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PROPAGATION PHENOMENA FOR MONO–WEAKLY HYPONORMAL OPERATOR PAIRS

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Abstract. In this note, we strengthen some of flatness results for mono-polynomially hyponormal and mono-weakly 2-hyponormal 2-variable weighted shifts in [15, 16, 17].

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be the algebra of bounded linear operators on H. An operator $T \in B(H)$ is called *normal* if $T^*T = TT^*$, it is called *subnormal* if there is a Hilbert space $K \supseteq H$ and a normal operator N on K such that $NH \subseteq H$ and $T = N|_H$, and it is called *hyponormal* if $[T^*, T] := T^*T - TT^* \geqslant 0$. The notions of hyponormal and subnormal operators were introduced by Halmos in 1950 (cf. [3, 18]). Note that if T is subnormal, then $p(T)$ is also subnormal for each $p \in \mathbb{C}[z]$, that is, subnormality is preserved under polynomial calculus. However, this is not the case for hyponormal operators, which can be easily seen from the kind of so called unilateral weighted shift operators. Recall that given a bounded sequence of positive real numbers $\alpha : \alpha_0, \alpha_1, \cdots$, the *unilateral weighted shift* W_α associated with α (called *weights*) is the operator on $l^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ $(n \geqslant 0)$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of $l^2(\mathbb{Z}_+)$. Given a hyponormal weighted shift W_α, there exists $p \in \mathbb{C}[z]$ such that $p(W_\alpha)$ is not hyponormal, see [13, 14] for such kind of examples. So hyponormality is not preserved under polynomial calculus. An operator T on $B(H)$ is called *polynomially hyponormal* if $p(T)$ is hyponormal for each $p \in \mathbb{C}[z]$, and is called *weakly k-hyponormal* if $p(T)$ is hyponormal for each $p \in \mathbb{C}[z]$, with degree no more than k. A nature problem asks whether each polynomially hyponormal operator is subnormal, which had been an open problem for a relatively long period and was answered negatively by Curto and Putinar in [11] via the so called Agler's dictionary [1] by establishing the relationship between positive linear functionals on specific convex cones of polynomials and bounded linear maps acting on a Hilbert space, with a distinguished cyclic vector.

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Before Curto and Putinar's remarkable work to prove that there exists a weighted shift that is polynomially hyponormal but not subnormal, a phenomenon for weighted shifts called flatness originated from Stampfli (cf. [21]) had attracted much attention, and been thought to provide an appropriate way to give a counterexample. Recall that a weighted shift W_α is called *flat* if $\alpha_{k+1} = \alpha_k$ for all $k \geqslant 1$. Stampfli showed that if a weighted shift W_α is subnormal and $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N}$, then W_α is flat. Joshi [14] and Fan [13] also constructed interesting related examples. Later, Curto [5] proved that if the weighted shift W_α is quadratically hyponormal (i.e. weakly 2-hyponormal), and if $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \in \mathbb{N}$, then W_α is flat. Moreover, when W_α is 2-hyponormal, the equality of any of the consecutive weights leads to the flatness of the weighted shift. The propagation phenomena for single weighted shifts are largely studied in the literature (see [5, 6, 4, 19] and the references therein) and the corresponding results and techniques are important in the theory of subnormal operators, relating to the study of dilations and extensions of operators on Hilbert spaces.

In [10, 8], the authors introduced the notion of flatness for 2-variable weighted shifts $\mathbf{T} = (T_1, T_2)$ which is the correct analogue of flatness for 1-variable weighted shifts. First let us recall some related notions. We denote by $\mathfrak{C}_0$ the class of commuting operator pairs on a given Hilbert space H. Recall that a k-tuple $\mathbf{T} = (T_1, \cdots, T_k)$ on the Hilbert space H is called *(jointly) hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := ([T_j^*, T_i])_{i,j=1}^k$$

is positive on the direct sum of k copies of H (cf. [2, 20]). A commuting pair $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is called *k-hyponormal* if

$$\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$$

is hyponormal, or equivalently, the operator matrix

$$M_k(\mathbf{T}) := ([(T_2^qT_1^p)^*, T_2^nT_1^m])_{1 \leqslant m+n \leqslant k, 1 \leqslant p+q \leqslant k}$$

is positive (cf. [7]). Recall that a commuting operator pair $\mathbf{T} = (T_1, T_2)$ is called *subnormal* if there is a Hilbert space $K \supseteq H$ and a commuting normal operator pair $\mathbf{N}$ on K such that H is the common invariant subspace of $\mathbf{N}$ and $\mathbf{T} = \mathbf{N}|_H$. For operator pairs in $\mathfrak{C}_0$, let us denote the class of subnormal pairs by $\mathfrak{H}_\infty$ and the class of k-hyponormal pairs by $\mathfrak{H}_k$ for each integer $k \geqslant 1$. Then we have $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_1$. An operator pair $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is called *mono-weakly k-hyponormal* (cf. [16]) if it holds that

$$\langle M_k(\mathbf{T}) \begin{pmatrix} \lambda_{(1,0)}x \\ \lambda_{(0,1)}x \\ \vdots \\ \lambda_{(0,k)}x \end{pmatrix}, \begin{pmatrix} \lambda_{(1,0)}x \\ \lambda_{(0,1)}x \\ \vdots \\ \lambda_{(0,k)}x \end{pmatrix} \rangle \geqslant 0, \ \forall\, \lambda_{(1,0)}, \lambda_{(0,1)}, \cdots, \lambda_{(0,k)} \in \mathbb{C}, \ \forall\, x \in H, \quad (1.1)$$

which is equivalent to

$$\langle [(\bar{\lambda}_{(1,0)}T_1 + \bar{\lambda}_{(0,1)}T_2 + \cdots \bar{\lambda}_{(0,k)}T_2^k)^*, (\bar{\lambda}_{(1,0)}T_1 + \bar{\lambda}_{(0,1)}T_2 + \cdots \bar{\lambda}_{(0,k)}T_2^k)]x, x \rangle \geqslant 0. \quad (1.2)$$

