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FREDHOLM WEIGHTED COMPOSITION OPERATORS

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Abstract. We show that Fredholm weighted composition operators on L^p-spaces with non-atomic measures are precisely the invertible ones. We also characterize the classes of Fredholm and invertible weighted composition operators on l^p. Furthermore, the closedness of ranges and Fredholmness of these operators on H^p-spaces of the unit disk are investigated.

Let B_1 and B_2 be Banach spaces over $\mathbb{C}$. A linear operator $T\colon B_1 \to B_2$ is said to be *Fredholm* if $\operatorname{ran}(T)$ is closed in B_2 and the dimensions of $\ker(T)$ and $B_2/\operatorname{ran}(T)$ are both finite, where $\ker(T)$ and $\operatorname{ran}(T)$ are the kernel and the range of T respectively. In this case, the *Fredholm index* of T, written as $\operatorname{ind} T$, is defined by $\operatorname{ind} T := \dim\ker(T) - \dim B_2/\operatorname{ran}(T)$.

In this paper, we study Fredholm weighted composition operators on Lebesgue spaces with non-atomic measures, on sequence spaces and on Hardy spaces of the unit disk. We also characterize those weighted composition operators on H^p with closed ranges.

1. Fredholm weighted composition operators on L^p

1.1. Preliminaries

Let (X,Σ,μ) and (Y,Γ,ν) be two σ-finite and complete measure spaces. The Lebesgue space consisting of all (equivalence classes of) p-integrable, where $1 \leqslant p < \infty$, complex-valued Σ-measurable (resp. Γ-measurable) functions on X (resp. on Y) is denoted by $L^p(\mu)$ (resp. by $L^p(\nu)$). The functions in $L^\infty(\mu)$ and $L^\infty(\nu)$ are essentially bounded. The norm of a function in $L^p(\mu)$ (resp. $L^p(\nu)$) is written as $\|\cdot\|_{L^p(\mu)}$ (resp. $\|\cdot\|_{L^p(\nu)}$).

If we take $X = \mathbb{N}$, $\Sigma = \mathscr{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and μ be the counting measure on $\mathscr{P}(\mathbb{N})$, then $L^p(\mu)$ is just the usual sequence space l^p. A Schauder basis for l^p $(1 \leqslant p < \infty)$ is given by $\{e_n\}_{n=1}^\infty$, where $e_n = \{e_{nk}\}_{k=1}^\infty$ and $e_{nk} = \delta_{nk}$ is the Kronecker delta.

Let u be a complex-valued Γ-measurable function and $\varphi\colon Y \to X$ be a point mapping such that $\varphi^{-1}(E) \in \Gamma$ for all $E \in \Sigma$. Assume that φ is also non-singular, which

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means the measure defined by $\nu\varphi^{-1}(E) := \nu(\varphi^{-1}(E))$ for $E \in \Sigma$, is absolutely continuous with respect to μ. We assume the corresponding Radon-Nikodym derivative h is finite-valued μ-a.e. on X.

The functions u and φ induce the *weighted composition operator* uC_φ from $L^p(\mu)$ $(1 \leqslant p \leqslant \infty)$ into the linear space of all Γ-measurable functions on Y by

$$uC_\varphi(f)(y) := u(y)f(\varphi(y)) \quad \text{for every } f \in L^p(\mu) \text{ and } y \in Y.$$

The non-singularity of φ guarantees that uC_φ is a well-defined mapping of equivalence classes of functions. When $u \equiv 1$ (resp. $(X,\Sigma,\mu) = (Y,\Gamma,\nu)$ and $\varphi(x) = x$ for all $x \in X$), the corresponding operator, denoted by C_φ (resp. by M_u), is called a *composition operator* (resp. a *multiplication operator*). Observe that $uC_\varphi = M_u \circ C_\varphi$.

If uC_φ maps $L^p(\mu)$ into $L^p(\nu)$, it follows from the closed graph theorem that uC_φ is bounded. Moreover, we say uC_φ is an operator on $L^p(\mu)$ if it maps $L^p(\mu)$ into itself. A main result of the next sub-section is that when (X,Σ,μ) is non-atomic, Fredholm weighted composition operators from $L^p(\mu)$ into $L^p(\nu)$ are precisely the invertible ones.

We introduce another notation. Let $\varphi^{-1}\Sigma$ be the relative completion of the σ-algebra generated by $\left\{\varphi^{-1}(E) : E \in \Sigma\right\}$, i.e.

$$\varphi^{-1}\Sigma := \left\{\varphi^{-1}(E)\Delta F : E \in \Sigma \text{ and } \nu(F) = 0\right\}.$$

In fact, the finiteness of h ensures that the measure space $\left(Y, \varphi^{-1}\Sigma, \nu\right)$ is σ-finite. To see this, write $X = \bigcup_{i=1}^\infty E_i$, where $E_i \in \Sigma$ and $\mu(E_i) < \infty$ for each $i \in \mathbb{N}$. For every $i, j \in \mathbb{N}$, define

$$G_i^j := \{x \in E_i : h(x) \leqslant j\}.$$

Then

$$\nu\varphi^{-1}\left(G_i^j\right) = \int_{G_i^j} h\,d\mu \leqslant j\mu\left(G_i^j\right) \leqslant j\mu(E_i) < \infty.$$

Since

$$Y = \left(\bigcup_{i=1}^\infty \bigcup_{j=1}^\infty \varphi^{-1}\left(G_i^j\right)\right) \cup \varphi^{-1}(\{x \in X : h(x) = \infty\})$$

and $\nu\varphi^{-1}(\{x \in X : h(x) = \infty\}) = 0$, the assertion follows.

Let g be a non-negative Γ-measurable function on Y. The measure given by $S \mapsto \int_S g\,d\nu$ for $S \in \varphi^{-1}\Sigma$, is absolutely continuous with respect to ν. Thus, there exists a unique (ν-a.e.) non-negative $\varphi^{-1}\Sigma$-measurable function on Y, denoted by $E(g)$, with

$$\int_S g\,d\nu = \int_S E(g)\,d\nu \quad \text{for each } S \in \varphi^{-1}\Sigma.$$

The function $E(g)$, which is called the *conditional expectation* of g with respect to $\varphi^{-1}\Sigma$, plays a crucial role in proving Lemma 1.1.

1.2. Main results

Assume that $1 \leqslant p < \infty$ in this sub-section. We first establish a lemma on the dimensions of $\ker uC_\varphi$ and $L^p(\nu)/\overline{\operatorname{ran}\left(uC_\varphi\right)}$, where $\overline{\operatorname{ran}\left(uC_\varphi\right)}$ is the norm-closure of $\operatorname{ran}\left(uC_\varphi\right)$ in $L^p(\nu)$. Similar results for composition operators were obtained in [6].

