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ALGEBRA–VALUED G–FRAMES IN HILBERT C^*–MODULES

MOHAMMAD B. ASADI AND Z. HASSANPOUR-YAKHDANI

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Abstract. In this paper, we consider the notion of algebra-valued G-frame as a special case of G-frame in a Hilbert C^*-module. It is shown that every Hilbert module over a commutative C^*-algebra A admits an algebra-valued G-frame iff A is a C^*-algebra of compact operators.

1. Introduction

M. Frank and D. R. Larson [3] generalized the classical frame theory in Hilbert spaces to Hilbert C^*-modules. They concluded from Kasparov's stabilization theorem that every countably (finitely) generated Hilbert C^*-module over a unital C^*-algebra admits a frame. However, as it is asked in Problem 8.1 of [3], the interesting open question is for which kind of C^*-algebra A, every Hilbert A-module admits a frame.

In 2010, Li characterized a commutative unital C^*-algebra that every Hilbert C^*-module over it has a frame as a finite dimensional C^*-algebra [6]. Li's result for non-unital commutative C^*-algebras is stated in [1] in the following way.

THEOREM 1.1. [1, Theorem. 1.4] *Let A be a commutative C^*-algebra. Then A is a C^*-algebra of compact operators (equivalently, it has discrete spectrum) if and only if every Hilbert A-module has a frame.*

As a generalization of frames, Sun in 2006 introduced G-frames in Hilbert spaces [8]. Also, this concept has been generalized to Hilbert C^*-modules in [5]. In this paper, we define the notion of algebra-valued G-frame in a Hilbert C^*-module, as a special case of G-frame.

We investigate the existence problem for algebra-valued G-frames in Hilbert C^*-modules. In fact, we use a generalization of Serre-Swan theorem [4] and prove that for A being a commutative C^*-algebra, every Hilbert A-module admits an algebra-valued G-frame iff A is a C^*-algebra of compact operators. In particular, for A being a unital commutative C^*-algebra, every Hilbert A-module admits an algebra-valued G-frame iff A is finite dimentional.

Assume that A is a C^*-algebra and X,Y are Hilbert A-modules. The family of all bounded A-linear maps from X into Y is denoted by $End(X,Y)$. Also, I is an arbitrary indexing set.

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DEFINITION 1.2. Let $\{Y_i : i \in I\}$ be a family of Hilbert A-modules. A family $\{\Lambda_i \in End(X,Y_i) : i \in I\}$ is called a G-frame for X with respect to $\{Y_i : i \in I\}$, if there exist constants $0 < C \leqslant D < \infty$ such that for every $x \in X$,

$$C\langle x,x\rangle \leqslant \sum_{i\in I}\langle \Lambda_i(x),\Lambda_i(x)\rangle \leqslant D\langle x,x\rangle, \tag{1.1}$$

where, by using the standard isometric embedding of A into its universal enveloping von Neumann algebra A^{**}, the value $\sum_{i\in I}\langle \Lambda_i(x),\Lambda_i(x)\rangle$ is the limit of the increasingly ordered net of its finite partial sums with respect to the ultraweak topology on A^{**}.

We remark that some authors use a slightly different definition so that each operator Λ_i is adjointable.

Obviously, every Hilbert A-module X has a G-frame. Indeed, one can consider the identity map on X, where I is a singleton set and $Y_i = X$, for $i \in I$. However, when we envisage a frame $\{h_i\}_{i\in I}$ in a Hilbert space H as a G-frame, we consider each h_i as a member of H^*, the dual of H. Due to this, algebra-valued G-frame can be a proper generalization of frame in Hilbert C^*-modules.

DEFINITION 1.3. A family $\{\Lambda_i \in End(X,A) : i \in I\}$ that satisfies the properties of a G-frame is called algebra-valued G-frame.

One can state an analogue of Proposition 3.1 in [6] for G-frames as follows.

PROPOSITION 1.4. *A family $\{(\Lambda_i,Y_i) : i \in I\}$ is a G-frame for X, with G-frame bounds C and D if and only if*

$$C\varphi(\langle x,x\rangle) \leqslant \Sigma_{i\in I}\varphi(\langle \Lambda_i(x),\Lambda_i(x)\rangle \leqslant D\varphi(\langle x,x\rangle), \tag{1.2}$$

for any $x \in X$ and any state φ of A.

2. Hilbert modules over commutative C^*-algebras

A generalization of Serre-Swan theorem states that the category of continuous fields of Hilbert spaces over a locally compact Hausdorff space Z is equivalent to the category of Hilbert C^*-modules over the commutative C^*-algebra $A = C_0(Z)$ [4, Theorem. 4.8.]. We have applied this fact to show that every Hilbert module over a commutative C^*-algebra A admits an algebra-valued G-frame iff A is a C^*-algebra of compact operators.

