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EXTENSIONS OF HIAI–LIN TYPE EIGENVALUE INEQUALITY

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Abstract. In this paper, we prove several extensions of Hiai-Lin type eigenvalue inequality which extends the relative result before.

1. Introduction and main results

A capital letter, such as T, stands for an $n \times n$ matrix. $T > 0$ means that T is a positive definite matrix. $\lambda_i(T)$ is the ith largest eigenvalue of T if T is Hermitian.

Let $A\sharp_t B$ stands for the weighted geometric mean. In other words,

$$A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

if $A, B > 0$ and $t \in [0,1]$. Similarly, $A\natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ if $A, B > 0$ and $t \notin [0,1]$.

In [1], F. Hiai and M. Lin proved the following eigenvalue inequality.

THEOREM 1.1. ([1]) *If $A, B > 0$, then*

$$\prod_{i=1}^{k} \lambda_i(AB) \geqslant \prod_{i=1}^{k} \lambda_i((A\sharp_t B)(A\sharp_{1-t} B)), \quad k = 1, 2, \cdots, n \tag{1.1}$$

holds for $t \in [0,1]$.

In this paper, we shall show extension of Theorem 1.1 as follows.

THEOREM 1.2. *If $A, B > 0$, then*

$$\prod_{i=1}^{k} \lambda_i(AB)^l \geqslant \prod_{i=1}^{k} \lambda_i\Big(\Big(A^{-r-1}\big(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\Big)\Big(A\sharp_{-(1-2t+r)\alpha}B\Big)\Big) \tag{1.2}$$

holds for $t \in [0, \frac{1}{2}]$, $\alpha \in [0,1]$ and $1 \geqslant -r > 1-t \geqslant \frac{1}{2}$, $k = 1, 2, \cdots, n$, where $l = -\frac{1-2t+r}{1-t+r} \cdot r\alpha$.

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THEOREM 1.3. *If* $A,B>0$, *then*

$$\prod_{i=1}^{k}\lambda_i(AB)^l \geqslant \prod_{i=1}^{k}\lambda_i\Big(\Big(A^{-r-1}\big(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\Big)\Big(A\sharp_{-\alpha r}B\Big)\Big) \tag{1.3}$$

holds for $t\in[\frac{1}{2},1]$, $\alpha\in[0,1]$ *and* $1\geqslant -r>1-t\geqslant 0$, $k=1,2,\cdots,n$, *where* $l=-\frac{\alpha r^2}{1-t+r}$.

In order to prove the main result we list a famous operator inequality – Tanahashi inequality here.

THEOREM 1.4. (Tanahashi inequality [2]) *If* $A\geqslant B\geqslant 0$ *with* $A>0$, *then*

$$A^{\frac{p'+2r'}{q'}} \geqslant (A^{r'}B^{p'}A^{r'})^{\frac{1}{q'}} \tag{1.4}$$

holds for $0\leqslant p'\leqslant 1$, $0<q'\leqslant 1$ *and* $-1\leqslant 2r'<0$ *satisfying*

$$-2r'(1-q')\leqslant p'\leqslant q'-2r'(1-q') \tag{1.5}$$

and

$$\frac{-2r'(1-q')-q'}{1-2q'}\leqslant p'\leqslant\frac{-2r'(1-q')}{1-2q'}\quad (when\ \ q'<1/2). \tag{1.6}$$

REMARK 1.1. If we put $r'=r/2$, $p'=p$, and $q'=\frac{p'+r'}{r'}$ in (1.4) and (1.5), then we can obtain that $A^r\geqslant(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$; If we put $r'=r/2$, $p'=p$, and $q'=\frac{p'+r'}{2p'-1+r'}$ in (1.4) and (1.6), then we can obtain that $A^{2p-1+r}\geqslant(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{2p-1+r}{p+r}}$. Thus, we can obtain the reformulations of Tanahashi inequality[2]: If $A\geqslant B\geqslant 0$ with $A>0$, $r<0$, then the following inequalities hold.

$$\begin{cases} \textit{Case } 1.\ A^r\geqslant(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}, & \text{if } 1\geqslant -r>p\geqslant 0 \text{ with } p\leqslant\frac{1}{2};\\ \textit{Case } 2.\ A^{2p-1+r}\geqslant(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{2p-1+r}{p+r}}, & \text{if } 1\geqslant -r>p\geqslant\frac{1}{2}. \end{cases}$$

2. Proofs of main results

In this section, we shall prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. By the well-known antisymmetric tensor power technique, we may need to prove that $B\leqslant A^{-1}$ ensures that

$$A^{-r-1}\big(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\leqslant\big(A\sharp_{-(1-2t+r)\alpha}B\big)^{-1}. \tag{2.1}$$

$B\leqslant A^{-1}$ is equivalent to $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$.

Let $p=1-t$ and apply $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$ to Tanahashi inequality(Case 2), we have

$$A^{-2(1-2t+r)}\geqslant(A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{1-2t+r}{1-t+r}}. \tag{2.2}$$

Because $-(1-2t+r)\in[0,1]$ and $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$,

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-2t+r}\geqslant A^{-2(1-2t+r)} \tag{2.3}$$

holds by Löwner-Heinz inequality.

