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ALL NON–COMMUTING SOLUTIONS OF THE YANG–BAXTER MATRIX EQUATION FOR A CLASS OF DIAGONALIZABLE MATRICES

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Abstract. Let A be a complex matrix such that its square equals the identity matrix. We solve the matrix equation $AXA = XAX$ to construct all the solutions. This finishes our task for finding all the non-commuting solutions, continued from the authors' previous work.

1. Introduction

Let A be an $n \times n$ complex matrix. The purpose of this paper is to solve the quadratic matrix equation

$$AXA = XAX \tag{1}$$

completely when A satisfies the equality $A^2 = I$. The equation for the same class of the matrix A has been studied in our recent paper [9] under some additional assumption for the solutions. The contribution of the current paper is that we shall find all the solutions without any condition for them. The quadratic matrix equation (1) is called the *Yang-Baxter matrix equation* since it has been extensively studied several years ago from the viewpoint of linear algebra. The reason for the name of the equation is that it has a similarity in format to the famous Yang-Baxter equation in statistical mechanics [1, 10] initiated by Yang in 1967 and by Baxter in 1972. For applications of the Yang-Baxter equation to various areas of mathematics and physics, see, e.g., the monographs [7, 11] and the references therein.

Since finding general solutions of the nonlinear matrix equation (1) is difficult, almost all the works so far have been toward constructing commuting solutions of the equation; see, e.g., [2, 3] and the references therein. In [5], all the commuting solutions were obtained when A is diagonalizable. In the more general case of A being non-diagonalizable, the structure of all commuting solutions has been obtained [6] for the class of nilpotent matrices. But still the explicit expression of all solutions of (1) is not available except for some special classes of matrices, such as that of idempotent matrices [8].

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For the matrices considered in this paper, whose inverses are themselves, since A is diagonalizable, all the commuting solutions can be obtained by the result of [5] (see also Section 2 of [9]). Thus, our purpose is to get all the non-commuting solutions. In an earlier paper [9], we found all the non-commuting solutions with the assumption that the off-diagonal blocks of the solution as a 2×2 block matrix are full ranked. In this paper, we use a different approach based a classic spectral perturbation result, so that we can drop the above assumption and find all the solutions unconditionally.

We first give a preliminary result in the next section concerning a rank relation between the off-diagonal blocks of the 2×2 block matrix solution. Then we prove our main result in Section 3. We conclude with Section 4.

2. A rank relation

Let A be an $n \times n$ complex matrix such that its square equals the identity matrix. From our previous paper [9] we know that the corresponding matrix equation (1) is equivalent to the matrix equation

$$JYJ = YJY, \tag{2}$$

where the Jordan form of A is

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix}.$$

Their equivalence is in the sense that X is a solution of (1) if and only if Y is a solution of (2) with the relation $X = SYS^{-1}$, where S is the nonsingular matrix satisfying $A = SJS^{-1}$. Here I_s denotes the $s \times s$ identity matrix. With Y partitioned into

$$Y = \begin{bmatrix} K & C \\ D & Z \end{bmatrix},$$

where K is $m \times m$ and Z is $(n-m) \times (n-m)$, (2) is equivalent to the following system of four equations for the sub-matrices of Y:

$$\begin{cases} K^2 - K &= CD, \\ Z^2 + Z &= DC, \\ (K+I)C &= CZ, \\ D(K+I) &= ZD. \end{cases} \tag{3}$$

By Theorem 2.1 of [9], all the commuting solutions of (2), i.e., the solutions Y satisfying $JY = YJ$, are $\operatorname{diag}(K,Z)$ such that $K^2 = K$ and $Z^2 = -Z$. It was also proved therein that any non-commuting solution of (2) must have $C \neq 0$ and $D \neq 0$. Furthermore, when $m = 1$ or $m = n-1$, all the non-commuting solutions have been constructed. And the case that both C and D are full ranked has also been solved completely. Hence, our only remaining case before completely solving (1) when $A^2 = I$ is that the off-diagonal sub-matrix C or D of the 2×2 block matrix Y is rank deficient.