$\mathbf{T}$ is called *mono-polynomially hyponormal* if (1.1) holds for each integer $k \geqslant 1$. Note in [12], the notion of mono-weakly k-hyponormal operator pair is also introduced, and called weakly k-hyponormal instead. One can see examples in [12, 9] that illustrated the relationship between mono-weakly hyponormal and hyponormal 2-variable weighted shifts. Compared with the one variable case, the notion of mono-weakly k-hyponormal operator pair is natural, as explained in [12, 16]. Clearly, from (1.2), the operator pair $\mathbf{T} \in \mathfrak{C}_0$ is mono-weakly k-hyponormal if and only if for each $p \in \mathbb{C}[z_1, z_2]$ with $deg\ p \leqslant k$, $p(T_1, T_2)$ is hyponormal, and from (1.1), k-hyponormal $\Rightarrow$ mono-weakly k-hyponormal for each k.

In [8, 10], the authors investigated the flatness for subnormal as well as k-hyponormal weighted shifts, and in [15, 16, 17], the authors investigated the flatness for mono-weakly k-hyponormal 2-variable weighted shifts. Based on the idea in [15, 16, 17], we can strengthen some of flatness results in [15, 16, 17], i.e., we can weaken the hypothesis leading to the flatness of 2-variable weighted shifts. We can do this by restricting operator pairs to two types of common invariant subspaces so as to obtain more information about weights.

Let $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbf{k} = \{k_1, k_2\} \in \mathbb{Z}_+^2$. Recall that a 2-variable weighted shift $\mathbf{T} = (T_1, T_2)$ on the Hilbert space $l^2(\mathbb{Z}_+^2)$ is defined by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}, \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}, \tag{1.3}$$

where $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}_+^2\}$ forms an orthonormal basis of $l^2(\mathbb{Z}_+^2)$, $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$, $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} > 0$, $\mathbf{k} \in \mathbb{Z}_+^2$ ($\{\alpha_{\mathbf{k}}\}, \{\beta_{\mathbf{k}}\} \in l^\infty(\mathbb{Z}_+^2)$ are called the *weight sequence*).

It is obvious that $T_1 T_2 = T_2 T_1$ is equivalent to

$$\alpha_{(k_1,k_2+1)} \beta_{(k_1,k_2)} = \beta_{(k_1+1,k_2)} \alpha_{(k_1,k_2)}, \text{ for all } (k_1,k_2) \in \mathbb{Z}_+^2. \tag{1.4}$$

The definition of flatness for commuting 2-variable weighted shifts was introduced in [8, 10]. A 2-variable weighted shift $W_{(\alpha,\beta)}$ is called *horizontally flat* (resp. *vertically flat*), if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geqslant 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geqslant 1$). Moreover, $W_{(\alpha,\beta)}$ is called *flat* if $W_{(\alpha,\beta)}$ is horizontally and vertically flat, and $W_{(\alpha,\beta)}$ is called *symmetrically flat* if $W_{(\alpha,\beta)}$ is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$.

First we review some basic results of flatness for 1-variable weighted shifts.

PROPOSITION 1.1. (Subnormality, see [22]) *Let W_α be a subnormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geqslant 0$, then W_α is flat.*

PROPOSITION 1.2. (2-hyponormality, see [5]) *Let W_α be a 2-hyponormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geqslant 0$, then W_α is flat.*

PROPOSITION 1.3. (Quadratic hyponormality, see [4]) *Let W_α be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$, and assume that W_α is quadratically hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geqslant 1$, then W_α is flat.*

PROPOSITION 1.4. (Polynomial hyponormality, see [4]) *Let W_α be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$, and assume that W_α is polynomially hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geqslant 0$, then W_α is flat.*

With respect to the 2-variable case, we can strengthen the corresponding results in [15, 16, 17]. The main results are the following theorems.

THEOREM 1.5. *Let $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ be a mono-polynomially hyponormal weighted shift. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ and $\beta_{(l_1,l_2+1)} = \beta_{(l_1,l_2)}$ for some $k_1, k_2, l_2 \geqslant 0$ and $l_1 \geqslant 1$, then $\mathbf{T}$ is flat.*

THEOREM 1.6. *Let $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ be a mono-weakly 2-hyponormal weighted shift. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ and $\beta_{(l_1,l_2)} = \beta_{(l_1,l_2+1)}$ for some $k_1, k_2, l_2 \geqslant 1$ and $l_1 \geqslant 2$, then $\mathbf{T}$ is flat.*

2. Proof of Theorem 1.5

We first note that the restriction of a joint hyponormal operator pair to a common invariant subspace is joint hyponormal. Also the restriction of a mono-polynomially hyponormal operator pair to a common invariant subspace is mono-polynomially hyponormal. The following results are frequently used throughout this paper.

LEMMA 2.1. (cf. [15]) *Given $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$. Then for any $m, n > 0$, it holds that $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-weakly k-hyponormal if and only if (mT_1, nT_2) is mono-weakly k-hyponormal $(k \geqslant 1)$.*

Before we prove Theorem 1.5, we give the following result.