LEMMA 1.1. *Suppose (X,Σ,μ) is non-atomic and let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^p(\nu)$.*

(a) The nullity of uC_φ (i.e. $\dim\ker uC_\varphi$) is either zero or infinite.

(b) The codimension of $\overline{\operatorname{ran}\left(uC_\varphi\right)}$ in $L^p(\nu)$ (i.e. $\dim L^p(\nu)/\overline{\operatorname{ran}\left(uC_\varphi\right)}$) is either zero or infinite.

Proof. We first prove (a). If uC_φ is injective, then $\dim\ker uC_\varphi = 0$. Otherwise, there is a non-zero function $f \in L^p(\mu)$ such that $uC_\varphi f = 0$. As (X,Σ,μ) is non-atomic and the set $E := \{x \in X : |f(x)| > 0\}$ is of positive μ-measure, we may choose a sequence $\{E_n\}_{n=1}^\infty$ of pairwise disjoint Σ-measurable sets in E with $0 < \mu(E_n) < \infty$. Let $f_n := f\chi_{E_n}$ for $n \in \mathbb{N}$. They are non-zero and linearly independent. Moreover,

$$\begin{aligned}\left\|uC_\varphi f_n\right\|_{L^p(\nu)}^p &= \int_Y |u|^p\,|f\chi_{E_n}\circ\varphi|^p\,d\nu = \int_Y |u|^p|f|^p\circ\varphi\,\chi_{\varphi^{-1}(E_n)}\,d\nu \\ &= \int_{\varphi^{-1}(E_n)} |u|^p|f|^p\circ\varphi\,d\nu \leqslant \int_Y |u|^p|f|^p\circ\varphi\,d\nu = \left\|uC_\varphi f\right\|_{L^p(\nu)}^p = 0,\end{aligned}$$

so that $f_n \in \ker uC_\varphi$ for all n. Thus, we have $\dim\ker uC_\varphi = \infty$.

For (b), suppose that $\dim L^p(\nu)/\overline{\operatorname{ran}\left(uC_\varphi\right)} \neq 0$. As

$$\dim L^p(\nu)/\overline{\operatorname{ran}\left(uC_\varphi\right)} = \dim\ker uC_\varphi{}^*,$$

there is a non-zero function $g \in L^q(\nu)$, where q is the conjugate exponent of p, such that

$$\int_Y \left(uC_\varphi f\right)\overline{g}\,d\nu = 0 \qquad \text{for all } f \in L^p(\mu).$$

When $1 < q < \infty$, we have

$$\int_Y E(|g|^q)\,d\nu = \int_Y |g|^q\,d\nu > 0,$$

so that the $\varphi^{-1}\Sigma$-measurable set $F := \{y \in Y : E(|g|^q) \geqslant \delta\}$ has positive ν-measure for some $\delta > 0$. We may also assume $\nu(F) < \infty$. The definition of $\varphi^{-1}\Sigma$ ensures that $F = \varphi^{-1}(E)$ for a Σ-measurable set E. Since (X,Σ,μ) is non-atomic, it follows from the lemma in [6] that there exists a sequence $\{E_n\}_{n=1}^\infty$ of pairwise disjoint Σ-measurable sets in E such that $0 < \nu\varphi^{-1}(E_n) < \infty$. The functionals $\phi_n \in L^p(\nu)^*$ represented by $g\chi_{\varphi^{-1}(E_n)}$, $n \in \mathbb{N}$, are all non-zero because

$$\begin{aligned}\int_Y |g\chi_{\varphi^{-1}(E_n)}|^q d\nu = \int_{\varphi^{-1}(E_n)} |g|^q\,d\nu &= \int_{\varphi^{-1}(E_n)} E(|g|^q)\,d\nu \\ &\geqslant \delta\nu\varphi^{-1}(E_n) > 0.\end{aligned}$$

As the sets $\{\varphi^{-1}(E_n)\}_{n=1}^{\infty}$ are pairwise disjoint, these functionals are also linearly independent. Moreover, we have

$$\phi_n\left(uC_\varphi f\right) = \int_Y \left(uC_\varphi f\right)\overline{g}\chi_{\varphi^{-1}(E_n)}\,d\nu = \int_Y \left(uC_\varphi f\chi_{E_n}\right)\overline{g}\,d\nu = 0$$

for every $f \in L^p(\mu)$, i.e. $\phi_n \in \ker uC_\varphi{}^*$ (for the case $q = \infty$, the preceding argument also applies with minor modifications). Hence $\dim\ker uC_\varphi{}^* = \infty$. $\square$

It has been shown in [14, Theorem 2.6] that Fredholm and invertible composition operators on $L^2(\mu)$ are equivalent. Takagi [15, Theorem 3] generalized this result to weighted composition operators on $L^p(\mu)$, by assuming boundedness of the corresponding multiplication operators. We prove that the same result is valid *without* this assumption and obtain measure-theoretic characterizations for invertible weighted composition operators from $L^p(\mu)$ onto $L^p(\nu)$.

THEOREM 1.2. *Suppose (X,Σ,μ) is non-atomic and let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^p(\nu)$. The following statements are equivalent:*

(i) uC_φ is invertible.

(ii) uC_φ is Fredholm.

(iii) (1) There exists a constant $\delta > 0$ such that $\int_{\varphi^{-1}(E)} |u|^p d\nu \geqslant \delta\,\mu(E)$ for every set $E \in \Sigma$ with $\mu(E) < \infty$, and

(2) For each set $F \in \Gamma$, there is a set $G \in \Sigma$ such that $\varphi^{-1}(G) = F$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. We first show that (ii) implies (iii).

To prove (iii)(1), assume uC_φ is Fredholm. It is injective by Lemma 1.1. Since the range of uC_φ is closed, there exists a number $c > 0$ such that

$$\left\|uC_\varphi f\right\|_{L^p(\nu)} \geqslant c\|f\|_{L^p(\mu)} \quad \text{for all } f \in L^p(\mu).$$

In particular, by choosing $f = \chi_E$, where $E \in \Sigma$ and $\mu(E) < \infty$, we obtain

$$\int_{\varphi^{-1}(E)} |u|^p d\nu = \left\|uC_\varphi\chi_E\right\|_{L^p(\nu)}^p \geqslant c^p\,\|\chi_E\|_{L^p(\mu)}^p = c^p\mu(E).$$

Thus, (iii)(1) follows. By Lemma 1.1 again, we have $\dim L^p(\nu)/\mathrm{ran}(uC_\varphi) = 0$ and so uC_φ is indeed surjective. We claim that $u \neq 0$ ν-a.e. on Y. Otherwise, there is a Γ-measurable set S such that $0 < \nu(S) < \infty$ and $u = 0$ on S. The surjectivity of uC_φ yields a function $f \in L^p(\mu)$ with $uC_\varphi f = \chi_S$. With the choice of S, however, this equality is invalid. The claim is justified.