From the analysis of [9], the above system (3) is equivalent to

$$\begin{cases} K^2 - K = CD, \\ Z^2 + Z = DC, \\ KC = -\frac{1}{2}C, \\ DK = -\frac{1}{2}D, \\ CZ = \frac{1}{2}C, \\ ZD = \frac{1}{2}D. \end{cases} \tag{4}$$

The new system indicates that all nonzero columns of C and D are eigenvectors of K and Z associated with eigenvalues $-1/2$ and $1/2$, respectively, and all nonzero rows of C and D are left eigenvectors of Z and K associated with eigenvalues $1/2$ and $-1/2$, respectively. Consequently for any non-commuting solution (K,Z,C,D) of the system (3), $-1/2$ and $1/2$ must be an eigenvalue of K and Z, respectively.

To find all non-commuting solutions, we need the following lemma. Let $r(M)$ denote the rank of matrix M.

LEMMA 1. *If (K,Z,C,D) forms a solution of (4), then*

$$r(C) = r(D).$$

Proof. Suppose (K,Z,C,D) satisfies (4). Multiplying C to the first equality from the right and using the third equality, we obtain $CDC = (K^2 - K)C = (3/4)C$, so

$$r(C) = r(CDC) \leqslant r(CD) \leqslant r(C),$$

from which $r(C) = r(CD)$. Similarly, multiplying D to the first equality from the left and using the fourth equality, we have $DCD = D(K^2 - K) = (3/4)D$, so

$$r(D) = r(DCD) \leqslant r(CD) \leqslant r(D),$$

from which $r(D) = r(CD)$. Hence, $r(C) = r(D)$. $\square$

REMARK 1. From the proof of Lemma 2.1, for all solutions (K,Z,C,D),

$$r(C) = r(D) = r(CD) = r(DC) = r(CDC) = r(DCD).$$

3. Solutions of the matrix equation

The proof of our main theorem below will need a spectral perturbation result from [4], which is stated below as a lemma. We use p_W to denote the characteristic polynomial of matrix W and $\sigma(W)$ the set of all eigenvalues of W which are zeros of p_W.

LEMMA 2. *Let $k \leqslant n$ be two positive integers, and let W and Λ be $n \times n$ and $k \times k$ matrices, respectively. If there is an $n \times k$ matrix U that satisfies $WU = U\Lambda$, then the identity*

$$p_{W+UV^T}(\lambda) \cdot p_\Lambda(\lambda) \equiv p_W(\lambda) \cdot p_{\Lambda+V^TU}(\lambda)$$

is true for any $n \times k$ matrix V.

We are ready to prove the main result of the paper:

THEOREM 1. *Let A be an $n \times n$ a complex matrix satisfying $A^2 = I$, and let m be the algebraic multiplicity of its eigenvalue 1. Then all the non-commuting solutions of the Yang-Baxter matrix equation (1) are exactly*

$$X = S \begin{bmatrix} K & C \\ D & Z \end{bmatrix} S^{-1},$$

where K is any $m \times m$ diagonalizable matrix and Z is any $(n-m) \times (n-m)$ diagonalizable matrix such that

(i) the nonzero matrices C and D have the same rank r such that

$$CDC = \frac{3}{4}C \text{ and } DCD = \frac{3}{4}D;$$

(ii) K and Z have eigenvalue $-1/2$ and $1/2$ of multiplicity r, respectively;

(iii) the nonzero columns of C and nonzero rows of D are eigenvectors and left eigenvectors of K respectively associated with eigenvalue $-1/2$, and the nonzero columns of D and nonzero rows of C are eigenvectors and left eigenvectors of Z respectively associated with eigenvalue $1/2$;

(iv) the other eigenvalues of K and Z belong to $\{0,1\}$ and $\{0,-1\}$, respectively.

Proof. We first show that any non-commuting solution (K,Z,C,D) of (4) must satisfy (i)–(iv). By Lemma 2.1, $r(C) = r(D) = r$. The first and second equalities of (4) give

$$CDC = \frac{3}{4}C \text{ and } DCD = \frac{3}{4}D,$$

so (i) is true. Since $C \neq 0$ and $D \neq 0$, the matrix K has eigenvalue $-1/2$ with nonzero columns of C as corresponding eigenvectors and the matrix Z has eigenvalue $1/2$ with nonzero columns of D as corresponding eigenvectors. We claim that $-1/2$ and $1/2$ are semi-simple eigenvalues of K and Z, respectively.