LEMMA 2.2. *Let $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ be a mono-polynomially hyponormal weighted shift. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ and $\beta_{(l_1,l_2)} = \beta_{(l_1,l_2+1)}$ for some $k_1, l_2 \geqslant 0$ and $k_2 \geqslant 1$, $l_1 \geqslant 2$, then $\mathbf{T}$ is flat.*

Proof. Given k_1, $l_2 \geqslant 0$, $k_2 \geqslant 1$, and $l_1 \geqslant 2$. According to Lemma 2.1, without loss of generality, we assume that $\alpha_{(k_1,1)} = \alpha_{(k_1+1,1)} = 1$ and $\beta_{(2,l_2)} = \beta_{(2,l_2+1)} = b > 0$. Since $\mathbf{T}$ is mono-polynomially hyponormal, it follows that T_1 and T_2 are both polynomially hyponormal. By Proposition 1.4, we have $\alpha_{(k_1,1)} = \alpha_{(k_1+1,1)} = 1$ and $\beta_{(2,l_2)} = \beta_{(2,l_2+1)} = b$ for all $k_1, l_2 \geqslant 0$. Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-polynomially hyponormal, we have

$$[(T_1 + T_2)^*, (T_1 + T_2)] \geqslant 0.$$

Let $M(n) := \bigvee \{e_{(k_1,k_2)} : k_1 + k_2 = n\}$ and P_n be the orthogonal projection from H to the subspace $M(n)$. Then it is easy to see that $M(n)$ is an invariant subspace of $[(T_1 + T_2)^*, (T_1 + T_2)]$.

By definition, we have

$$\begin{aligned}
&[(T_1+T_2)^*,(T_1+T_2)]e_{(k_1,k_2)}\\
&=[T_1^*,T_1]e_{(k_1,k_2)}+[T_2^*,T_2]e_{(k_1,k_2)}+[T_1^*,T_2]e_{(k_1,k_2)}+[T_2^*,T_1]e_{(k_1,k_2)}\\
&=[\alpha^2_{(k_1,k_2)}-\alpha^2_{(k_1-1,k_2)}]e_{(k_1,k_2)}+[\beta^2_{(k_1,k_2)}-\beta^2_{(k_1,k_2-1)}]e_{(k_1,k_2)}\\
&\quad+[\alpha_{(k_1-1,k_2+1)}\beta_{(k_1,k_2)}-\alpha_{(k_1-1,k_2)}\beta_{(k_1-1,k_2)}]e_{(k_1-1,k_2+1)}\\
&\quad+[\alpha_{(k_1,k_2)}\beta_{(k_1+1,k_2-1)}-\alpha_{(k_1,k_2-1)}\beta_{(k_1,k_2-1)}]e_{(k_1+1,k_2-1)},
\end{aligned}\tag{2.1}$$

where $\alpha_{(k_1,k_2)}=0$ and $\beta_{(l_1,l_2)}=0$ when any of k_1,l_1,k_2,l_2 is smaller than 0.

Note that $[(T_1+T_2)^*,(T_1+T_2)]M(n)\subset M(n)$, and consider the operator

$$M_1:=P_3[(T_1+T_2)^*,(T_1+T_2)]P_3,$$

then it has the following matrix representation to the ordered basis $\{e_{(3,0)},e_{(2,1)},e_{(1,2)},e_{(0,3)}\}$,

$$M_1=\begin{bmatrix} a_{11} & b_{21} & 0 & 0\\ b_{21} & a_{22} & b_{32} & 0\\ 0 & b_{32} & a_{33} & b_{43}\\ 0 & 0 & b_{43} & a_{44}\end{bmatrix},$$

where

$$\begin{aligned}
a_{ii}&:=\alpha^2_{(4-i,i-1)}-\alpha^2_{(3-i,i-1)}+\beta^2_{(4-i,i-1)}-\beta^2_{(4-i,i-2)},\quad 1\leqslant i\leqslant 4,\\
b_{i_{+1}i}&:=\alpha_{(3-i,i)}\beta_{(4-i,i-1)}-\alpha_{(3-i,i-1)}\beta_{(3-i,i-1)},\quad 1\leqslant i\leqslant 3.
\end{aligned}$$

On the other hand, $\alpha_{(k_1,1)}=1$ and $\beta_{(2,l_2)}=b$ for all $k_1,l_2\geqslant 0$. Hence, the matrix M_1 can be written as follows,

$$M_1=\begin{bmatrix} \alpha^2_{(3,0)}-\alpha^2_{(2,0)}+\beta^2_{(3,0)} & \beta_{(3,0)}-b\alpha_{(2,0)} & 0 & 0\\ \beta_{(3,0)}-b\alpha_{(2,0)} & 0 & b\alpha_{(1,2)}-\beta_{(1,1)} & 0\\ 0 & b\alpha_{(1,2)}-\beta_{(1,1)} & a_{33} & b_{43}\\ 0 & 0 & b_{43} & a_{44}\end{bmatrix}.$$

Suppose that Q_1 is the orthogonal projection onto $\bigvee\{e_{(3,0)},e_{(2,1)}\}$. Then $M_2:=Q_1M_1Q_1$ is clearly positive, that is,

$$M_2=\begin{bmatrix} \alpha^2_{(3,0)}-\alpha^2_{(2,0)}+\beta^2_{(3,0)} & \beta_{(3,0)}-b\alpha_{(2,0)}\\ \beta_{(3,0)}-b\alpha_{(2,0)} & 0\end{bmatrix}\geqslant 0.$$

Hence, we get

$$\begin{aligned}
detM_2 &:= -[\beta_{(3,0)}-b\alpha_{(2,0)}]^2\geqslant 0,\\
&\Rightarrow \beta_{(3,0)}=b\alpha_{(2,0)}.
\end{aligned}$$

On the other hand, the commuting property of $\mathbf{T}$ gives that

$$\beta_{(3,0)}\alpha_{(2,0)}=\alpha_{(2,1)}\beta_{(2,0)}.$$

So

$$\beta_{(3,0)} = b, \ \ \alpha_{(2,0)} = 1. \tag{2.2}$$

Suppose that Q_2 is the orthogonal projection onto $\bigvee\{e_{(2,1)}, e_{(1,2)}\}$. Then $M_3 := Q_2 M_1 Q_2$ is clearly positive, that is,

$$M_3 = \begin{bmatrix} 0 & b\alpha_{(1,2)} - \beta_{(1,1)} \\ b\alpha_{(1,2)} - \beta_{(1,1)} & \alpha^2_{(1,2)} - \alpha^2_{(0,2)} + \beta^2_{(1,2)} - \beta^2_{(1,1)} \end{bmatrix} \geqslant 0.$$