DEFINITION 2.1. Let Z be a locally compact Hausdorff space. Consider $((H_z)_{z\in Z},\Gamma)$, where $(H_z)_{z\in Z}$ is a family of Hilbert spaces and Γ is a subset of $\prod_{z\in Z} H_z$. Also, we set

$$C_0 - \prod_{z\in Z} H_z = \{x \in \prod_{z\in Z} H_z : \ [z \mapsto \|x(z)\|] \in C_0(Z)\}.$$

The pair $((H_z)_{z\in Z},\Gamma)$ satisfying the following properties is said to be a continuous field of Hilbert spaces.

1) Γ is a linear subspace of $C_0 - \prod_{z\in Z} H_z$.

2) The set $\{x(z) : x \in \Gamma\}$ equals to H_z, for every $z \in Z$.

3) If $x \in C_0 - \prod_{z\in Z} H_z$ and for every $z \in Z$ and every $\varepsilon > 0$ there is a $x' \in \Gamma$ such that $\|x(s) - x'(s)\| < \varepsilon$ in some neighborhood of z, then $x \in \Gamma$.

The space $\mathscr{H} = \prod_{z\in Z} H_z$ is called the total space.

REMARK 2.2. a) Note that the function $z \mapsto \langle x(z), y(z)\rangle$ is an element of $C_0(Z)$, for every $x, y \in \Gamma$. Also, if the topological space Z is discrete, then $\Gamma = C_0 - \prod_{z\in Z} H_z$ [2].

b) A morphism $\psi : ((H_z)_{z\in Z}, \Gamma) \longrightarrow ((K_z)_{z\in Z}, \Gamma')$ of continuous fields of Hilbert spaces is a family of linear maps $\{\psi_z : H_z \longrightarrow K_z : z \in Z\}$ such that the induced map $\psi : \mathscr{H} \longrightarrow \mathscr{K}$ on the total spaces satisfies $\{\psi \circ x : x \in \Gamma\} \subseteq \Gamma'$ and also the map $z \mapsto \|\psi_z\|$ is locally bounded. By [4, Proposition 4.7.], Γ has a structure of Hilbert $C_0(Z)$-module with pointwise multiplication and inner product

$$\langle x, y\rangle(z) = \langle x(z), y(z)\rangle \qquad (x, y \in \Gamma, z \in Z).$$

Indeed, the category of Hilbert $C_0(Z)$-modules is equivalent to the category of continuous fields of Hilbert spaces. In particular, if $((H_z)_{z\in Z}, \Gamma)$ and $((K_z)_{z\in Z}, \Gamma')$ are the corresponding continuous fields of Hilbert spaces to Hilbert $C_0(Z)$-modules X and Y, then for each Λ in $End(X,Y)$, the map $\Lambda_z : H_z \longrightarrow K_z$ defined by $\Lambda_z(x(z)) = (\Lambda(x))(z)$ is a well-defined bounded linear operator, for every $z \in Z$ [4].

c) If we consider $A = C_0(Z)$ as a Hilbert A-module, in the natural way, then the corresponding continuous field of Hilbert spaces to Hilbert A-module A is $((\mathbb{C}_z)_{z\in Z}, \Gamma_A)$, where $\mathbb{C}_z = \mathbb{C}$, for every $z \in Z$ and $\Gamma_A = \{(f(z))_{z\in Z} : f \in C_0(Z)\}$. In particular, when Z is discrete then $\Gamma_A = C_0 - \prod_{z\in Z} \mathbb{C}_z$.

THEOREM 2.3. *If Z is a discrete topological space, then every Hilbert $C_0(Z)$-module admits a algebra-valued G-frame.*

Proof. Let X be a Hilbert C^*-module over a commutative C^*-algebra $A = C_0(Z)$, where Z is a discrete topological space. There is a continuous field of Hilbert spaces $((H_z)_{z\in Z}, \Gamma)$ that X is of the form Γ. Since Z is discrete, then $\Gamma = C_0 - \prod_{z\in Z} H_z$, by part (a) of Remark 2.2.

Let $\{f_i^z : i \in I_z\}$ be an orthonormal basis for H_z and $\mathbb{C}_z = \mathbb{C}$, for every $z \in Z$. For each $i \in I = \cup_{z\in Z} I_z$, we define $\Lambda_i : \Gamma \longrightarrow C_0 - \prod_{z\in Z} \mathbb{C}_z$ by $\Lambda_i((x_z)_{z\in Z}) = (\lambda_{i,z})_{z\in Z}$, where $x_z \in H_z$ and $\lambda_{i,z} = \langle x_z, f_i^z\rangle$ if $i \in I_z$ and $\lambda_{i,z} = 0$ otherwise. Clearly, for every $x = (x_z)_{z\in Z} \in \Gamma$, we have

$$\langle x, x\rangle(z) = \langle x_z, x_z\rangle = \Sigma_{i\in I} \langle \Lambda_i(x), \Lambda_i(x)\rangle(z).$$

Hence, $\{\Lambda_i\}_{i\in I}$ is a G-frame for Γ. On the other hand, by part (c) of Remark 2.2, there is an A-module isomorphism Ψ from $C_0 - \prod_{z\in Z} \mathbb{C}_z$ onto A. Obviously, $\{\Psi \circ \Lambda_i\}_{i\in I}$ is an algebra-valued G-frame for $\Gamma = X$. $\square$

The following proposition is a generalization of [6, Proposition 2.4].