If eigenvalue $-1/2$ of K is not semi-simple, then there exists a nonzero vector v satisfying $u \equiv (K + (1/2)I)v \neq 0$ and $(K + (1/2)I)u = (K + (1/2)I)^2 v = 0$. The eigenvector u and the generalized eigenvector v must be linearly independent. In fact, if $\alpha u + \beta v = 0$ for some $\alpha, \beta \in \mathbb{C}$, then via multiplying this equality by $K + (1/2)I$ from the left we get $\beta(K + (1/2)I)v = \beta u = 0$, which implies that $\beta = 0$ and so $\alpha = 0$.

Since

$$CDv = (K^2 - K)v = \left(K + \frac{1}{2}I\right)^2 v - 2\left(K + \frac{1}{2}I\right)v + \frac{3}{4}v = -2u + \frac{3}{4}v,$$

$u = (3/8)v - (1/2)CDv$, and thus

$$0 = \left(K + \frac{1}{2}I\right)^2 v = \left(K + \frac{1}{2}I\right) u = \left(K + \frac{1}{2}I\right)\left(\frac{3}{8}v - \frac{1}{2}CDv\right) = \frac{3}{8}u \neq 0.$$

This is a contradiction. Therefore $-1/2$ is a semi-simple eigenvalue of K. Similarity, eigenvalues $1/2$ of Z is semi-simple. This means that (iii) is true.

To show that (iv) is also valid, we apply Lemma 3.1 to the first equality of (4). It follows that

$$p_{K+CD}(\lambda) \cdot p_{-\frac{1}{2}I_{n-m}}(\lambda) \equiv p_K(\lambda) \cdot p_{DC-\frac{1}{2}I_{n-m}}(\lambda).$$

Since $K + CD = K^2$, using the fact that the eigenvalues of the square of a matrix are the squares of the eigenvalues of the matrix, counting algebraic multiplicity, we can write the above identity as

$$\prod_{i=1}^{m}(\lambda - \lambda_i^2) \cdot \left(\lambda + \frac{1}{2}\right)^{n-m} \equiv \prod_{j=1}^{m}(\lambda - \lambda_j) \cdot \prod_{k=1}^{n-m}\left[\lambda - \left(\mu_k^2 + \mu_k - \frac{1}{2}\right)\right],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of K and $\mu_1, \mu_2, \ldots, \mu_{n-m}$ are the eigenvalues of Z, all counting algebraic multiplicity, where $\lambda_i = -1/2$ and $\mu_i = 1/2$ for $i = 1, \ldots, r$. Dividing both sides by $(\lambda + 1/2)^r$, we have

$$\prod_{i=1}^{m}(\lambda - \lambda_i^2) \cdot \left(\lambda + \frac{1}{2}\right)^{n-m-r} \equiv \prod_{j=r+1}^{m}(\lambda - \lambda_j) \cdot \prod_{k=1}^{n-m}\left[\lambda - \left(\mu_k^2 + \mu_k - \frac{1}{2}\right)\right],$$

which, since $\lambda_i^2 = \mu_k^2 + \mu_k - 1/2 = 1/4$ for $i, k = 1, \ldots, r$, can be further simplified to

$$\prod_{i=r+1}^{m}(\lambda - \lambda_i^2) \cdot \left(\lambda + \frac{1}{2}\right)^{n-m-r} \equiv \prod_{j=r+1}^{m}(\lambda - \lambda_j) \cdot \prod_{k=r+1}^{n-m}\left[\lambda - \left(\mu_k^2 + \mu_k - \frac{1}{2}\right)\right].$$

Consequently, the above identity implies that

$$\lambda_i^2 = \lambda_i,\ i = r+1, \ldots, m;\ \ \mu_k^2 = -\mu_k,\ k = r+1, \ldots, n-m.$$

Hence, $\lambda_i = 0$ or 1 for $i = r+1, \ldots, m$ and $\mu_k = 0$ or -1 for $k = r+1, \ldots, n-m$. The remaining thing is to show that such eigenvalues are semi-simple.