Hence, we get

$$det M_3 := -[b\alpha_{(1,2)} - \beta_{(1,1)}]^2 \geqslant 0,$$
$$\Rightarrow \beta_{(1,1)} = b\alpha_{(1,2)}.$$

Moreover, with the commuting property of $\mathbf{T}$, we have

$$\alpha_{(1,1)}\beta_{(2,1)} = \alpha_{(1,2)}\beta_{(1,1)}$$

which yields that

$$\alpha_{(1,2)} = 1, \ \ \beta_{(1,1)} = b. \tag{2.3}$$

Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-polynomially hyponormal, we have

$$[(T_1^2 + T_2)^*, (T_1^2 + T_2)] \geqslant 0.$$

An easy computation gives that

$$\begin{aligned} &[(T_1^2 + T_2)^*, (T_1^2 + T_2)]e_{(k_1,k_2)} \\ &= [T_1^{2*}, T_1^2]e_{(k_1,k_2)} + [T_2^*, T_2]e_{(k_1,k_2)} + [T_1^{2*}, T_2]e_{(k_1,k_2)} + [T_2^*, T_1^2]e_{(k_1,k_2)} \\ &= [\alpha^2_{(k_1,k_2)}\alpha^2_{(k_1+1,k_2)} - \alpha^2_{(k_1-1,k_2)}\alpha^2_{(k_1-2,k_2)}]e_{(k_1,k_2)} + [\beta^2_{(k_1,k_2)} - \beta^2_{(k_1,k_2-1)}]e_{(k_1,k_2)} \\ &\quad + [\beta_{(k_1,k_2)}\alpha_{(k_1-1,k_2+1)}\alpha_{(k_1-2,k_2+1)} - \beta_{(k_1-2,k_2)}\alpha_{(k_1-1,k_2)}\alpha_{(k_1-2,k_2)}]e_{(k_1-2,k_2+1)} \\ &\quad + [\beta_{(k_1+2,k_2-1)}\alpha_{(k_1,k_2)}\alpha_{(k_1+1,k_2)} - \beta_{(k_1,k_2-1)}\alpha_{(k_1,k_2-1)}\alpha_{(k_1+1,k_2-1)}]e_{(k_1+2,k_2-1)}. \end{aligned} \tag{2.4}$$

Given nonnegative integers n, k_1, k_2, let $\tilde{M}(n) := \bigvee\left\{e_{(k_1,k_2)} : k_1 + 2k_2 = n\right\}$ be an invariant subspace of $[(T_1^2 + T_2)^*, (T_1^2 + T_2)]$, $\tilde{P}_n$ be the orthogonal projection from H onto $\tilde{M}(n)$, and $M_4 := \tilde{P}_4[(T_1^2 + T_2)^*, (T_1^2 + T_2)]\tilde{P}_4$. Then with respect to the ordered basis $\{e_{(4,0)}, e_{(2,1)}, e_{(0,2)}\}$, M_4 has the following matrix representation,

$$M_4 = \begin{bmatrix} a_{11} & \beta_{(4,0)} - b\alpha_{(3,0)} & 0 \\ \beta_{(4,0)} - b\alpha_{(3,0)} & 0 & b\alpha_{(0,2)} - \beta_{(0,1)} \\ 0 & b\alpha_{(0,2)} - \beta_{(0,1)} & a_{33} \end{bmatrix},$$

where

$$a_{ii} := \alpha^2_{(7-2i,i-1)}\alpha^2_{(6-2i,i-1)} - \alpha^2_{(5-2i,i-1)}\alpha^2_{(4-2i,i-1)} + \beta^2_{(6-2i,i-1)} - \beta^2_{(6-2i,i-2)}, \ \ 1 \leqslant i \leqslant 3,$$

$$b_{i+1i} := \beta_{(6-2i,i-1)}\alpha_{(5-2i,i)}\alpha_{(4-2i,i)} - \beta_{(4-2i,i-1)}\alpha_{(5-2i,i-1)}\alpha_{(4-2i,i-1)}, \quad 1 \leqslant i \leqslant 2.$$

Let $\tilde{Q}_1$ be the orthogonal projection onto $\bigvee\{e_{(4,0)}, e_{(2,1)}\}$. Then $M_5 := \tilde{Q}_1 M_4 \tilde{Q}_1$ is clearly positive, that is,

$$M_5 = \begin{bmatrix} a_{11} & \beta_{(4,0)} - b\alpha_{(3,0)} \\ \beta_{(4,0)} - b\alpha_{(3,0)} & 0 \end{bmatrix} \geqslant 0.$$

Hence, we get

$$det M_5 := -[\beta_{(4,0)} - b\alpha_{(3,0)}]^2 \geqslant 0,$$
$$\Rightarrow \beta_{(4,0)} = b\alpha_{(3,0)}.$$

On the other hand, from the commuting property of $\mathbf{T}$, it follows that $\alpha_{(3,0)}\beta_{(4,0)} = \beta_{(3,0)}\alpha_{(3,1)}$, so we have

$$\beta_{(4,0)} = b, \quad \alpha_{(3,0)} = 1. \tag{2.5}$$

Suppose that $\tilde{Q}_2$ is the orthogonal projection onto $\bigvee\{e_{(2,1)}, e_{(0,2)}\}$. Then $M_6 := \tilde{Q}_2 M_4 \tilde{Q}_2$ is clearly positive, that is,

$$M_6 = \begin{bmatrix} 0 & b\alpha_{(0,2)} - \beta_{(0,1)} \\ b\alpha_{(0,2)} - \beta_{(0,1)} & a_{33} \end{bmatrix} \geqslant 0.$$

