ALGEBRA–VALUED G–FRAMES IN HILBERT $C^*$–MODULES

MOHAMMAD B. ASADI AND Z. HASSANPOUR-YAKHDANI

(Communicated by D. Bakić)

Abstract. In this paper, we consider the notion of algebra-valued G-frame as a special case of G-frame in a Hilbert $C^*$-module. It is shown that every Hilbert module over a commutative $C^*$-algebra $A$ admits an algebra-valued G-frame iff $A$ is a $C^*$-algebra of compact operators.

1. Introduction

M. Frank and D. R. Larson [3] generalized the classical frame theory in Hilbert spaces to Hilbert $C^*$-modules. They concluded from Kasparov’s stabilization theorem that every countably (finitely) generated Hilbert $C^*$-module over a unital $C^*$-algebra admits a frame. However, as it is asked in Problem 8.1 of [3], the interesting open question is for which kind of $C^*$-algebra $A$, every Hilbert $A$-module admits a frame.

In 2010, Li characterized a commutative unital $C^*$-algebra that every Hilbert $C^*$-module over it has a frame as a finite dimensional $C^*$-algebra [6]. Li’s result for non-unital commutative $C^*$-algebras is stated in [1] in the following way.

THEOREM 1.1. [1, Theorem. 1.4] Let $A$ be a commutative $C^*$-algebra. Then $A$ is a $C^*$-algebra of compact operators (equivalently, it has discrete spectrum) if and only if every Hilbert $A$-module has a frame.

As a generalization of frames, Sun in 2006 introduced G-frames in Hilbert spaces [8]. Also, this concept has been generalized to Hilbert $C^*$-modules in [5]. In this paper, we define the notion of algebra-valued G-frame in a Hilbert $C^*$-module, as a special case of G-frame.

We investigate the existence problem for algebra-valued G-frames in Hilbert $C^*$-modules. In fact, we use a generalization of Serre-Swan theorem [4] and prove that for $A$ being a commutative $C^*$-algebra, every Hilbert $A$-module admits an algebra-valued G-frame iff $A$ is a $C^*$-algebra of compact operators. In particular, for $A$ being a unital commutative $C^*$-algebra, every Hilbert $A$-module admits an algebra-valued G-frame iff $A$ is finite dimensional.

Assume that $A$ is a $C^*$-algebra and $X, Y$ are Hilbert $A$-modules. The family of all bounded $A$-linear maps from $X$ into $Y$ is denoted by $End(X, Y)$. Also, $I$ is an arbitrary indexing set.


Keywords and phrases: Hilbert $C^*$-modules, continuous field of Hilbert spaces, frames, G-frames.
DEFINITION 1.2. Let \( \{ Y_i : i \in I \} \) be a family of Hilbert \( A \)-modules. A family \( \{ \Lambda_i \in \text{End}(X, Y_i) : i \in I \} \) is called a G-frame for \( X \) with respect to \( \{ Y_i : i \in I \} \), if there exist constants \( 0 < C \leq D < \infty \) such that for every \( x \in X \),

\[
C \langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \leq D \langle x, x \rangle,
\]

(1.1)

where, by using the standard isometric embedding of \( A \) into its universal enveloping von Neumann algebra \( A^{**} \), the value \( \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \) is the limit of the increasingly ordered net of its finite partial sums with respect to the ultraweak topology on \( A^{**} \).

We remark that some authors use a slightly different definition so that each operator \( \Lambda_i \) is adjointable. Obviously, every Hilbert \( A \)-module \( X \) has a G-frame. Indeed, one can consider the identity map on \( X \), where \( I \) is a singleton set and \( Y_i = X \), for \( i \in I \). However, when we envisage a frame \( \{ h_i \}_{i \in I} \) in a Hilbert space \( H \) as a G-frame, we consider each \( h_i \) as a member of \( H^* \), the dual of \( H \). Due to this, algebra-valued G-frame can be a proper generalization of frame in Hilbert \( C^* \)-modules.

DEFINITION 1.3. A family \( \{ \Lambda_i \in \text{End}(X, A) : i \in I \} \) that satisfies the properties of a G-frame is called algebra-valued G-frame.