If 0 is an eigenvalue of K that is not semi-simple, then there is a vector $v \neq 0$ such that $u \equiv Kv \neq 0$ and $K^2 v = 0$. From $-u = (K^2 - K)v = CDv$ we obtain that

$$0 = -Ku = KCDv = (-1/2)CDv = (1/2)u \neq 0,$$

a contradiction. If 1 is an eigenvalue of K that is not semi-simple, then there is a vector $q \neq 0$ satisfying $p \equiv (K - I)q \neq 0$ and $(K - I)p = 0$. Since $CDb = (K^2 - K)q = (K - I)^2 q + (K - I)q = p$,

$$0 = (K - I)^2 q = (K - I)p = (K - I)CDq = -\frac{3}{2}CDp = -\frac{3}{2}p \neq 0,$$

another contradiction. Therefore the eigenvalues 0 and 1 are semi-simple. Similarity, the eigenvalues 0 and -1 of Z are semi-simple. Hence, K and Z are diagonalizable. This proves the necessity part of the theorem.

Conversely, suppose K is an $m\times m$ diagonalizable matrix, Z an $(n-m)\times(n-m)$ diagonalizable matrix, C an $m\times(n-m)$ matrix, and D an $(n-m)\times m$ matrix such that (K,Z,C,D) satisfies (i)–(iv). We show that it is a solution of (4). It is enough to show that the first two equations of (4) are satisfied. Let $\hat{C}$ be the $m\times r$ matrix consisting of r linearly independent columns of C. Then from $(K^2-K)C=(3/4)C=(CD)C$, we obtain

$$(K^2-K)\hat{C}=(CD)\hat{C}.$$

On the other hand, let E be an $m\times(m-r)$ matrix whose columns are linearly independent eigenvectors of K associated with other eigenvalues which belong to $\{0,1\}$. Then $DE=0$ from matrix theory since all rows of D are left eigenvectors of K associated with eigenvalue $-1/2$. Thus,

$$0=(K^2-K)E=C(DE)=(CD)E.$$

Since the columns of $\hat{C}$ and E form a basis of $\mathbb{C}^m$, we see that $K^2-K=CD$.

By the same token and with the help of the other assumption that $DCD=(3/4)D$, we deduce that $Z^2+Z=DC$. Therefore, (K,Z,C,D) solves (4). This proves the theorem. $\square$

The non-commuting solutions obtained from Theorem 3.1 can be constructed in an easy way from a similarity consideration. By Lemma 3.1 of [9], if (K,Z,C,D) is a solution of (3), then $(K',Z',C',D,)$ is also a solution of the same system, where

$$K'=PKP^{-1},\ \ Z'=QZQ^{-1},\ \ C'=PCQ^{-1},\ \ D'=QDP^{-1}$$

with P and Q any $m\times m$ and $(n-m)\times(n-m)$ nonsingular matrices, respectively. Note that

$$C'D'C'=PCQ^{-1}QDP^{-1}PCQ^{-1}=PCDCQ^{-1}=\frac{3}{4}PCQ^{-1}=\frac{3}{4}C'$$

if $CDC=(3/4)C$ and

$$D'C'D'=QDP^{-1}PCQ^{-1}QDP^{-1}=QDCDP^{-1}=\frac{3}{4}QDP^{-1}=\frac{3}{4}D'$$

if $DCD=(3/4)D$. Thus, all the non-commuting solutions of (4) can be constructed in the following way:

Construction of All Non-Commuting Solutions:

Step I. Pick any integer r between 1 and $\min\{m,n-m\}$.

Step II. Let

$$K_0=\mathrm{diag}(-1/2,\ldots,-1/2,\alpha_{r+1},\ldots,\alpha_m)$$

and

$$Z_0 = \operatorname{diag}(1/2, \ldots, 1/2, \beta_{r+1}, \ldots, \beta_{n-m}),$$

where $\alpha_i = 0$ or 1 for $i = r+1, \ldots, m$ and $\beta_j = 0$ or -1 for $j = r+1, \ldots, n-m$.

Step III. Let

$$C_0 = \begin{bmatrix} \frac{\sqrt{3}}{2} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } D_0 = \begin{bmatrix} \frac{\sqrt{3}}{2} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Step IV. Pick any $m \times m$ and $(n-m) \times (n-m)$ nonsingular matrices P and Q, respectively.