Hence, we get

$$det M_6 := -[b\alpha_{(0,2)} - \beta_{(0,1)}]^2 \geqslant 0,$$
$$\Rightarrow \beta_{(0,1)} = b\alpha_{(0,2)}.$$

Moreover, with the commuting property of $\mathbf{T}$, we have

$$\beta_{(0,1)}\alpha_{(0,2)} = \alpha_{(0,1)}\beta_{(1,1)},$$
$$\Rightarrow \beta_{(0,1)} = b, \alpha_{(0,2)} = 1.$$

Since $\mathbf{T}$ is mono-polynomially hyponormal, T_1 is polynomially hyponormal. From Proposition 1.4 and $\alpha_{(0,2)} = \alpha_{(1,2)} = 1$, it follows that

$$\alpha_{(k_1,2)} = 1, \text{ for all } k_1 \geqslant 0. \tag{2.6}$$

From (2.2) and (2.5), it follows that $\alpha_{(2,0)} = \alpha_{(3,0)} = 1$. So by Proposition 1.4 again, it follows that

$$\alpha_{(k_1,0)} = 1, \text{ for all } k_1 \geqslant 0. \tag{2.7}$$

Moreover, from (2.2) and (2.3), we have

$$\beta_{(3,0)} = \beta_{(1,1)} = b. \tag{2.8}$$

From (2.6), (2.7), (2.8), the assumption that $\alpha_{(k_1,1)} = 1$ for all $k_1 \geqslant 0$ and the commuting property of $\mathbf{T}$, we conclude that

$$\beta_{(l_1,0)} = \beta_{(l_1,1)} = b, \text{ for all } l_1 \geqslant 0.$$

So by the polynomial hyponormality of T_2, we have

$$\beta_{(l_1,l_2)} = b.$$

Combining with the commuting property of $\mathbf{T}$, we conclude that

$$\alpha_{(k_1,k_2)} = 1,$$

as desired. $\square$

LEMMA 2.3. *Let $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ be a mono-polynomially hyponormal weighted shift. If $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2)}$ and $\beta_{(l_1,l_2+1)} = \beta_{(l_1,l_2)}$ for some $k_1, l_2 \geqslant 0$ and $k_2, l_1 \geqslant 1$, then $\mathbf{T}$ is flat.*

Proof. From Lemma 2.2, we need only to show that if there exists a weighted shift $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ which is mono-polynomially hyponormal and

$$\alpha_{(k_1,1)} = \alpha_{(k_1+1,1)} = 1, \ \ \beta_{(1,l_2)} = \beta_{(1,l_2+1)} = b \ (b > 0), \tag{2.9}$$

for some $l_2 \geqslant 0$ and $k_1 \geqslant 0$, then $\mathbf{T}$ is flat. On the contrary, we assume that $\mathbf{T}$ is not flat.

Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-polynomially hyponormal, we have

$$[(T_1 + T_2)^*, (T_1 + T_2)] \geqslant 0.$$

Restrict the operator $[(T_1 + T_2)^*, (T_1 + T_2)]$ to the invariant subspace $M(2)$, and by the same reasoning as in Lemma 2.2, we conclude that

$$\beta_{(2,0)} = \beta_{(0,1)} = b, \ \ \alpha_{(1,0)} = \alpha_{(0,2)} = 1. \tag{2.10}$$

Let $\alpha_{(1,2)} = x_0$. By the commuting property of $\mathbf{T}$, we have $bx_0 = \beta_{(2,1)}$. From the hyponormality of T_1 and (2.10), we have $x_0 \geqslant 1$. Now we claim that $x_0 > 1$. Otherwise, $x_0 = 1$. Combining (2.10) with the commuting property of $\mathbf{T}$, we get $\alpha_{(0,2)} = \alpha_{(1,2)} = 1$ and $\beta_{(2,0)} = \beta_{(2,1)} = b$. By Lemma 2.2, we show that $\mathbf{T}$ is flat, which is contradicting to the assumption. Hence, $x_0 > 1$, as desired.

Let $\alpha_{(k_1,2)} = x_{k_1-1}$ $(k_1 \geqslant 1)$. We conclude that x_{k_1-1} is strictly increasing as k_1 is increasing. Otherwise, with the same reasoning as in the preceding paragraph that leads to $x_0 > 1$, we can get that $\mathbf{T}$ is flat, which is contradicting to the assumption.

Therefore, with the commuting property of $\mathbf{T}$, $\beta_{(2,1)} = bx_0$, $\beta_{(3,1)} = bx_0x_1, \cdots,$ $\beta_{(k_1,1)} = bx_0x_1 \cdots x_{k_1-2}$. So we can get $\beta_{(k_1,1)} > bx_0^{k_1-1}$, and thus, $\lim\limits_{k_1 \to \infty} \beta_{(k_1,1)} = \infty$, which is contradicting to the boundedness of T_2.

