})$ with a sequence of imbricated cubes (not necessarily concentric), and reduce the decay analysis of the resolvent between $\mathbf{x}$ and $\partial\mathbf{B}_L(\mathbf{u})$ to that of the resolvents of smaller-size cubes, such as $\mathbf{B}_\ell(\mathbf{x})$ in (76).

Proof of assertion **(B)** *of Theorem* 1. According to Corollary 3, the exponent $\mathfrak{s} > 0$ in the RHS of (75) can be made arbitrarily large, provided $A, A^{\#} > 0$ are large enough. Let $\boldsymbol{\psi} = \boldsymbol{\psi}(\omega)$ be a nontrivial generalized eigenfunction of $\mathbf{H}(\omega)$ with $E \in I_*$, then there exists $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}(\omega) \in (\mathbb{R}^d)^N \cap (\mathbb{Z}^d)^N$ such that $\|\boldsymbol{\psi}\mathbf{1}_{\mathbf{C}_{\widehat{\mathbf{x}}}}\|_2 \neq 0$; fix such a point $\widehat{\mathbf{x}}$ for the rest of the proof. Here and below, $(\mathbb{Z}^d)^N$ is unrelated to $\mathcal{Z}^N$ and, more generally, to the structure of the potential (recall that in our model $\mathcal{Z}$ appears in the definition of the external random potential $V(x,\omega)$).

For notational brevity, it is convenient to set $\gamma := 1 + 2\tau_N$, with N fixed through the proof. The exact structure of the exponent $1 + 2\tau_N$, extensively used in Section 3, is of little importance for the arguments given below. All we need to know is that $\gamma > 1$.

Recall that we introduced in Section 5.1 the functions $E \mapsto \mathbf{F}_{\mathbf{x},L}(E)$. For any $\mathbf{v} \in (\mathbb{Z}^d)^N$ and $k \geqslant 1$ denote, with the same $\mathrm{C}_{\mathrm{ws}} > 0$ as in Theorem 4,

$$\mathcal{A}_k(\mathbf{y}) := \mathbf{B}_{\mathrm{C}_{\mathrm{ws}} L_{k+1}^{\gamma}}(\mathbf{y}) \setminus \mathbf{B}_{\mathrm{C}_{\mathrm{ws}} L_k^{\gamma}}(\mathbf{y}) \tag{78}$$

and introduce the events

$$\mathscr{E}_k(\mathbf{y}) := \left\{ \omega \,\middle|\, \exists \mathbf{v} \in \mathcal{A}_k(\mathbf{y}) \cap (\mathbb{Z}^d)^N : \sup_{E \in I_*} \min_{\mathbf{z} \in \{\mathbf{y},\mathbf{v}\}} \mathbf{F}_{\mathbf{z},L}(E) > L_k^{-\mathfrak{b}} \right\}$$

(i.e., both cubes $\mathbf{B}_{L_k}(\mathbf{y})$ and $\mathbf{B}_{L_k}(\mathbf{v})$ are singular for at least one common energy $E \in I_*$). Recalling that $\mathfrak{s} > 0$ in the RHS of (75) can be assumed arbitrarily large provided $A, A^{\#}$ are large enough, under the latter condition it follows easily from (75) that $\sum_{k \geqslant 1} \mathbb{P}\{\mathscr{E}_k\} < +\infty$ for any d and N fixed before the choice of A and $A^{\#}$. Thus by the Borel–Cantelli lemma, with probability one, only a finite number of the events $\mathscr{E}_k(\mathbf{y})$ occur. Since $\mathbf{y}$ is chosen from a countable set, it also holds that for $\mathbb{P}$-a.e. ω there exists $k_1(\omega)$ such that for $k \geqslant k_1(\omega)$ and any $\mathbf{y} \in (\mathbb{Z}^d)^N$ neither of the events $\mathscr{E}_k(\mathbf{y})$ occurs; this includes $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}(\omega)$ the position of which is random.

The next standard observation (cf. [14]) is that for $\mathbb{P}$-a.e. ω there exists $k_2(\omega) \geqslant k_1(\omega)$ such that for all $k \geqslant k_2(\omega)$ all the cubes $\mathbf{B}_{L_k}(\widehat{\mathbf{x}})$ must be $(E, L_k^{-\mathfrak{b}})$-singular.