Step V. Let

$$K = PK_0P^{-1},\ Z = QZ_0Q^{-1},\ C = PC_0Q^{-1},\ D = QD_0P^{-1}.$$

Then (K, Z, C, D) is a solution of (4).

With all possible choices of r, P, and Q, we get all the non-commuting solutions of (4), from which all the non-commuting solutions of the Yang-Baxter matrix equation (1) are thus obtained.

Next, we present a 5×5 example to illustrate our main results. Let

$$A = \begin{bmatrix} 5 & 0 & -2 & -4 & 0 \\ 18 & 1 & -10 & -20 & 2 \\ 4 & 0 & -1 & -4 & 0 \\ 4 & 0 & -2 & -3 & 0 \\ 6 & 0 & -2 & -4 & -1 \end{bmatrix},$$

then $A^2 = I$ and its Jordan form is

$$J = \operatorname{diag}(1, 1, 1, -1, -1)$$

with

$$S = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \text{ and } S^{-1} = \begin{bmatrix} -7 & -1 & 4 & 9 & -1 \\ 17 & 2 & -10 & -19 & 2 \\ -8 & -1 & 5 & 9 & -1 \\ -7 & 0 & 4 & 8 & -1 \\ 5 & 0 & -3 & -6 & 1 \end{bmatrix}.$$

By the above discussion, we can pick up $r = 1$,

$$K_0 = \operatorname{diag}(-1/2, 0, 1), \quad Z_0 = \operatorname{diag}(1/2, -1),$$

$$C_0 = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_0 = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

Then

$$K = PK_0P^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} & \frac{7}{4} \\ 0 & -1 & \frac{1}{2} \\ 0 & -4 & 2 \end{bmatrix}, \quad C = PC_0Q^{-1} = \begin{bmatrix} \frac{3\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D = QD_0P^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} \\ \sqrt{3} & -\sqrt{3} & -\frac{\sqrt{3}}{2} \end{bmatrix}, \quad Z = QZ_0Q^{-1} = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 9 & -4 \end{bmatrix}$$

and (K,Z,C,D) is a solution of (4). Therefore we obtain one particular non-commuting solution of (1),

$$X = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} & \frac{7}{4} & \frac{3\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & -4 & 2 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} & \frac{7}{2} & -\frac{3}{2} \\ \sqrt{3} & -\sqrt{3} & -\frac{\sqrt{3}}{2} & 9 & -4 \end{bmatrix} \begin{bmatrix} -7 & -1 & 4 & 9 & -1 \\ 17 & 2 & -10 & -19 & 2 \\ -8 & -1 & 5 & 9 & -1 \\ -7 & 0 & 4 & 8 & -1 \\ 5 & 0 & -3 & -6 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -1848-172\sqrt{3} & -165-15\sqrt{3} & 1093+99\sqrt{3} & 2083+201\sqrt{3} & -237-23\sqrt{3} \\ -1008-160\sqrt{3} & -50-20\sqrt{3} & 590+92\sqrt{3} & 1150+188\sqrt{3} & -142-20\sqrt{3} \\ -1972-120\sqrt{3} & -180-15\sqrt{3} & 1166+69\sqrt{3} & 2224+141\sqrt{3} & -252-15\sqrt{3} \\ -1092-172\sqrt{3} & -75-15\sqrt{3} & 643+99\sqrt{3} & 1237+201\sqrt{3} & -147-23\sqrt{3} \\ -3560-612\sqrt{3} & -165-70\sqrt{3} & 2083+352\sqrt{3} & 4063+718\sqrt{3} & -505-78\sqrt{3} \end{bmatrix}.$$

4. Conclusions

As a continuation of our previous work [9], we have found all the solutions of the Yang-Baxter matrix equation (1) for the class of matrices A such that $A^2 = I$. In doing so, we found all the non-commuting solutions of the equation for the general case that the nonzero sub-matrices C and D in the solution matrix Y have no artificial restriction. Our main result Theorem 3.1 followed from a general spectral perturbation result and the fact that C and D have the same rank for all the solutions.

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