This contradiction shows that $\mathbf{T}$ is flat, as desired. $\square$

Now we can complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\alpha_{(k_1,0)} = \alpha_{(k_1+1,0)} = 1$ for some $k_1 \geqslant 0$ and $\beta_{(1,l_2)} = \beta_{(1,l_2+1)} = b$ for some $l_2 \geqslant 0$. Proposition 1.4 shows that $\alpha_{(k_1,0)} = 1$ and $\beta_{(1,l_2)} = b$ for all $k_1, l_2 \geqslant 0$. Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-polynomially hyponormal, we get

$$[(T_1T_2)^*, (T_1T_2)] = T_2^*T_1^*T_1T_2 - T_1T_2T_2^*T_1^* \geqslant 0.$$

Moreover,

$$[(T_1T_2)^*, (T_1T_2)]e_{(k_1,k_2)} = (\alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2)}\beta^2_{(k_1-1,k_2-1)})e_{(k_1,k_2)},$$

where $\alpha_{(k_1,k_2)} = 0$ and $\beta_{(l_1,l_2)} = 0$ when any of k_1, l_1, k_2, l_2 is smaller than 0. So we get

$$\alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2)}\beta^2_{(k_1-1,k_2-1)} \geqslant 0.$$

With the commuting property of $\mathbf{T}$, we have

$$\alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2)}\beta^2_{(k_1-1,k_2-1)} = \alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2-1)}\beta^2_{(k_1,k_2-1)}.$$

Therefore,

$$\begin{aligned} \alpha^2_{(1,2)}\beta^2_{(1,1)} - \alpha^2_{(0,0)}\beta^2_{(1,0)} &\geqslant 0, \\ \Rightarrow \alpha^2_{(1,2)}b^2 - b^2 \cdot 1 &\geqslant 0, \\ \Rightarrow \alpha_{(1,2)} &\geqslant 1. \end{aligned}$$

Since $\alpha_{(2,2)} \geqslant \alpha_{(1,2)}$, it follows that $\alpha_{(2,2)} \geqslant 1$. If $\alpha_{(2,2)} > 1$, let $\alpha_{(k_1,2)} = x_{k_1}$, $\beta_{(k_1,0)} = z_{k_1}$, $\beta_{(k_1,1)} = y_{k_1}$. Clearly, $y_{k_1} \geqslant z_{k_1}$ for each $k_1 \geqslant 1$. With the commuting property of $\mathbf{T}$, we obtain

$$b^2\Pi_{k_1=1}^{n-1} x_{k_1} = y_n z_n.$$

Therefore, $y_n^2 \geqslant b^2\Pi_{k_1=1}^{n-1} x_{k_1}$. Since $x_{k_1} > 1$, $y_n \to +\infty$. Clearly, it is contradicting to the boundedness of T_2. So we conclude that $\alpha_{(2,2)} = 1$, and thus, $\alpha_{(1,2)} = 1$. Then from Lemma 2.3, we show that $\mathbf{T}$ is flat. $\square$

3. Proof of Theorem 1.6

Recall that the restriction of a mono-weakly k-hyponormal operator pair to an invariant subspace is also mono-weakly k-hyponormal. We first prove the following results.

LEMMA 3.1. *Let $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ be a mono-weakly 2-hyponormal weighted shift. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ and $\beta_{(l_1,l_2)} = \beta_{(l_1,l_2+1)}$ for some k_1, $l_2 \geqslant 1$ and $k_2 \geqslant 2$, $l_1 \geqslant 3$ or $k_2 \geqslant 3$, $l_1 \geqslant 2$, then $\mathbf{T}$ is flat.*

Proof. We assume that $\alpha_{(k_1,2)} = \alpha_{(k_1+1,2)} = 1$ and $\beta_{(3,l_2)} = \beta_{(3,l_2+1)} = b$ for some $k_1, l_2 \geqslant 1$ and $b > 0$. Since T_1 and T_2 are both weakly 2-hyponormal, by Proposition 1.3, we have

$$\alpha_{(k_1,2)} = 1 \text{ and } \beta_{(3,l_2)} = b\,, \text{ for all } k_1, l_2 \geqslant 1. \tag{3.1}$$

Since $\mathbf{T}=(T_1,T_2)\in\mathfrak{C}_0$ is mono-weakly 2-hyponormal, we have

$$[(T_1+T_2)^*,(T_1+T_2))]\geqslant 0.$$

Recall from the proof of Lemma 2.2, $M(n)=\bigvee\left\{e_{(k_1,k_2)}:k_1+k_2=n\right\}$ and it holds that $[(T_1+T_2)^*,(T_1+T_2)]M(n)\subset M(n)$. We consider the operator

$$M_7:=P_5[(T_1+T_2)^*,(T_1+T_2)]P_5.$$

From (2.1), with respect to the ordered basis $\{e_{(5,0)},e_{(4,1)},e_{(3,2)},e_{(2,3)},e_{(1,4)},e_{(0,5)}\}$, M_7 can be written as follows,

$$M_7=\begin{bmatrix} a_{11} & \bar{b}_{21} & 0 & 0 & 0 & 0\\ b_{21} & a_{22} & \beta_{(4,1)}-b\alpha_{(3,1)} & 0 & 0 & 0\\ 0 & \beta_{(4,1)}-b\alpha_{(3,1)} & 0 & b\alpha_{(2,3)}-\beta_{(2,2)} & 0 & 0\\ 0 & 0 & b_{43} & a_{44} & \bar{b}_{54} & 0\\ 0 & 0 & 0 & b_{54} & a_{55} & \bar{b}_{65}\\ 0 & 0 & 0 & 0 & b_{65} & a_{66}\end{bmatrix},$$

$$a_{ii}:=\alpha^2_{(6-i,i-1)}-\alpha^2_{(5-i,i-1)}+\beta^2_{(6-i,i-1)}-\beta^2_{(6-i,i-2)},\ \ 1\leqslant i\leqslant 6,$$
$$b_{i+1i}:=\beta_{(6-i,i-1)}\alpha_{(5-i,i)}-\alpha_{(5-i,i-1)}\beta_{(5-i,i-1)},\ \ 1\leqslant i\leqslant 5.$$