To prove (iii)(2), take any set $F \in \Gamma$ with $\nu(F) < \infty$. Let $g \in L^p(\mu)$ be the function such that $uC_\varphi g = \chi_F$, or $C_\varphi g = \frac{1}{u}\chi_F$. Let $\mathscr{E} := \{\varphi^{-1}(E) : E \in \Sigma\}$. As $C_\varphi g$ is $\mathscr{E}$-measurable, so is $\frac{1}{u}\chi_F$. By writing $Y = \bigcup_{i=1}^{\infty} F_i$, where $\{F_i\}_{i=1}^{\infty}$ is an increasing sequence of Γ-measurable sets with finite ν-measures, we have $\frac{1}{u} = \lim_{i\to\infty}\frac{1}{u}\chi_{F_i}$ on

Y. It follows that $\frac{1}{u}$ is $\mathscr{E}$-measurable. Hence χ_F is also $\mathscr{E}$-measurable for each $F \in \Gamma$ satisfying $\nu(F) < \infty$.

It remains to show that (iii) implies (i). We may express (iii)(1) as

$$\left\| uC_\varphi \chi_E \right\|_{L^p(\nu)}^p \geqslant \delta \left\| \chi_E \right\|_{L^p(\mu)}^p \qquad \text{for every } E \in \Sigma \text{ with } \mu(E) < \infty.$$

The operator uC_φ maps functions with disjoint cozero sets into functions with disjoint cozero sets (the cozero set of a function $f \in L^p(\mu)$ is the set of all $x \in X$ on which f does not vanish). This, together with the fact that simple functions (with finite μ-measure cozero sets) are dense in $L^p(\mu)$, implies the above inequality holds for all $f \in L^p(\mu)$. Thus, uC_φ is injective and has closed range.

It remains to show that $uC_\varphi{}^*$ is injective, which is equivalent to the surjectivity of uC_φ. Let $\phi \in L^p(\nu)^*$ be a functional represented by the function $h \in L^q(\nu)$, where q is the conjugate exponent of p, such that

$$\int_Y h\left(uC_\varphi f\right) d\nu = 0 \qquad \text{for all } f \in L^p(\mu).$$

If $G \in \Sigma$ and $\mu(G) < \infty$, then $\int_{\varphi^{-1}(G)} hu\, d\nu = 0$. By (iii)(2), we see that

$$\int_F hu\, d\nu = 0 \qquad \text{for every } F \in \Gamma.$$

The injectivity of $uC_\varphi{}^*$ follows immediately provided that $u \neq 0$ ν-a.e. on Y. To justify the latter, assume the contrary that the set $N := \{y \in Y : u(y) = 0\}$ has positive ν-measure. From (iii)(2) and σ-finiteness of (X, Σ, μ), there exists a set $M \in \Sigma$ such that $\varphi^{-1}(M) \subset N$ and $0 < \mu(M) < \infty$. Then,

$$0 = \int_N |u|^p d\nu \geqslant \int_{\varphi^{-1}(M)} |u|^p d\nu \geqslant \delta\, \mu(M) > 0,$$

which is impossible. The proof of the theorem is now complete. □

In [7, Theorem 3.2], Jabbarzadeh claimed that when (X, Σ, μ) is non-atomic, the operator uC_φ is Fredholm on $L^p(\mu)$ if and only if $J \geqslant \delta$ μ-a.e. on X for some constant $\delta > 0$, where J can be shown to be the Radon-Nikodym derivative of the measure $E \mapsto \int_{\varphi^{-1}(E)} |u|^p d\mu$ $(E \in \Sigma)$ with respect to μ [9, p.5]. The latter condition, however, is not sufficient for the Fredholmness of uC_φ. The fallacy in the proof is that M_u is not necessarily injective even if J is bounded away from zero. To illustrate this, let $X = [0,1]$ be equipped with the Lebesgue measure μ on the σ-algebra Σ of Borel sets in X. With

$$u(x) = x\chi_{[\frac{1}{2},1]}(x) \qquad \text{and} \qquad \varphi(x) = 2x\chi_{[0,\frac{1}{2})}(x) + (2-2x)\chi_{[\frac{1}{2},1]}(x),$$

we have

$$\frac{1}{2}\left(x - \frac{x^2}{4}\right) = \int_{\varphi^{-1}([0,x))} |u|\, d\mu = \int_{[0,x)} J\, d\mu.$$

Hence $J = \frac{1}{2}\left(1 - \frac{x}{2}\right) \geqslant \frac{1}{4}$ for every $0 < x \leqslant 1$. The operator M_u is not injective, for $\ker M_u$ is non-trivial (for example, $\chi_{[0,\frac{1}{2})} \in \ker M_u$). In fact, since $\ker uC_\varphi{}^*$ is also non-trivial (so that $\dim\ker uC_\varphi{}^* = \infty$ by Lemma 1.1), uC_φ is not Fredholm at all.

EXAMPLE 1.1. The composition operator C_φ on l^2 induced by

$$\varphi(n) := \begin{cases} 1 & \text{if } n = 1,2, \\ n-1 & \text{if } n = 3,4,\dots, \end{cases}$$

is Fredholm, since $\dim\ker C_\varphi = 0$ and $\dim l^2/\operatorname{ran}(C_\varphi) = \dim\ker {C_\varphi}^* = 1$. However, it is not invertible. This example shows that when (X,Σ,μ) contains atoms, a Fredholm (weighted) composition operator on $L^p(\mu)$ is *not* necessarily invertible.

EXAMPLE 1.2. Let $X = [1,\infty)$ and Σ be the σ-algebra of Borel sets in X with the Lebesgue measure μ. Define $\varphi(x) = \sqrt{x}$ for all $x \in X$. By taking $u_1(x) = \frac{1}{1+x}$ and $u_2(x) = \frac{1}{1+\sqrt{x}}$, we have

$$\frac{\int_{\varphi^{-1}([1,x))} u_1\, d\mu}{\mu([1,x))} = \frac{\log\left(\frac{1+x^2}{2}\right)}{x-1} \to 0 \quad \text{as } x \to \infty,$$

and

$$\frac{\int_{\varphi^{-1}([1,x))} u_2\, d\mu}{\mu([1,x))} = \frac{\int_1^{x^2} \frac{1}{1+\sqrt{t}}\, dt}{x-1} \geqslant 1 \quad \text{for each } x > 1.$$

From Theorem 1.2, $u_2 C_\varphi$ is a Fredholm (and invertible) operator on $L^1(\mu)$, whereas $u_1 C_\varphi$ is not. Since $\varphi^{-1}\Sigma = \Sigma$ and $u_1 \neq 0$ on X, the range of $u_1 C_\varphi$ is dense in $L^1(\mu)$.