One can state an analogue of Proposition 3.1 in [6] for G-frames as follows.

PROPOSITION 1.4. A family \( \{ (\Lambda_i, Y_i) : i \in I \} \) is a G-frame for \( X \), with G-frame bounds \( C \) and \( D \) if and only if

\[
C \varphi(\langle x, x \rangle) \leq \sum_{i \in I} \varphi(\langle \Lambda_i(x), \Lambda_i(x) \rangle) \leq D \varphi(\langle x, x \rangle),
\]

(1.2)

for any \( x \in X \) and any state \( \varphi \) of \( A \).

2. Hilbert modules over commutative \( C^* \)-algebras

A generalization of Serre-Swan theorem states that the category of continuous fields of Hilbert spaces over a locally compact Hausdorff space \( Z \) is equivalent to the category of Hilbert \( C^* \)-modules over the commutative \( C^* \)-algebra \( A = C_0(Z) \) [4, Theorem. 4.8.]. We have applied this fact to show that every Hilbert module over a commutative \( C^* \)-algebra \( A \) admits an algebra-valued G-frame iff \( A \) is a \( C^* \)-algebra of compact operators.

DEFINITION 2.1. Let \( Z \) be a locally compact Hausdorff space. Consider \( ( (H_z)_{z \in Z}, \Gamma ) \), where \( (H_z)_{z \in Z} \) is a family of Hilbert spaces and \( \Gamma \) is a subset of \( \prod_{z \in Z} H_z \). Also, we set

\[
C_0 - \prod_{z \in Z} H_z = \{ x \in \prod_{z \in Z} H_z : \Gamma \rightarrow ||x(z)|| \in C_0(Z) \}.
\]

The pair \( ( (H_z)_{z \in Z}, \Gamma ) \) satisfying the following properties is said to be a continuous field of Hilbert spaces.
1) \( \Gamma \) is a linear subspace of \( C_0 - \prod_{z \in Z} H_z \).

2) The set \( \{x(z) : x \in \Gamma \} \) equals to \( H_z \), for every \( z \in Z \).

3) If \( x \in C_0 - \prod_{z \in Z} H_z \) and for every \( z \in Z \) and every \( \varepsilon > 0 \) there is a \( x' \in \Gamma \) such that \( \|x(s) - x'(s)\| < \varepsilon \) in some neighborhood of \( z \), then \( x \in \Gamma \).

The space \( \mathcal{H} = \prod_{z \in Z} H_z \) is called the total space.

**Remark 2.2.** a) Note that the function \( z \mapsto \langle x(z), y(z) \rangle \) is an element of \( C_0(Z) \), for every \( x, y \in \Gamma \). Also, if the topological space \( Z \) is discrete, then \( \Gamma = C_0 - \prod_{z \in Z} H_z \) [2].

b) A morphism \( \psi : ((H_z)_{z \in Z}, \Gamma) \rightarrow ((K_z)_{z \in Z}, \Gamma') \) of continuous fields of Hilbert spaces is a family of linear maps \( \{\psi_z : H_z \rightarrow K_z : z \in Z \} \) such that the induced map \( \psi : \mathcal{H} \rightarrow \mathcal{K} \) on the total spaces satisfies \( \{\psi \circ x : x \in \Gamma \} \subseteq \Gamma' \) and also the map \( z \mapsto \|\psi_z\| \) is locally bounded. By [4, Proposition 4.7.], \( \Gamma \) has a structure of Hilbert \( C_0(Z) \)-module with pointwise multiplication and inner product

\[
\langle x, y \rangle(z) = \langle x(z), y(z) \rangle \quad (x, y \in \Gamma, z \in Z).
\]

Indeed, the category of Hilbert \( C_0(Z) \)-modules is equivalent to the category of continuous fields of Hilbert spaces. In particular, if \( ((H_z)_{z \in Z}, \Gamma) \) and \( ((K_z)_{z \in Z}, \Gamma') \) are the corresponding continuous fields of Hilbert spaces to Hilbert \( C_0(Z) \)-modules \( X \) and \( Y \), then for each \( \Lambda \in \text{End}(X, Y) \), the map \( \Lambda_z : H_z \rightarrow K_z \) defined by \( \Lambda_z(x(z)) = (\Lambda(x))(z) \) is a well-defined bounded linear operator, for every \( z \in Z \) [4].