Let Q_3, Q_4 be the orthogonal projections onto $\bigvee\{e_{(4,1)},e_{(3,2)}\}$ and $\bigvee\{e_{(3,2)},e_{(2,3)}\}$ respectively. Then the matrix $M_8:=Q_3M_7Q_3$ is clearly positive, that is,

$$M_8=\begin{bmatrix}\alpha^2_{(4,1)}-\alpha^2_{(3,1)}+\beta^2_{(4,1)}-\beta^2_{(4,0)} & \beta_{(4,1)}-b\alpha_{(3,1)}\\ \beta_{(4,1)}-b\alpha_{(3,1)} & 0\end{bmatrix}\geqslant 0.$$

Therefore, $detM_8:=-[\beta_{(4,1)}-b\alpha_{(3,1)}]^2\geqslant 0$, which implies

$$\beta_{(4,1)}=b\alpha_{(3,1)}. \tag{3.2}$$

With the commuting property of $\mathbf{T}$, we have

$$\alpha_{(3,2)}\beta_{(3,1)}=\alpha_{(3,1)}\beta_{(4,1)}.$$

From (3.1), it follows that $\alpha_{(3,2)}\beta_{(3,1)}=b$ and $b=\alpha_{(3,1)}\beta_{(4,1)}$. Hence,

$$\alpha_{(3,1)}=1,\beta_{(4,1)}=b. \tag{3.3}$$

In the same way, we have

$$\alpha_{(2,3)}=1,\ \ \beta_{(2,2)}=b. \tag{3.4}$$

Recall that $\tilde{M}(n)=\bigvee\left\{e_{(k_1,k_2)}:k_1+2k_2=n\right\}$ and $\tilde{P}_n$ is the orthogonal projection onto $\tilde{M}(n)$. Then $\tilde{M}(n)$ is an invariant subspace of $[(T_1^2+T_2)^*,(T_1^2+T_2)]$. Since $\mathbf{T}=(T_1,T_2)\in\mathfrak{C}_0$ is mono-weakly 2-hyponormal, we have

$$[(T_1^2+T_2)^*,(T_1^2+T_2)]\geqslant 0.$$

Then $M_9 := \tilde{P}_7[(T_1^2+T_2)^*, (T_1^2+T_2)]\tilde{P}_7$ is positive. From (2.4), with respect to the ordered basis $\{e_{(7,0)}, e_{(5,1)}, e_{(3,2)}, e_{(1,3)}\}$, M_9 can be written as follows,

$$M_9 = \begin{bmatrix} a_{11} & b_{21} & 0 & 0 \\ b_{21} & a_{22} & b_{32} & 0 \\ 0 & b_{32} & a_{33} & b_{43} \\ 0 & 0 & b_{43} & a_{44} \end{bmatrix},$$

where

$$a_{ii} := \alpha^2_{(10-2i,i-1)}\alpha^2_{(9-2i,i-1)} - \alpha^2_{(8-2i,i-1)}\alpha^2_{(7-2i,i-1)} + \beta^2_{(9-2i,i-1)} - \beta^2_{(9-2i,i-2)}, \quad 1 \leqslant i \leqslant 4,$$
$$b_{i+1i} := \alpha_{(8-2i,i)}\alpha_{(7-2i,i)}\beta_{(9-2i,i-1)} - \alpha_{(8-2i,i-1)}\alpha_{(7-2i,i-1)}\beta_{(7-2i,i-1)}, \quad 1 \leqslant i \leqslant 3.$$

Let Q_5, Q_6 be the orthogonal projections onto $\bigvee\{e_{(3,2)}, e_{(1,3)}\}$ and $\bigvee\{e_{(5,1)}, e_{(3,2)}\}$ respectively. Then the matrix $M_{10} := Q_5 M_9 Q_5$ is clearly positive, that is,

$$M_{10} = \begin{bmatrix} 0 & b\alpha_{(1,3)} - \beta_{(1,2)} \\ b\alpha_{(1,3)} - \beta_{(1,2)} & a_{44} \end{bmatrix} \geqslant 0,$$

Hence, $det M_{10} := -[b\alpha_{(1,3)} - \beta_{(1,2)}]^2 \geqslant 0$, which implies

$$\beta_{(1,2)} = b\alpha_{(1,3)}.$$

With the commuting property of $\mathbf{T}$, we have

$$\beta_{(1,2)}\alpha_{(1,3)}\alpha_{(2,3)} = \alpha_{(1,2)}\alpha_{(2,2)}\beta_{(3,2)}.$$

Combining the above equality with (3.1) and (3.4), we have $b = \beta_{(1,2)}\alpha_{(1,3)}$, which shows that

$$\alpha_{(1,3)} = 1, \ \ \beta_{(1,2)} = b. \tag{3.5}$$

Then $\alpha_{(1,3)} = \alpha_{(2,3)} = 1$. Moreover, from the assumption, T_1 is quadratically hyponormal. Then by Proposition 1.3, we conclude that

$$\alpha_{(k_1,3)} = 1, \text{ for all } k_1 \geqslant 1.$$

Since $Q_6 M_9 Q_6 \geqslant 0$, the same reasoning imposes $\alpha_{(4,1)} = 1$ and $\beta_{(5,1)} = b$.

From $\alpha_{(4,1)} = \alpha_{(3,1)}$ and T_1 is quadratically hyponormal, by Proposition 1.3 again, it follows that

$$\alpha_{(k_1,1)} = 1, \text{ for all } k_1 \geqslant 1.$$

With the commuting property of $\mathbf{T}$, we get $\beta_{(l_1,1)} = \beta_{(l_1,2)} = b$ for all $l_1 \geqslant 1$. Since T_2 is quadratically hyponormal, we have

$$\beta_{(l_1,l_2)} = b, \text{ for all } l_1, l_2 \geqslant 1.$$

With the commuting property of $\mathbf{T}$, we obtain $\beta_{(l_1,l_2)} = b$ and $\alpha_{(k_1,k_2)} = 1$, for all $k_1, k_2, l_1, l_2 \geqslant 1$.