In light of Example 1.1, we now characterize the classes of Fredholm and invertible weighted composition operators on l^p by generalizing the methods in [5] and [13]. For every $n \in \mathbb{N}$, define

$$S_n := \varphi^{-1}(\{n\}) \cap \operatorname{coz} u,$$

where $\operatorname{coz} u$ is the cozero set of u on $\mathbb{N}$, i.e. $\operatorname{coz} u := \{k \in \mathbb{N} : u(k) \neq 0\}$. Observe that $S_n \neq \emptyset$ if $n \in \varphi(\operatorname{coz} u)$.

The cardinality of a subset C of $\mathbb{N}$ is denoted by $|C|$. It is useful to compute the dimensions of both $\dim\ker uC_\varphi$ and $\dim\ker u{C_\varphi}^*$ first.

LEMMA 1.3. *Let uC_φ be a weighted composition operator on l^p. Then*

(a) $\dim\ker uC_\varphi = |\mathbb{N} \setminus \varphi(\operatorname{coz} u)|$.

(b) $\dim\ker u{C_\varphi}^* = |\mathbb{N} \setminus \operatorname{coz} u| + \sum_{n \in \varphi(\operatorname{coz} u)} (|S_n| - 1)$.

Proof. We first prove (a). Let $x = \{x_k\}_{k=1}^\infty$ be a sequence in l^p such that $uC_\varphi x = 0$, the zero sequence. Then $u(k)x_{\varphi(k)} = 0$ for all $k \in \mathbb{N}$. If $k \in \operatorname{coz} u$, we have $x_{\varphi(k)} = 0$. Thus,

$$\ker uC_\varphi = \{\{x_k\}_{k=1}^\infty \in l^p : x_k = 0 \text{ if } k \in \varphi(\operatorname{coz} u)\}.$$

A basis for $\ker uC_\varphi$ is $\{e_n : n \notin \varphi(\operatorname{coz} u)\}$ and so $\dim\ker uC_\varphi = |\mathbb{N} \setminus \varphi(\operatorname{coz} u)|$.

To prove (b), suppose that $\{w_k\}_{k=1}^{\infty}$ is a sequence in l^q, where q is the conjugate exponent of p, for which

$$\sum_{k=1}^{\infty} u(k)x_{\varphi(k)}\overline{w_k} = 0 \qquad \text{for all } x = \{x_k\}_{k=1}^{\infty} \in l^p.$$

Then

$$\begin{aligned} 0 &= \sum_{k\in \operatorname{coz} u} u(k)x_{\varphi(k)}\overline{w_k} \\ &= \sum_{n\in\varphi(\operatorname{coz} u)} \sum_{k\in S_n} u(k)x_{\varphi(k)}\overline{w_k} \\ &= \sum_{n\in\varphi(\operatorname{coz} u)} \left(\sum_{k\in S_n} u(k)\overline{w_k}\right) x_n. \end{aligned}$$

By taking $x = e_n$ for each $n \in \varphi(\operatorname{coz} u)$, we have

$$\sum_{k\in S_n} u(k)\overline{w_k} = 0.$$

Hence

$$\ker uC_\varphi{}^* = \left\{ \{w_k\}_{k=1}^{\infty} \in l^q : \sum_{k\in S_n} \overline{u(k)}w_k = 0 \text{ for every } n \in \varphi(\operatorname{coz} u) \right\}$$

(here we identify a linear functional in $\ker uC_\varphi{}^*$ with the representing sequence in l^q) and $\dim\ker uC_\varphi{}^* = |\mathbb{N}\setminus \operatorname{coz} u| + \sum_{n\in\varphi(\operatorname{coz} u)}(|S_n|-1)$. $\square$

LEMMA 1.4. *A weighted composition operator uC_φ on l^p has closed range if and only if there exists a constant $\delta > 0$ such that*

$$\sum_{k\in S_n} |u(k)|^p \geqslant \delta \qquad \text{for each } n \in \varphi(\operatorname{coz} u). \tag{1}$$

Proof. Let

$$l_1^p := \{\{x_k\}_{k=1}^{\infty} \in l^p : x_k = 0 \text{ if } k \in \varphi(\operatorname{coz} u)\}$$

and

$$l_2^p := \{\{x_k\}_{k=1}^{\infty} \in l^p : x_k = 0 \text{ if } k \in \mathbb{N}\setminus\varphi(\operatorname{coz} u)\}$$

be two closed subspaces of l^p. Assume that (1) holds. If $x = \{x_k\}_{k=1}^{\infty} \in l_2^p$, then

$$\begin{aligned} \left\|uC_\varphi x\right\|_{l^p}^p &= \sum_{k\in\operatorname{coz} u} |u(k)|^p \left|x_{\varphi(k)}\right|^p = \sum_{n\in\varphi(\operatorname{coz} u)} \left(\sum_{k\in S_n} |u(k)|^p\right) |x_n|^p \\ &\geqslant \delta \sum_{n\in\varphi(\operatorname{coz} u)} |x_n|^p = \delta\|x\|_{l^p}^p. \end{aligned}$$

The above inequality, together with the facts that $l_p = l_1^p \oplus l_2^p$ and $\ker uC_\varphi = l_1^p$, implies $uC_\varphi(l^p)$ is closed in l^p.

Conversely, suppose $uC_\varphi(l^p)$ is closed in l^p. Since uC_φ is injective on l_2^p and $uC_\varphi\left(l_2^p\right)$ is also closed in l^p, it follows that there is a constant $c > 0$ for which

$$\left\|uC_\varphi x\right\|_{l^p} \geqslant c\|x\|_{l^p} \quad \text{for all } x \in l_2^p.$$

In particular, with $x = e_n$ for every $n \in \varphi(\operatorname{coz} u)$, we have

$$c^p = c^p\|e_n\|_{l^p}^p \leqslant \left\|uC_\varphi e_n\right\|_{l^p}^p = \sum_{k\in S_n} |u(k)|^p.$$

The proof of the lemma is now complete. □

THEOREM 1.5. *A weighted composition operator uC_φ on l^p is Fredholm if and only if the following conditions are all satisfied:*

(i) Both sets $\mathbb{N}\setminus\operatorname{coz} u$ and $\mathbb{N}\setminus\varphi(\operatorname{coz} u)$ are finite.

(ii) φ is one-to-one on the complement of a finite subset of $\operatorname{coz} u$.

(iii) There exists a constant $\delta > 0$ such that $\sum_{k\in S_n} |u(k)|^p \geqslant \delta$ for every $n \in \varphi(\operatorname{coz} u)$.