Indeed, assuming the opposite and applying Lemma 9 to an infinite sequence of cubes $\mathbf{B}_{L_{k_i}}(\widehat{\mathbf{x}})$ with $L_{k_i} \to +\infty$, we would get in the RHS of (76) (with $\mathbf{u} = \widehat{\mathbf{x}}$) a sequence of upper bounds for $\|\boldsymbol{\psi}\mathbf{1}_{\mathbf{C}_{\widehat{\mathbf{x}}}}\|$ converging to 0, and thus conclude that $\|\boldsymbol{\psi}(\omega)\mathbf{1}_{\mathbf{C}_{\widehat{\mathbf{x}}}}\|_2 = 0$, which contradicts the construction of $\widehat{\mathbf{x}}$.

At the same time, by the choice of $k_1(\omega)$ and knowing that $\mathbf{B}_{L_k}(\widehat{\mathbf{x}})$ is singular for $k \geqslant k_2(\omega)$ at the energy $E = E(\boldsymbol{\psi}(\omega))$, no cube $\mathbf{B}_{L_k}(\mathbf{v})$ with $\mathbf{v} \in \mathcal{A}_k(\widehat{\mathbf{x}})$ can be singular at the same energy E.

From this point on, as in [14], the arguments become deterministic: probabilistic considerations served us to ensure the a.s. existence of $k_1(\omega)$ and $k_2(\omega)$ with the prescribed properties.

The complement $(\mathbb{R}^d)^N \setminus \mathbf{B}_{\mathrm{C_{ws}} L_{k_2}^{\gamma}}$ is the union of the disjoint annuli $\mathcal{A}_j(\widehat{\mathbf{x}})$ over $j \geqslant k_2$. Pick any $\mathbf{x}$ from this complement, then there is a unique k such that $\mathbf{x} \in \mathcal{A}_k(\widehat{\mathbf{x}})$. Let $R = |\mathbf{x} - \widehat{\mathbf{x}}|$, then $R \lesssim L_{k+1}^{\gamma}$, so $L_{k+1} \gtrsim R^{\gamma^{-1}}$.

Since $\mathrm{C_{ws}} \geqslant 1$ (cf. Theorem 4), $\gamma > 1$, and therefore $2\mathrm{C_{ws}} L_k^{\gamma} - 2L_k > \mathrm{C_{ws}} L_k^{\gamma}$ for $L_{N,0}$ large enough, we can apply Lemma 2 to the cube $\mathbf{B}_{L_k^{\gamma}}(\mathbf{x})$ and conclude that this cube is (E, δ)-NS with $\delta = (L_k^{-\mathfrak{b}})^M$, $M = (2L_{k+1}/L_k) > L_{k+1}^{1-\alpha_N^{-1}}$. Thus by (76)

$$\|\boldsymbol{\psi}\mathbf{1}_{\mathbf{x}}\| \lesssim (L_k^{-\mathfrak{b}})^M \leqslant \mathrm{e}^{C L_{k+1}^{1-\alpha_N^{-1}}} \leqslant \mathrm{e}^{-C' R^{\zeta'}}, \quad \zeta' := \frac{1-\alpha^{-1}}{\gamma} > 0.$$

To recover exactly the RHS of the inequality (10), it suffices to take any $\zeta \in (0, \zeta')$ and thus absorb the pre-factor C' in the smaller exponent ζ, assuming that $L_{N,0}$ (thus $R \geqslant L_{k_2}^{\gamma}$, too) is sufficiently large. This proves (10) for all $\mathbf{x}$ except in a finite cube around $\widehat{\mathbf{x}}$. To take care of the remaining values of $\mathbf{x}$ and make the bound uniform, it suffices to put a sufficiently small positive factor $C_{\boldsymbol{\psi}}(\omega)$ in front of $\mathrm{e}^{-R^{\zeta}}$. □

Appendix A Verification of the assumptions on the parameters made in proofs

For the proof of Lemma 5, the parameter s_1 has to fulfill the conditions

$$2(A+d)\alpha_1^{-1} < \underbrace{s_1}_{=c_1\alpha_1 A} < \frac{(A-d)\alpha_1}{1+\sigma}, \tag{79}$$

while for $N = 2, 3$ we need

$$\left(2 + 2NdA^{-1}\right)\alpha^{-1} A < \underbrace{s_N}_{=Ac_1\alpha_1/\varrho^{N-1}} < \frac{1-dA^{-1}}{1+\sigma}\alpha A. \tag{80}$$