Therefore, $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is flat. $\square$

LEMMA 3.2. *Let* $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ *be a mono-weakly 2-hyponormal weighted shift. If* $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ *and* $\beta_{(l_1,l_2)} = \beta_{(l_1,l_2+1)}$ *for some* $k_1, l_2 \geqslant 1$ *and* $k_2, l_1 \geqslant 2$, *then* $\mathbf{T}$ *is flat.*

Proof. From Lemma 3.1, we need only to show that if there exists a weighted shift $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ which is mono-weakly 2-hyponormal and

$$\alpha_{(k_1,2)} = \alpha_{(k_1+1,2)} = 1, \ \ \beta_{(2,l_2)} = \beta_{(2,l_2+1)} = b \ (b > 0), \tag{3.6}$$

for some $l_2 \geqslant 1$ and $k_1 \geqslant 1$, then $\mathbf{T}$ is flat. On the contrary, we assume that $\mathbf{T}$ is not flat, then we will show that it will lead to a contradiction.

Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-weakly 2-hyponormal, we have

$$[(T_1 + T_2)^*, (T_1 + T_2)] \geqslant 0.$$

Restrict the operator $[(T_1 + T_2)^*, (T_1 + T_2)]$ to the invariant subspace $M(4)$, and by the same reasoning as in Lemma 3.1, we conclude that

$$\beta_{(3,1)} = \beta_{(1,2)} = b, \alpha_{(2,1)} = \alpha_{(1,3)} = 1. \tag{3.7}$$

Assume that $\alpha_{(2,3)} = x_0$, by the commuting property of $\mathbf{T}$, we have $bx_0 = \beta_{(3,2)}$. From the hyponormality of T_1 and (3.7), we have $x_0 \geqslant 1$. Now we claim that $x_0 > 1$. Otherwise, $x_0 = 1$. Combining (3.7) with the commuting property of $\mathbf{T}$, we get $\alpha_{(1,3)} = \alpha_{(2,3)} = 1$ and $\beta_{(3,1)} = \beta_{(3,2)} = b$. By Lemma 3.1, we conclude that $\mathbf{T}$ is flat, which is contradicting to the assumption. Thus, $x_0 > 1$.

Let $\alpha_{(k_1,3)} = x_{k_1-2}$ $(k_1 \geqslant 2)$. We conclude that the sequence $\{x_{k_1-2}\}$ is strictly increasing as k_1 is increasing. Otherwise, with the same reasoning as in the preceding paragraph that leads to $x_0 > 1$, we obtain $\mathbf{T}$ is flat, which is contradicting to the assumption.

Therefore, with the commuting property of $\mathbf{T}$, $\beta_{(3,2)} = bx_0$, $\beta_{(4,2)} = bx_0x_1, \cdots$, $\beta_{(k_1,2)} = bx_0x_1 \cdots x_{k_1-3}$ and we get $\beta_{(k_1,2)} > bx_0^{k_1-3}$. Thus, $\lim\limits_{k_1 \to \infty} \beta_{(k_1,2)} = \infty$, which is contradicting to the boundedness of T_2.

This contradiction shows that $\mathbf{T}$ is flat, as desired. □

Now we can complete the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $\alpha_{(k_1,1)} = \alpha_{(k_1+1,1)} = 1$ for some $k_1 \geqslant 1$ and $\beta_{(2,l_2)} = \beta_{(2,l_2+1)} = b$ for some $l_2 \geqslant 1$. Then by Proposition 1.3, we have $\alpha_{(k_1,1)} = 1$ and $\beta_{(2,l_2)} = b$ for all k_1, $l_2 \geqslant 1$. Since $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$ is mono-weakly 2-hyponormal, we get

$$[(T_1T_2)^*, (T_1T_2)] = T_2^*T_1^*T_1T_2 - T_1T_2T_2^*T_1^* \geqslant 0.$$

So

$$[(T_1T_2)^*, (T_1T_2)]e_{(k_1,k_2)} = (\alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2)}\beta^2_{(k_1-1,k_2-1)})e_{(k_1,k_2)}.$$

With the commuting property of $\mathbf{T}$, we know that

$$\alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2)}\beta^2_{(k_1-1,k_2-1)} = \alpha^2_{(k_1,k_2+1)}\beta^2_{(k_1,k_2)} - \alpha^2_{(k_1-1,k_2-1)}\beta^2_{(k_1,k_2-1)}.$$

Therefore, from $[(T_1T_2)^*,(T_1T_2)] \geqslant 0$, it follows that

$$\alpha^2_{(2,3)}\beta^2_{(2,2)} - \alpha^2_{(1,2)}\beta^2_{(1,1)} = \alpha^2_{(2,3)}b^2 - b^2\cdot 1 \geqslant 0,$$

which implies

$$\alpha_{(2,3)} \geqslant 1.$$

So $\alpha_{(3,3)} \geqslant \alpha_{(2,3)}$. If $\alpha_{(3,3)} > 1$, we let $\alpha_{(k_1,3)} = x_{k_1}$, $\beta_{(k_1,1)} = z_{k_1}$, $\beta_{(k_1,2)} = y_{k_1}$. Clearly, $y_{k_1} \geqslant z_{k_1}$ for each $k_1 \geqslant 2$. With the commuting property of $\mathbf{T}$, we know that

$$b^2\Pi_{k_1=2}^{n-1}x_{k_1} = y_nz_n.$$

Therefore, $y_n^2 \geqslant b^2\Pi_{k_1=2}^{n-1}x_{k_1}$. Since $x_{k_1} > 1$, $y_n \to +\infty$. Clearly, it is contradicting to the boundedness of T_2. This shows that $\alpha_{(3,3)} = 1$. Then from Lemma 3.2, we know that $\mathbf{T}$ is flat. $\square$

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