Proof. By Lemma 1.4, the closedness of range of uC_φ is equivalent to (iii). It is evident from Lemma 1.3 that the condition $\dim\ker uC_\varphi < \infty$ is just equivalent to the finiteness of $\mathbb{N}\setminus\varphi(\operatorname{coz} u)$. An appeal to Lemma 1.3 also shows that the other condition $\dim\ker {uC_\varphi}^* < \infty$ can be expressed as the finiteness of $\mathbb{N}\setminus\operatorname{coz} u$ and the existence of the finite set $E := \bigcup_{\substack{n\in\varphi(\operatorname{coz} u)\\ |S_n|>1}} S_n$ for which φ is one-to-one on $\operatorname{coz} u\setminus E$. □

Both conditions in (iii) of Theorem 1.2 actually characterize invertible weighted composition operators from $L^p(\mu)$ onto $L^p(\nu)$ for an arbitrary (σ-finite and complete) measure space (X,Σ,μ), which is *not* necessarily non-atomic. When the L^p-spaces are sequence spaces in particular, not only the characterizations for invertible weighted maps are simpler, but also the invertibility of uC_φ and φ are related. Furthermore, the inverse of uC_φ (provided that it exists) is a weighted composition operator. While the first statement of the following result can be deduced from Theorem 1.2, it is also a straightforward consequence of Lemmas 1.3 and 1.4.

THEOREM 1.6. *A weighted composition operator uC_φ on l^p is invertible if and only if $\inf_{k\in\mathbb{N}} |u(k)| > 0$ and φ is invertible. In this case, $\left(uC_\varphi\right)^{-1} = \frac{1}{u\circ\varphi^{-1}} C_{\varphi^{-1}}$, where $\left(uC_\varphi\right)^{-1}$ and φ^{-1} are the inverses of uC_φ and φ respectively.*

Proof. We only prove the formula for $\left(uC_\varphi\right)^{-1}$. Let $T := \frac{1}{u\circ\varphi^{-1}}C_{\varphi^{-1}}$. For every $x = \{x_k\}_{k=1}^{\infty} \in l^p$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left(uC_\varphi \circ T\right)(x)(n) &= uC_\varphi\left(\left\{\frac{x_{\varphi^{-1}(k)}}{u\left(\varphi^{-1}(k)\right)}\right\}_{k=1}^{\infty}\right)(n) = u(n)\frac{x_{\varphi\left(\varphi^{-1}(n)\right)}}{u\left(\varphi\left(\varphi^{-1}(n)\right)\right)}\\ &= x_n = \frac{u\left(\varphi^{-1}(n)\right)}{u\left(\varphi^{-1}(n)\right)}x_{\varphi\left(\varphi^{-1}(n)\right)}\\ &= T\left(\left\{u(k)x_{\varphi(k)}\right\}_{k=1}^{\infty}\right)(n) = \left(T\circ uC_\varphi\right)(x)(n).\end{aligned}$$

Hence $T = \left(uC_\varphi\right)^{-1}$. $\square$

The invertibility of φ in general does not guarantee uC_φ is invertible on general L^p-spaces, and vice versa. For example, the weighted operator u_1C_φ in Example 1.2 is not invertible on $L^1(\mu)$, whereas φ is invertible on $[1,\infty)$. Another illustration is given by [12, Example 2.1]. Let $\varphi(n) := \begin{cases} n & \text{if } n \text{ is odd,}\\ n-1 & \text{if } n \text{ is even.}\end{cases}$ Then the operator C_φ is invertible on $L^2(\mathbb{N},\Sigma,\mu)$, where μ is the counting measure on $\Sigma := \{\varphi^{-1}(E) : E \in \mathscr{P}(\mathbb{N})\}$. However, φ is not onto.

2. Fredholm weighted composition operators on H^p

2.1. Preliminaries

Let D be the unit disk $\{z\in\mathbb{C} : |z|<1\}$ in the complex plane $\mathbb{C}$ and T be the unit circle $\{z\in\mathbb{C} : |z|=1\}$. The Hardy space H^p, where $1\leqslant p<\infty$, of D consists of all analytic functions f on D such that

$$\sup_{0\leqslant r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta < \infty.$$

We define H^∞ to be the set of all functions f which are analytic and bounded on D.

Let m be the normalized Lebesgue measure on T, i.e. $dm := \frac{d\theta}{2\pi}$, and write $L^p = L^p(m)$ in the sequel. Norms of H^p and L^p are both denoted by $\|\cdot\|_p$. Given that $f\in H^p$ for $1\leqslant p\leqslant\infty$, its radial limit

$$\hat{f}(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

exists m-a.e. on T, and $\hat{f}\in L^p$ with $\|\hat{f}\|_p = \|f\|_p$. If, in addition, $f\not\equiv 0$, then $\hat{f}\neq 0$ m-a.e. on T. Suppose that $z = re^{it}$ for $0\leqslant r<1$ and $0\leqslant t<2\pi$. The functions f and $\hat{f}$ are related by the equality

$$f(z) = \int_0^{2\pi} P_r(t-\theta)\hat{f}(e^{i\theta})\,dm,$$

where P_r is the Poisson kernel defined by $P_r(\theta) := \frac{1-r^2}{1-2r\cos\theta+r^2}$.

We may consider the extension of f to $\overline{D} := \{z \in \mathbb{C} : |z| \leqslant 1\}$, also denoted by f, such that $f|_T = \hat{f}$.

Fix an arbitrary point ω in D. The evaluation functional at $z = \omega$, denoted by δ_ω, is given by

$$\delta_\omega(f) := f(\omega) \quad \text{for each } f \in H^p.$$

It is bounded, and $\|\delta_\omega\| = \left(\frac{1}{1-|\omega|^2}\right)^{1/p}$ if $1 \leqslant p < \infty$. Thus, if $f \in H^p$, then

$$|f(\omega)| \leqslant \frac{\|f\|_p}{(1-|\omega|^2)^{1/p}}.$$

It can also be shown that if $f \in H^p$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence in D such that $|z_n| \to 1$, then $\left(1-|z_n|^2\right)^{1/p} f(z_n) \to 0$.

Let u and φ be two analytic functions on D such that $\varphi(D) \subset D$. They induce a *weighted composition operator* uC_φ from H^p into the linear space of all analytic functions on D by

$$uC_\varphi(f)(z) := u(z)f(\varphi(z)) \quad \text{for every } f \in H^p \text{ and } z \in D.$$

When $u \equiv 1$ (resp. $\varphi(z) = z$ for all $z \in D$), the corresponding operator, denoted by C_φ (resp. by M_u), is known as a *composition operator* (resp. a *multiplication operator*). To avoid triviality, we assume both u and φ are non-constant functions. All the three operators C_φ, M_u and uC_φ are then injective.