Further, s_N must also obey (cf. (53)–(54), (57))

$$s_N > \max\left[4Nd, \alpha_N d\right]. \tag{81}$$

Set $\alpha_1 = \alpha^4$, $\alpha = \varrho^2$, $c_1 = \frac{1}{2\varrho}$. Then $s_N = \frac{1}{2}\varrho^{10-3N} A$, $1 \leqslant N \leqslant 3$. Explicitly,

$$s_1 = \frac{1}{2}\varrho^7 A, \quad s_2 = \frac{1}{2}\varrho^4 A, \quad s_3 = \frac{1}{2}\varrho A. \tag{82}$$

For $N = 1$ we have

$$4\alpha_1^{-1} A \equiv 4\alpha^{-2} A < \underbrace{s_1}_{=2\alpha A} < \frac{1}{2}\alpha^2 A \equiv \frac{1}{2}\alpha_1 A, \tag{83}$$

and for $N = 2, 3$, with $\alpha \geqslant 8\varrho$,

$$\left(4\varrho^{-2} A \equiv\right) \; 4\alpha^{-1} A \leqslant \underbrace{s_N}_{=2\varrho^{4-N} A} \leqslant \frac{1}{2}\alpha A \quad \left(\equiv \frac{1}{2}\varrho^2 A\right). \tag{84}$$

With $A > 24d \geqslant 8Nd$ and $\sigma \in (0, \frac{1}{4})$, (83)–(84) imply (79)–(80). (81) is also guaranteed for $A > 2\,\varrho\,d$.

One can see that a straightforward extension even to $N = 4$ seems problematic, but perhaps there is a better way to organize the scale induction.

Acknowledgements. It is a pleasure to thank Misha Goldstein, Günter Stolz and Ivan Veselić for fruitful discussions of the Anderson models with long-range interactions. I also thank the team of the Isaac Newton Institute (Cambridge, UK) and the organizers of the program "Periodic and Ergodic Spectral Problems" for their warm hospitality and the opportunity to work in the stimulating atmosphere of the Institute, where a part of the present project has been carried out.

REFERENCES

[1] P. W. ANDERSON, *Absence of diffusion in certain random lattices*, Phys. Rev. **109** (5), 1492–1505 (1958).
[2] M. AIZENMAN, S. WARZEL, *Localization bounds for multiparticle systems*, Commun. Math. Phys. **290**, 903–934 (2009).
[3] J. M. BARBAROUX, J. M. COMBES AND P. D. HISLOP, *Localization near band edges for random Schrödinger operators*, Helv. Phys. Acta **70**, 16–43 (1997).
[4] V. BEAUD, S. WARZEL, *Low-energy Fock-space localization for attractive hard-core particles in disorder*, Ann. Henri Poincaré **18** (10), 3143–3166 (2017).
[5] R. CARMONA, J. LACROIX, *Spectral theory of random Schrödinger operators*, Birkhäuser Boston Basel Berlin Inc. (1990).
[6] V. CHULAEVSKY, *On resonances in disordered multi-particle systems*, C.R.A.S. Acad. Sci. Paris, Ser. I **350**, 81–85 (2012).
[7] V. CHULAEVSKY, *From fixed-energy localization analysis to dynamical localization: An elementary path*, J. Stat. Phys. **154**, 1391–1429 (2014).
[8] V. CHULAEVSKY, *Efficient localization bounds in a continuous N-particle Anderson model with long-range interaction*, Lett. Math. Phys. **106** (4), 509–533 (2016).
[9] V. CHULAEVSKY, *Universality of smoothness of Density of States in arbitrary higher-dimensional disorder under non-local interactions I*, From Viéte–Euler identity to Anderson localization, `ArXiv:math-ph/1604.03312` (2016).
[10] V. CHULAEVSKY, A. BOUTET DE MONVEL, Y. SUHOV, *Dynamical localization for a multi-particle model with an alloy-type external random potential*, Nonlinearity **24** (5), 1451–1472 (2011).
[11] V. CHULAEVSKY, Y. SUHOV, *Multi-particle Anderson localisation: Induction on the number of particles*, Math. Phys. Anal. Geom. **12**, 117–139 (2009).
[12] V. CHULAEVSKY, Y. SUHOV, *Efficient Anderson localization bounds for large multi-partilce systems*, J. Spec. Theory **7**, 269–320 (2017).
[13] J. M. COMBES, L. THOMAS, *Asymptotic behaviour of eigenfunctions for multi-particle Schrödinger operators*, Commun. Math. Phys. **34**, 251–263 (1973).
[14] H. VON DREIFUS, A. KLEIN, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124**, 285–299 (1989).
[15] H. VON DREIFUS, A. KLEIN, *Localization for random Schrödinger operators with correlated potentials*, Commun. Math. Phys. **140**, 133–147 (1991).
[16] A. A. ELGART KLEIN, G. STOLZ, *Droplet localization in the random XXZ model and its manifestations*, J. Phys. A **51** (1), 01LT02 (2018).
[17] A. ELGART, A. KLEIN, G. STOLZ, *Manifestations of dynamical localization in the disordered XXZ spin chain*, Commun. Math. Phys. **361**, 1083–1113 (2018).
[18] A. ELGART, A. KLEIN, G. STOLZ, *Many-body localization in the droplet spectrum of the random XXZ quantum spin chain*, J. Funct. Anal. **275** (1), 211–258 (2018).
[19] A. ELGART, M. TAUTENHAHN, I. VESELIĆ, *Localization via fractional moments for models on $\boldsymbol{Z}$ with single-site potentials of finite support*, J. Phys. A **43** (8), 474, 021 (2010).