c) If we consider \( A = C_0(Z) \) as a Hilbert \( A \)-module, in the natural way, then the corresponding continuous field of Hilbert spaces to Hilbert \( A \)-module \( A \) is \( ((C_z)_{z \in Z}, \Gamma_A) \), where \( \mathbb{C}_z = \mathbb{C} \), for every \( z \in Z \) and \( \Gamma_A = \{(f(z))_{z \in Z} : f \in C_0(Z)\} \). In particular, when \( Z \) is discrete then \( \Gamma_A = C_0 - \prod_{z \in Z} \mathbb{C}_z \).

**Theorem 2.3.** If \( Z \) is a discrete topological space, then every Hilbert \( C_0(Z) \)-module admits a algebra-valued \( G \)-frame.

**Proof.** Let \( X \) be a Hilbert \( C^* \)-module over a commutative \( C^* \)-algebra \( A = C_0(Z) \), where \( Z \) is a discrete topological space. There is a continuous field of Hilbert spaces \( ((H_z)_{z \in Z}, \Gamma) \) that \( X \) is of the form \( \Gamma \). Since \( Z \) is discrete, then \( \Gamma = C_0 - \prod_{z \in Z} H_z \), by part (a) of Remark 2.2.

Let \( \{f_i^z : i \in I_z\} \) be an orthonormal basis for \( H_z \) and \( \mathbb{C}_z = \mathbb{C} \), for every \( z \in Z \). For each \( i \in I = \bigcup_{z \in Z} I_z \), we define \( \Lambda_i : \Gamma \rightarrow C_0 - \prod_{z \in Z} \mathbb{C}_z \) by \( \Lambda_i((x_z)_{z \in Z}) = (\lambda_i z)_{z \in Z} \), where \( x_z \in H_z \) and \( \lambda_i z = \langle x_z, f_i^z \rangle \) if \( i \in I_z \) and \( \lambda_i z = 0 \) otherwise. Clearly, for every \( x = (x_z)_{z \in Z} \in \Gamma \), we have

\[
\langle x, x \rangle(z) = \langle x_z, x_z \rangle = \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle(z).
\]

Hence, \( \{\Lambda_i\}_{i \in I} \) is a \( G \)-frame for \( \Gamma \). On the other hand, by part (c) of Remark 2.2, there is an \( A \)-module isomorphism \( \Psi \) from \( C_0 - \prod_{z \in Z} \mathbb{C}_z \) onto \( A \). Obviously, \( \{\Psi \circ \Lambda_i\}_{i \in I} \) is an algebra-valued \( G \)-frame for \( \Gamma = X \).

The following proposition is a generalization of [6, Proposition 2.4].
PROPOSITION 2.4. [1, Proposition 1.3] Let Z be an infinite non-discrete locally compact Hausdorff space. Then there exist a continuous field of Hilbert spaces \((H_z)_{z \in Z}, \Gamma\) over Z, a countable subset \(W \subseteq Z\) and a point \(z_\infty \in \overline{W} \setminus W\) that \(H_z\) is separable for every \(z \in W\) and \(H_{z_\infty}\) is non-separable.

THEOREM 2.5. Let Z be an infinite non-discrete locally compact Hausdorff space. Then there is a Hilbert \(C_0(Z)\)-module that admits no algebra-valued G-frame.