It is well-known that C_φ is always bounded on H^p for $1 \leqslant p \leqslant \infty$. This is not necessarily true for weighted composition operators. If uC_φ maps H^p into itself, an appeal to the closed graph theorem yields its boundedness. We say uC_φ is a weighted composition operator on H^p. Moreover,

$$\left(uC_\varphi{}^*\delta_\omega\right)(f) = \delta_\omega(uC_\varphi f) = u(\omega)f(\varphi(\omega)) = u(\omega)\delta_{\varphi(\omega)}(f)$$

for all $f \in H^p$, i.e.

$$uC_\varphi{}^*\delta_\omega = u(\omega)\delta_{\varphi(\omega)}.$$

Suppose $1 \leqslant p < \infty$. Then

$$|u(\omega)|^p \|\delta_{\varphi(\omega)}\|^p = \left\|uC_\varphi{}^*\delta_\omega\right\|^p \leqslant \left\|uC_\varphi{}^*\right\|^p \|\delta_\omega\|^p,$$

which gives

$$|u(\omega)|^p \leqslant \left(\frac{1-|\varphi(\omega)|^2}{1-|\omega|^2}\right) \left\|uC_\varphi{}^*\right\|^p. \tag{2}$$

2.2. Main results

Assume that $1 \leqslant p < \infty$ in this sub-section. We first characterize invertible weighted composition operators on H^p.

THEOREM 2.1. *Let uC_φ be a weighted composition operator on H^p. Then it is invertible if and only if both the following conditions hold:*

(i) φ *is an automorphism of* D.

(ii) There exists a constant $\delta > 0$ *such that* $|u| \geqslant \delta$ *on* D.

Proof. Assume uC_φ is invertible on H^p. As $1 \in \operatorname{ran}(uC_\varphi)$, we have $u \neq 0$ on D. To prove (i), it suffices to show that φ is univalent and surjective. If φ were *not* univalent, then there exist distinct points a, b in D with $\varphi(a) = \varphi(b)$. Let

$$\phi := \frac{1}{u(a)}\delta_a - \frac{1}{u(b)}\delta_b,$$

where δ_a and δ_b are the evaluation functionals (on H^p) at $z = a$ and $z = b$ respectively. Note that $\phi \not\equiv 0$ for

$$\phi(z-b) = \frac{1}{u(a)}\delta_a(z-b) - \frac{1}{u(b)}\delta_b(z-b) = \frac{a-b}{u(a)} \neq 0.$$

However,

$$uC_\varphi{}^*\phi = \frac{1}{u(a)}uC_\varphi{}^*\delta_a - \frac{1}{u(b)}uC_\varphi{}^*\delta_b = \frac{1}{u(a)}\cdot u(a)\delta_{\varphi(a)} - \frac{1}{u(b)}\cdot u(b)\delta_{\varphi(b)} \equiv 0.$$

This contradicts the injectivity of $uC_\varphi{}^*$. Thus, φ is univalent.

Next we prove φ is also surjective. Assuming the contrary, i.e. $\varphi(D) \neq D$, one may exhibit a point α in $D \backslash \varphi(D)$ and a sequence $\{z_n\}_{n=1}^\infty$ in D such that this sequence converges and $\varphi(z_n) \to \alpha$. In fact, $|z_n| \to 1$. Define

$$\phi_n := \left(1 - |z_n|^2\right)^{1/p} \delta_{z_n}$$

for $n \in \mathbb{N}$. Then, $\|\phi_n\| = 1$ and

$$\left\|uC_\varphi{}^*\phi_n\right\| = \left(1 - |z_n|^2\right)^{1/p} \left\|uC_\varphi{}^*\delta_{z_n}\right\| = \frac{|u(z_n)|\left(1-|z_n|^2\right)^{1/p}}{\left(1-|\varphi(z_n)|^2\right)^{1/p}} \to 0.$$

On the other hand, the surjectivity of uC_φ implies there is a constant $c > 0$ with

$$\left\|uC_\varphi{}^*\phi_n\right\| \geqslant c\,\|\phi_n\| = c \qquad \text{for all } n. \tag{3}$$

This contradiction shows that φ maps D onto D.

It remains to prove (ii). Fix any $\omega \in D$. With the constant c in (3), we have

$$\left\|uC_\varphi{}^*\delta_\omega\right\| \geqslant c\,\|\delta_\omega\|.$$

Thus,

$$|u(w)|^p \geqslant \frac{1-|\varphi(\omega)|^2}{1-|\omega|^2}c^p.$$

In view of (i), we may write $\varphi(\omega)=\zeta\frac{\beta-\omega}{1-\overline{\beta}\omega}$ for some $\beta\in D$ and $\zeta\in T$. Then

$$1-|\varphi(\omega)|^2=\frac{(1-|\beta|^2)(1-|\omega|^2)}{|1-\overline{\beta}\omega|^2}.$$

It follows that

$$\frac{1-|\varphi(\omega)|^2}{1-|\omega|^2}=\frac{1-|\beta|^2}{|1-\overline{\beta}\omega|^2}\geqslant\frac{1-|\beta|^2}{(1+|\beta|)^2}=\frac{1-|\beta|}{1+|\beta|}.$$

Therefore,

$$|u(\omega)|\geqslant c\left(\frac{1-|\beta|}{1+|\beta|}\right)^{1/p}.$$

Conversely, suppose both (i) and (ii) are satisfied. It suffices to show uC_φ is surjective. The first condition ensures the operator C_φ is surjective. Choose any function $g\in H^p$. Thanks to (ii), we also have $\frac{g}{u}\in H^p$. Then, there exists a function $f\in H^p$ with $C_\varphi f=\frac{g}{u}$, or $uC_\varphi f=g$. The proof of the theorem is now complete. □

Gunatillake [4, Theorem 2.0.1] also obtained a similar characterization for invertible weighted composition operators on H^2 with a slightly different method. In [2, Theorem 1], Cima et al. showed that a composition operator on H^2 is Fredholm if and only if it is invertible, i.e. it is induced by an automorphism. Bourdon [1] proved the same result by characterizing finite co-dimensional invariant subspaces of H^p as follows.

LEMMA 2.2. *Let $h\in H^\infty$. The following two statements are equivalent:*

(i) h is univalent on D.

(ii) Every closed finite co-dimensional subspace of H^p that is invariant under M_h has the form BH^p, where B is a finite Blaschke product.

Applying this lemma and Theorem 2.1, we generalize the characterizations for Fredholm weighted composition operators in [16, Theorems 1.1 and 1.2] to any H^p-space. The Fredholm indices of these operators are also determined.

THEOREM 2.3. *Let uC_φ be a weighted composition operator on H^p. Then it is Fredholm if and only if both the following conditions hold:*

(i) φ is an automorphism of D.

(ii) $\liminf_{|z|\to1^-}|u(z)|>0$

In this case, the Fredholm index of uC_φ is $-n$, where n is the number of zeros of u on D counting multiplicities.