[20] M. FAUSER, S. WARZEL, *Multiparticle localization for disordered systems on continuous space via the fractional moment method*, Rev. Math. Phys. **27** (4), 1550, 010 (2015).
[21] J. FRÖHLICH, F. MARTINELLI, E. SCOPPOLA, T. SPENCER, *Constructive proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **101**, 21–46 (1985).
[22] J. FRÖHLICH, T. SPENCER, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88**, 151–184 (1983).
[23] F. GERMINET, A. KLEIN, *Bootstrap multiscale analysis and localization in random media*, Commun. Math. Phys. **222**, 415–448 (2001).
[24] F. GERMINET, A. KLEIN, *A characterization of the Anderson metal-insulator transport transition*, Duke Math. J. **124** (2), 309–350 (2004).
[25] F. GERMINET, A. KLEIN, *New characterization of the region of complete localization for random Schrödinger operators*, J. Stat. Phys. **122**, 73–94 (2006).
[26] F. GERMINET, A. KLEIN, *A comprehensive proof of localization for continuous Anderson model with singular random potentials*, J. Spec. Theory **15** (1), 55–143 (2013).
[27] W. KIRSCH, P. STOLLMANN, G. STOLZ, *Anderson localization for random Schrödinger operators with long range interactions*, Commun. Math. Phys. **195**, 495–507 (1998).
[28] A. KLEIN, S. T. NGUYEN, *Bootstrap multiscale analysis for the multi-particle Anderson model*, J. Stat. Phys. **151** (5), 938–973 (2013).
[29] A. KLEIN, S. T. NGUYEN, *Bootstrap multiscale analysis for the multi-particle continuous Anderson Hamiltonians*, J. Spec. Theory **5** (2), 399–444 (2016).
[30] A. KLEIN, S. T. NGUYEN, C. ROJAS-MOLINA, *Characterization of the metal-insulator transport transition for the two-particle Anderson model*, Ann. Henri Poincaré **18**, 2327–2365 (2017).
[31] F. MARTINELLI, E. SCOPPOLA, *Remark on the absence of absolutely continuous spectrum for d-dimensional Schrödinger operators with random potential for large disorder or low energy*, Commun. Math. Phys. **97**, 465–471 (1985).
[32] P. STOLLMANN, *Caught by disorder*, Progress in Mathematical Physics, vol. 20, Birkhäuser Boston Inc., Boston, MA (2001), bound states in random media.
[33] M. REED, B. SIMON, *Methods of modern mathematical physics*, vol. 4, Academic Press, Inc., San Diego Boston Toronto (1979).
[34] F. WEGNER, *Bounds on the density of states in disordered systems*, Z. Phys. B. Condensed Matter **44**, 9–15 (1981).

(Received February 7, 2018)

Victor Chulaevsky
Université de Reims
Département de Mathématiques
Moulin de la Housse, B. P. 1039, 51687 Reims Cedex, France
e-mail: victor.tchoulaevski@univ-reims.fr