Continuing applying Löwner-Heinz inequality for $\alpha\in[0,1]$ to (2.2) and (2.3), we have

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{(1-2t+r)\alpha}\geqslant (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{(1-2t+r)\alpha}{1-t+r}}. \tag{2.4}$$

Notice that

$$\begin{aligned}&A^{-r-1}\big(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\\ =&A^{-r-1}A^{r+\frac{1}{2}}(A^{-r-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{\frac{1}{2}}A^{-r-\frac{1}{2}})^{\frac{(1-2t+r)\alpha}{1-t+r}}A^{r+\frac{1}{2}}A^{-r-1}\\ =&A^{-\frac{1}{2}}(A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{(1-2t+r)\alpha}{1-t+r}}A^{-\frac{1}{2}}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\big(A\sharp_{-(1-2t+r)\alpha}B\big)^{-1}\\ =&\big(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-(1-2t+r)\alpha}A^{\frac{1}{2}}\big)^{-1}\\ =&A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{(1-2t+r)\alpha}A^{-\frac{1}{2}}.\end{aligned} \tag{2.6}$$

Together with (2.4), (2.5) and (2.6), (2.1) holds obviously. □

Proof of Theorem 1.3. We only need to prove that $B\leqslant A^{-1}$ ensures that

$$A^{-r-1}\big(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\leqslant \big(A\sharp_{-\alpha r}B\big)^{-1}. \tag{2.7}$$

Notice that $B\leqslant A^{-1}\iff A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$. Let $p=1-t$ and apply $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$ to Tanahashi inequality(Case 1), we have

$$A^{-2r}\geqslant (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{r}{1-t+r}}. \tag{2.8}$$

Because $-r\in[0,1]$ and $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant A^{-2}$, then

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r}\geqslant A^{-2r} \tag{2.9}$$

holds by Löwner-Heinz inequality.

Continuing applying Löwner-Heinz inequality for $\alpha\in[0,1]$ to (2.8) and (2.9), we have

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha r}\geqslant (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{\alpha r}{1-t+r}}. \tag{2.10}$$

Notice that

$$\begin{aligned}&A^{-r-1}\big(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B)\big)A^{-r-1}\\ =&A^{-r-1}A^{r+\frac{1}{2}}(A^{-r-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{\frac{1}{2}}A^{-r-\frac{1}{2}})^{\frac{\alpha r}{1-t+r}}A^{r+\frac{1}{2}}A^{-r-1}\\ =&A^{-\frac{1}{2}}(A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{\alpha r}{1-t+r}}A^{-\frac{1}{2}}\end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\left(A\sharp_{-\alpha r}B\right)^{-1}\\ =&\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\alpha r}A^{\frac{1}{2}}\right)^{-1}\\ =&A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha r}A^{-\frac{1}{2}}. \end{aligned} \tag{2.12}$$

Together with (2.10), (2.11) and (2.12), (2.7) holds obviously. □

3. Some corollaries of main results

In this section, we show some corollaries of main results.

By computing,

$$\begin{aligned} &\prod_{i=1}^{n}\lambda_i\left(\left(A^{-r-1}\left(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\right)A^{-r-1}\right)\left(A\sharp_{-(1-2t+r)\alpha}B\right)\right)\\ =&\det\left(\left(A^{-r-1}\left(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\right)A^{-r-1}\right)\left(A\sharp_{-(1-2t+r)\alpha}B\right)\right)\\ =&\det\left(A^{-r-1}\left(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\right)A^{-r-1}\right)\cdot\det\left(A\sharp_{-(1-2t+r)\alpha}B\right)\\ =&\det A^{-r-1}\cdot\det\left(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\right)\cdot\det A^{-r-1}\cdot\det\left(A\sharp_{-(1-2t+r)\alpha}B\right)\\ =&\det A^{-2r-2}\cdot\det A^{(2r+1)(1-\frac{(1-2t+r)\alpha}{1-t+r})}\cdot\det(A\sharp_{1-t}B)^{\frac{(1-2t+r)\alpha}{1-t+r}}\cdot\det\left(A\sharp_{-(1-2t+r)\alpha}B\right)\\ =&\det A^{-2r-2+(2r+1)(1-\frac{(1-2t+r)\alpha}{1-t+r})}\cdot\det A^{\frac{(1-2t+r)\alpha t}{1-t+r}}\cdot\det B^{\frac{(1-2t+r)\alpha(1-t)}{1-t+r}}\\ &\cdot\det A^{1+(1-2t+r)\alpha}\cdot\det B^{-(1-2t+r)\alpha}\\ =&\det A^{-\frac{1-2t+r}{1-t+r}\cdot r\alpha}\cdot\det B^{-\frac{1-2t+r}{1-t+r}\cdot r\alpha}\\ =&\det(AB)^{-\frac{1-2t+r}{1-t+r}\cdot r\alpha}=\prod_{i=1}^{n}\lambda_i(AB)^{-\frac{1-2t+r}{1-t+r}\cdot r\alpha}, \end{aligned}$$

we have the following corollary.