Proof. We first observe that since polynomials are dense in H^p and $C_\varphi(zf) = \varphi C_\varphi f$ for all polynomials f, the norm-closure of $\operatorname{ran}(uC_\varphi)$ is an invariant subspace of H^p under multiplication by φ. Suppose uC_φ is Fredholm. Then φ must be univalent on D. Otherwise, there exist two distinct points a and b in D with $\varphi(a) = \varphi(b)$. Following the argument of the lemma in [1], we choose some $\varepsilon > 0$ for which both sets $\{z \in \mathbb{C} : |z-a| \leqslant \varepsilon\}$ and $\{z \in \mathbb{C} : |z-b| \leqslant \varepsilon\}$ are contained in D. Moreover, we may extract two sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ in D such that $a_i \neq b_j$ whenever $i \neq j$ and $\varphi(a_n) = \varphi(b_n)$ for all n.

The analyticity of u implies that $u(a_n) = u(b_n) = 0$ for finitely many a_n's and b_n's only. Without loss of generality, we assume $u(a_n), u(b_n) \neq 0$ for all n. Define

$$\phi_n := \frac{1}{u(a_n)}\delta_{a_n} - \frac{1}{u(b_n)}\delta_{b_n} \qquad \text{for } n \in \mathbb{N}.$$

These ϕ_n's are linearly independent. As in the proof of Theorem 2.1, we have $\phi_n \in \ker uC_\varphi{}^*$. This contradicts the assumption that $\dim H^p/\operatorname{ran}\left(uC_\varphi\right) < \infty$.

By Lemma 2.2, there is a finite Blaschke product B such that $\operatorname{ran}(uC_\varphi) = BH^p$. In particular, $u = Bg$ for a function $g \in H^p$. Thus, $\operatorname{ran}(gC_\varphi) = H^p$. If g is constant on D, then both (i) and (ii) follow immediately. When g is non-constant, it follows from Theorem 2.1 that φ is also surjective and there is a constant $\delta > 0$ such that $|g| \geqslant \delta$ on D. With $\lim_{|z|\to 1^-} |B(z)| = 1$, we thus obtain $\liminf_{|z|\to 1^-} |u(z)| \geqslant \delta > 0$.

Conversely, assume both (i) and (ii) hold. By (ii), there exist constants $c, r > 0$ such that $|u(z)| \geqslant c$ if $r < |z| < 1$. Moreover, the number of zeros of u on $\{z \in \mathbb{C} : |z| \leqslant r\}$ is finite. We claim that

$$\operatorname{ran}(uC_\varphi) = BH^p,$$

where B is the finite Blaschke product associated with the zeros of u on D. To verify this, we write $u = Bh$ for some $h \in H^p$ with $h \neq 0$ on D. Then $\operatorname{ran}(hC_\varphi) \subset H^p$. As h is continuous for $|z| \leqslant r$ and $|h| \geqslant c$ for $r < |z| < 1$, we see that h is bounded away from zero on D. By Theorem 2.1, we conclude that $\operatorname{ran}(hC_\varphi) = H^p$. The claim now follows.

It remains to consider the codimension of BH^p in H^p. Assume the zeros of u on D, namely $z_1, z_2, \ldots, z_n$, are all simple (in case u has multiple zeros, we may modify the argument slightly by using a Hermite interpolating polynomial). The kernel and the range of the linear map on H^p given by $f \mapsto \sum_{i=1}^n f(z_i)z^i$ are BH^p and the linear span of $z, z^2, \ldots, z^n$ respectively. Therefore, $\dim H^p/BH^p = \dim \operatorname{span}\left\{z, z^2, \ldots, z^n\right\} = n$. This, together with the injectivity of uC_φ, yields $\operatorname{ind} uC_\varphi = -n$. $\square$

NOTE 2.1. Two simple necessary conditions for Fredholmness of uC_φ on H^p are

(a) $u \in H^\infty$ and

(b) the number of zeros of u on D is finite.

That (b) holds has been shown in the proof of Theorem 2.3. For (a), since φ is a disk automorphism, an argument similar to the proof of Theorem 2.1 gives

$$\frac{1-|\varphi(\omega)|^2}{1-|\omega|^2} \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

From the above inequality and that in (2), we have

$$\|u\|_\infty \leqslant \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/p} \left\|uC_\varphi{}^*\right\|.$$

In view of Theorems 2.1, 2.3 and the above note, the operator uC_φ is Fredholm (resp. invertible) on H^p if and only if both M_u and C_φ are Fredholm (resp. invertible) on H^p. We also remark that a Fredholm weighted composition operator uC_φ on H^p is not necessarily invertible (compare this with Theorem 1.2). The weight function u of a Fredholm weighted composition operator is bounded away from zero *near* T, and it may vanish on D; while that of an invertible weighted map is to be bounded away from zero *on* D.

Similar characterizations for Fredholm (resp. invertible) weighted composition operators on H^∞ have been obtained by Ohno et al. in [11, Theorems 2.3 and 2.4]. In this paper, they also characterized weighted composition operators on H^∞ with closed ranges by applying the Banach algebra structure of H^∞. We now study the closedness of ranges of weighted composition operators on H^p à la the method of Cima et al. [2, Theorem 2], who characterized those composition operators on H^2 with closed ranges. To this end, define a measure m_p on $\overline{D}$ by

$$m_p(E) := \int_{\varphi^{-1}(E)\cap T} |u|^p\, dm$$

for every measurable subset E of $\overline{D}$. By [3, Lemma 2.1],

$$\int_T |u|^p (f\circ\varphi)\, dm = \int_{\overline{D}} f\, dm_p,$$

where f is an arbitrary measurable positive function on $\overline{D}$. If we restrict m_p to all the measurable subsets of T, then $m_p(E) = \int_{\varphi^{-1}(E)} |u|^p dm$ for all such sets E. This measure, denoted by m_p as well, is absolutely continuous with respect to m:

PROPOSITION 2.4. *Let uC_φ be a weighted composition operator on H^p. Then, m_p is absolutely continuous with respect to m and $\left[\frac{dm_p}{dm}\right] \in L^\infty$, where $\left[\frac{dm_p}{dm}\right]$ is the corresponding Radon-Nikodym derivative.*

Proof. In view of [10, Lemma 1.3], it suffices to prove that there exists a constant $c > 0$ such that

$$m_p(Q(\zeta, r)) \leqslant cr$$

for all $\zeta \in T$ and $0 < r < 1$, where $Q(\zeta, r) := \{z \in T : |z-\zeta| \leqslant r\}$. By the boundedness of uC_φ, we have $\left\|uC_\varphi f\right\|_p^p \leqslant \left\|uC_\varphi\right\|^p \|f\|_p^p$, i.e.

$$\int_{\overline{D}} |f|^p dm_p = \int_T |u|^p |f|^p \circ \varphi\, dm \leqslant \left\|uC_\varphi\right\|^p \|f\|_p^p \quad \text{for every } f \in H^p. \tag{4}$$