Proof. With the notations in Proposition 2.4, we show that the Hilbert \(C_0(Z)\)-module \(\Gamma\) admits no algebra-valued G-frames. To get a contradiction, assume that \(A = C_0(Z)\) and \(\{\Lambda_i : \Gamma \rightarrow A : i \in I\}\) is an algebra-valued G-frame for \(\Gamma\), with bounds \(C\) and \(D\). Hence, for every \(x \in \Gamma\) and every \(z \in Z\),

\[
C\|x(z)\|^2 \leq \Sigma_{i \in I}\langle \Lambda_i(x), \Lambda_i(x) \rangle(z) \leq D\|x(z)\|^2.
\]

On the other hand, the Hilbert \(A\)-module \(A\) is isomorphic to \(\Gamma_A\). Hence, by part (b) of Remark 2.2, for every \(i \in I\), \(\Lambda_i\) corresponds to \(\{\Lambda_{i,z} : H_z \rightarrow C_z : z \in Z\}\). By Riesz representation theorem, for every \(z \in Z\) we can find a subset \(\{f_i^z : i \in I\}\) of \(H_z\) such that for every \(x \in \Gamma\),

\[
\Lambda_{i,z}(x(z)) = \langle x(z), f_i^z \rangle. \tag{2.1}
\]

Therefore, for every \(x \in \Gamma\),

\[
C\|x(z)\|^2 \leq \Sigma_{i \in I}\langle \Lambda_{i,z}(x(z)), \Lambda_{i,z}(x(z)) \rangle \leq D\|x(z)\|^2.
\]

By Equation 2.1 and Axiom 2 of Definition 2.1, for every \(z \in Z\) and every \(w \in H_z\),

\[
C\langle w, w \rangle \leq \Sigma_{i \in I}\langle w, f_i^z \rangle \langle f_i^z, w \rangle \leq D\langle w, w \rangle. \tag{2.2}
\]

The remaining part of the proof is the same as the proof of Lemma 3.2 in [6]. In fact, for each \(z \in Z\), we first choose an orthonormal basis \(S_z\) for \(H_z\). By (2.2), for every \(w \in S_z\), \(F_w = \{i \in I : \langle w, f_i^z \rangle \neq 0\}\) is countable. For every \(z \in W\), \(S_z\) is countable, so \(F_z = \{i \in I : f_i^z \neq 0\}\), which is equal to \(\bigcup_{w \in S_z} F_w\), is countable. Hence, \(F = \bigcup_{z \in W} F_z\) is countable and also for every \(i \in I \setminus F\) and every \(z \in W\), \(f_i^z = 0\).

On the other hand, by Remark 2.2, the map \(z \mapsto \langle f_i^z, f_i^z \rangle = \Lambda_{i,z}(f_i^z)\) is continuous. Hence, for every \(i \in I \setminus F\), \(f_i^{z_\infty} = 0\), because \(z_\infty \in \overline{W}\).

Since \(H_{z_\infty}\) is nonseparable, there is a non-zero \(w \in H_{z_\infty}\) that is orthogonal to \(f_i^{z_\infty}\), for every \(i \in F\). Therefore, for every \(i \in I\), we have \(\langle f_i^{z_\infty}, w \rangle = 0\). By (2.2), \(w\) is equal to zero, that is a contradiction. \(\square\)

COROLLARY 2.6. Every Hilbert \(C^*\)-module over a commutative \(C^*\)-algebra \(A\) admits an algebra-valued G-frame iff \(A\) is \(C^*\)-algebra of compact operators.

Proof. By Theorems 2.3 and 2.5, every Hilbert \(C^*\)-module over a commutative \(C^*\)-algebra \(A = C_0(Z)\) admits an algebra-valued G-frame iff \(Z\) is discrete. On the other hand, \(C_0(Z)\) is a \(C^*\)-algebra of compact operators iff \(Z\) is discrete [2, 4.7.20]. \(\square\)
Acknowledgement. The authors would like to thank the referee for her/his valuable comments and suggestions to improve the quality of the paper.

The research of the first author was supported by Iran National Science Foundation (INSF) (No. 97006005).

REFERENCES


(Received June 11, 2018)

Mohammad B. Asadi
School of Mathematics, Statistics and Computer Science
College of Science, University of Tehran
Tehran, Iran
e-mail: mb.asadi@khayam.ut.ac.ir

Z. Hassannpour-Yakhdani
School of Mathematics, Statistics and Computer Science
Collage of Science, University of Tehran
Tehran, Iran
e-mail: z.hasanpour@ut.ac.ir