PROPOSITION 2.4. [1, Proposition 1.3] *Let Z be an infinite non-discrete locally compact Hausdorff space. Then there exist a continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$ over Z, a countable subset $W\subseteq Z$ and a point $z_\infty\in\overline{W}\setminus W$ that H_z is separable for every $z\in W$ and H_{z_∞} is non-separable.*

THEOREM 2.5. *Let Z be an infinite non-discrete locally compact Hausdorff space. Then there is a Hilbert $C_0(Z)$-module that admits no algebra-valued G-frame.*

Proof. With the notations in Proposition 2.4, we show that the Hilbert $C_0(Z)$-module Γ admits no algebra-valued G-frames. To get a contradiction, assume that $A=C_0(Z)$ and $\{\Lambda_i:\Gamma\longrightarrow A : i\in I\}$ is an algebra-valued G-frame for Γ, with bounds C and D. Hence, for every $x\in\Gamma$ and every $z\in Z$,

$$C\|x(z)\|^2\leqslant\Sigma_{i\in I}\langle\Lambda_i(x),\Lambda_i(x)\rangle(z)\leqslant D\|x(z)\|^2.$$

On the other hand, the Hilbert A-module A is isomorphic to Γ_A. Hence, by part (b) of Remark 2.2, for every $i\in I$, Λ_i corresponds to $\{\Lambda_{i,z}:H_z\longrightarrow\mathbb{C}_z : z\in Z\}$. By Riesz representation theorem, for every $z\in Z$ we can find a subset $\{f_i^z : i\in I\}$ of H_z such that for every $x\in\Gamma$,

$$\Lambda_{i,z}(x(z))=\langle x(z),f_i^z\rangle. \tag{2.1}$$

Therefore, for every $x\in\Gamma$,

$$C\|x(z)\|^2\leqslant\Sigma_{i\in I}\langle\Lambda_{i,z}(x(z)),\Lambda_{i,z}(x(z))\rangle\leqslant D\|x(z)\|^2.$$

By Equation 2.1 and Axiom 2 of Definition 2.1, for every $z\in Z$ and every $w\in H_z$,

$$C\langle w,w\rangle\leqslant\Sigma_{i\in I}\langle w,f_i^z\rangle\langle f_i^z,w\rangle\leqslant D\langle w,w\rangle. \tag{2.2}$$

The remaining part of the proof is the same as the proof of Lemma 3.2 in [6]. In fact, for each $z\in Z$, we first choose an orthonormal basis S_z for H_z. By (2.2), for every $w\in S_z$, $F_w=\{i\in I:\langle w,f_i^z\rangle\neq 0\}$ is countable. For every $z\in W$, S_z is countable, so $F_z=\{i\in I: f_i^z\neq 0\}$, which is equal to $\bigcup_{w\in S_z}F_w$, is countable. Hence, $F=\bigcup_{z\in W}F_z$ is countable and also for every $i\in I\setminus F$ and every $z\in W$, $f_i^z=0$.

On the other hand, by Remark 2.2, the map $z\mapsto\langle f_i^z,f_i^z\rangle=\Lambda_{i_z}(f_i^z)$ is continuous. Hence, for every $i\in I\setminus F, f_i^{z_\infty}=0$, because $z_\infty\in\overline{W}$.

Since H_{z_∞} is nonseparable, there is a non-zero $w\in H_{z_\infty}$ that is orthogonal to $f_i^{z_\infty}$, for every $i\in F$. Therefore, for every $i\in I$, we have $\langle f_i^{z_\infty},w\rangle=0$. By (2.2), w is equal to zero, that is a contradiction. □

COROLLARY 2.6. *Every Hilbert C^*-module over a commutative C^*-algebra A admits an algebra-valued G-frame iff A is C^*-algebra of compact operators.*

Proof. By Theorems 2.3 and 2.5, every Hilbert C^*-module over a commutative C^*-algebra $A=C_0(Z)$ admits an algebra-valued G-frame iff Z is discrete. On the other hand, $C_0(Z)$ is a C^*-algebra of compact operators iff Z is discrete [2, 4.7.20]. □

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Mohammad B. Asadi
School of Mathematics, Statistics and Computer Science
College of Science, University of Tehran
Tehran, Iran
e-mail: mb.asadi@khayam.ut.ac.ir

Z. Hassanpour-Yakhdani
School of Mathematics, Statistics and Computer Science
Collage of Science, University of Tehran
Tehran, Iran
e-mail: z.hasanpour@ut.ac.ir