With the above ζ and r, we let $\omega = (1-r)\zeta$. Consider the function $g(z) := \frac{1}{(1-\overline{w}z)^{4/p}}$. A direct computation gives

$$\|g\|_p^p = \frac{1+(1-r)^2}{r^3(2-r)^3}.$$

Since

$$|1-\overline{w}z| = |1-(1-r)\overline{\zeta}z| \leqslant |\overline{\zeta}||z-\zeta| + |r\overline{\zeta}z| \leqslant 2r \quad \text{for } z \in Q(\zeta,r),$$

we see that

$$|g| \geqslant \frac{1}{(2r)^{4/p}} \quad \text{on } Q(\zeta,r).$$

Now, it follows from (4) that

$$\begin{aligned}\frac{m_p(Q(\zeta,r))}{(2r)^4} &\leqslant \int_{Q(\zeta,r)} |g|^p dm_p \leqslant \int_{\overline{D}} |g|^p dm_p \\ &\leqslant \left\|uC_\varphi\right\|^p \|g\|_p^p = \left\|uC_\varphi\right\|^p \cdot \frac{1+(1-r)^2}{r^3(2-r)^3}.\end{aligned}$$

Thus,

$$m_p(Q(\zeta,r)) \leqslant 16\left\|uC_\varphi\right\|^p \cdot \frac{1+(1-r)^2}{(2-r)^3} r \leqslant 32\left\|uC_\varphi\right\|^p r. \quad \square$$

THEOREM 2.5. *Let uC_φ be a weighted composition operator on H^p. The following statements are equivalent:*

(i) uC_φ has closed range.

(ii) There exists a constant $\delta > 0$ such that $\left[\frac{dm_p}{dm}\right] \geqslant \delta$ m-a.e. on T, where $\left[\frac{dm_p}{dm}\right]$ is defined in Proposition 2.4.

(iii) There exists a constant $c > 0$ such that $\int_{\varphi^{-1}(E)} |u|^p dm \geqslant c\, m(E)$ for all measurable sets E of T.

Proof. The equivalence of (ii) and (iii) is clear. Moreover, (i) follows from (ii) because

$$\left\|uC_\varphi f\right\|_p^p = \int_T |u|^p |f|^p \circ \varphi\, dm \geqslant \int_T |f|^p dm_p = \int_T \left[\frac{dm_p}{dm}\right] |f|^p dm \geqslant \delta \|f\|_p^p$$

for each $f \in H^p$.

It remains to show that (i) implies (ii). Assume (ii) does not hold. Then the sets

$$E_k := \left\{ z \in T : \left[\frac{dm_p}{dm}\right](z) < \frac{1}{k} \right\} \quad \text{where } k \in \mathbb{N},$$

are of positive m-measures. We may also assume $m(T\setminus E_k) > 0$ for each k. Let $f_k\colon D \to \mathbb{C}$ be an outer function in H^p such that

$$|f_k| = \begin{cases} 1 & \text{on } E_k, \\ \dfrac{1}{2} & \text{on } T\setminus E_k. \end{cases}$$

Let n and k be positive integers with k fixed. Then

$$\|f_k^n\|_p^p = m(E_k) + \left(\frac{1}{2}\right)^{np} m(T\setminus E_k) \to m(E_k) \quad \text{as } n \to \infty. \tag{5}$$

Moreover,

$$\begin{aligned} \left\|uC_\varphi f_k^n\right\|_p^p &= \int_{E_k} |f_k|^{np}\, dm_p + \int_{T\setminus E_k} |f_k|^{np}\, dm_p + \int_D |f_k|^{np}\, dm_p \\ &\leqslant m_p(E_k) + \left(\frac{1}{2}\right)^{np} m_p(T\setminus E_k) + \int_D |f_k|^{np}\, dm_p. \end{aligned}$$

Note that

$$|f_k(z)| = \exp\left\{\log\frac{1}{2}\left[\int_{T\setminus E_k} P_r(t-\theta)\, dm\right]\right\},$$

where $z = re^{it}$ and P_r is the Poisson kernel. Since $0 < \int_{T\setminus E_k} P_r(t-\theta)\, dm < 1$, we have $|f_k(z)| < 1$ on D. From the dominated convergence theorem,

$$\int_D |f_k|^{np}\, dm_p \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\limsup_{n\to\infty} \left\|uC_\varphi f_k^n\right\|_p^p \leqslant m_p(E_k). \tag{6}$$

In view of (5) and (6), we choose a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\left\|f_k^{n_k}\right\|_p^p > \frac{1}{2} m(E_k) \quad \text{and} \quad \left\|uC_\varphi f_k^{n_k}\right\|_p^p < 2m_p(E_k) \quad \text{for all } k.$$

Hence

$$\frac{\left\|uC_\varphi f_k^{n_k}\right\|_p^p}{\left\|f_k^{n_k}\right\|_p^p} < \frac{4m_p(E_k)}{m(E_k)} = \frac{4}{m(E_k)} \int_{E_k} \left[\frac{dm_p}{dm}\right] dm \leqslant \frac{4}{k} \to 0 \quad \text{as } k \to \infty.$$

This shows that the range of uC_φ is not closed. $\square$

The above characterization of a weighted composition operator on H^p with closed range involves the Radon-Nikodym derivative of the measure m_p. It is desirable to characterize its closedness of range more explicitly in terms of function-theoretic properties (for example, ranges) of the symbol functions u and φ. While this awaits further investigation, the corresponding problem for composition operators has been considered in [8, Theorem 5.1]. It was shown that a composition operator C_φ on H^p has closed range if and only if there exists a constant $c>0$ such that if $0<r<1$ and $\zeta \in T$, then

$$\frac{1}{A(S(\zeta,r))}\int_{S(\zeta,r)} N_\varphi(z)\,dA(z) \geqslant c\,r,$$

where

(a) $S(\zeta,r) := \{z \in D : |z-\zeta| \leqslant r\}$;

(b) A is the normalized Lebesgue area measure on D, i.e. $dA = \frac{1}{\pi} r dr d\theta$; and

(c) N_φ is the Nevanlinna counting function given by

$$N_\varphi(\omega) := \begin{cases} \sum\limits_{z\in\varphi^{-1}\{\omega\}} \log\dfrac{1}{|z|} & \text{if } \omega \in \varphi(D)\setminus\{\varphi(0)\}, \\ 0 & \text{if } \omega \notin \varphi(D), \end{cases}$$

and $\varphi^{-1}\{\omega\}$ denotes the sequence of φ-preimages of ω with each point occurring as many times as its multiplicity.

For the case of composition operators, it is interesting to see the measure-theoretic conditions (ii) and (iii) in Theorem 2.5 are equivalent to the above function-theoretic conditions.

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