COROLLARY 3.1. *If* $A,B>0$, *then*

$$\begin{aligned} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{l}\underset{(\log)}{\succ}&\left(A\sharp_{-(1-2t+r)\alpha}B\right)^{\frac{1}{2}}\left(A^{-r-1}\left(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B)\right)A^{-r-1}\right)\\ &\times\left(A\sharp_{-(1-2t+r)\alpha}B\right)^{\frac{1}{2}} \end{aligned}$$

holds for $t\in[0,\frac{1}{2}]$, $\alpha\in[0,1]$ *and* $1\geqslant -r>1-t\geqslant\frac{1}{2}$, *where* $l=-\frac{1-2t+r}{1-t+r}\cdot r\alpha$.

Similarly, we can obtain the following corollary.

COROLLARY 3.2. *If $A,B>0$, then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{l} \underset{(\log)}{\succ} \left(A\sharp_{-\alpha r}B\right)^{\frac{1}{2}} \left(A^{-r-1}\left(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B)\right)A^{-r-1}\right) \left(A\sharp_{-\alpha r}B\right)^{\frac{1}{2}}$$

holds for $t\in[\frac{1}{2},1]$, $\alpha\in[0,1]$ and $1\geqslant -r>1-t\geqslant 0$, where $l=-\frac{\alpha r^2}{1-t+r}$.

Next, we show some simple corollaries directly from main results.

COROLLARY 3.3. *If $A,B>0$, then*

$$\prod_{i=1}^{k}\lambda_i\left(AB\right)^{-r} \geqslant \prod_{i=1}^{k}\lambda_i\left(\left(A^{-r-1}(A\sharp_{1-t}B)A^{-r-1}\right)\left(A\sharp_{-(1-t+r)}B\right)\right)$$

holds for $t\in[0,\frac{1}{2}]$ and $1\geqslant -r>1-t\geqslant \frac{1}{2}$, where $k=1,2,\cdots,n$.

Proof. Notice that $\frac{1-t+r}{1-2t+r}\in[0,1]$. Put $\alpha=\frac{1-t+r}{1-2t+r}$ in Theorem 1.2. □

COROLLARY 3.4. *If $A,B>0$, then*

$$\prod_{i=1}^{k}\lambda_i\left(AB\right)^{-r} \geqslant \prod_{i=1}^{k}\lambda_i\left(\left(A^{-r-1}(A\sharp_{1-t}B)A^{-r-1}\right)\left(A\sharp_{-(1-t+r)}B\right)\right)$$

holds for $t\in[\frac{1}{2},1]$ and $1\geqslant -r>1-t\geqslant 0$, where $k=1,2,\cdots,n$.

Proof. Notice that $\frac{1-t+r}{r}\in[0,1]$. Put $\alpha=\frac{1-t+r}{r}$ in Theorem 1.3. □

REMARK 3.1. If we put $r=-1$ in Corollary 3.3 and Corollary 3.4, they are just Theorem 1.1.

REMARK 3.2. Together with Corollary 3.3 and Corollary 3.4, it is obvious that

$$\prod_{i=1}^{k}\lambda_i\left(AB\right)^{-r} \geqslant \prod_{i=1}^{k}\lambda_i\left(\left(A^{-r-1}(A\sharp_{1-t}B)A^{-r-1}\right)\left(A\sharp_{-(1-t+r)}B\right)\right)$$

holds for $1\geqslant -r>1-t\geqslant 0$, where $A,B>0$, $k=1,2,\cdots,n$.

COROLLARY 3.5. *If $A,B>0$, then*

$$\prod_{i=1}^{k}\lambda_i\left(AB\right)^{2\alpha} \geqslant \prod_{i=1}^{k}\lambda_i\left(\left(A^{-1}\natural_{2\alpha}(A\sharp_{1-t}B)\right)\left(A\sharp_{2t\alpha}B\right)\right)$$

holds for $t\in[0,\frac{1}{2}]$ and $\alpha\in[0,1]$, where $k=1,2,\cdots,n$.

Proof. Put $r=-1$ in Theorem 1.2. □

COROLLARY 3.6. *If $A,B>0$, then*

$$\prod_{i=1}^{k}\lambda_i(AB)^{\frac{\alpha}{t}} \geqslant \prod_{i=1}^{k}\lambda_i\Big(\big(A^{-1}\natural_{\frac{\alpha}{t}}(A\sharp_{1-t}B)\big)\big(A\sharp_{\alpha}B\big)\Big)$$

holds for $t\in[\frac{1}{2},1]$ and $\alpha\in[0,1]$, where $k=1,2,\cdots,n$.

Proof. Put $r=-1$ in Theorem 1.3. □

REMARK 3.3. If we put $\alpha=1/2$ in Corollary 3.5 and $\alpha=t$ in Corollary 3.6, they are just Theorem 1.1.

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