Operators and Matrices
Volume 13, Number 1 (2019), 209–220
doi:10.7153/oam-2019-13-14

COMMUTATIVITY AND SPECTRAL PROPERTIES OF k^{th} –ORDER SLANT LITTLE HANKEL OPERATORS ON THE BERGMAN SPACE

ANURADHA GUPTA AND BHAWNA GUPTA

(*Communicated by V. V. Peller*)

Abstract. In this paper, we introduce the notion of k^{th}-order slant little Hankel operator on the Bergman space with essentially bounded harmonic symbols on the unit disc and obtain its algebraic and spectral properties. We have also discussed the conditions under which k^{th}-order slant little Hankel operators commute.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and $dA = dxdy$ denotes the Lebesgue area measure on $\mathbb{D}$, normalised so that the measure of $\mathbb{D}$ is 1. Let $L^2(\mathbb{D}, dA)$ be the space of all Lebesgue measurable functions f on $\mathbb{D}$ for which

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty.$$

It forms a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z).$$

The Bergman space $L_a^2(\mathbb{D}, dA)$ consists of all analytic functions f on $\mathbb{D}$ such that $f \in L^2(\mathbb{D}, dA)$ and is closed subspace of Hilbert space $L^2(\mathbb{D}, dA)$. It is well known that $\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ is an orthonormal basis for $L_a^2(\mathbb{D}, dA)$ and the orthogonal projection $P : L^2(\mathbb{D}, dA) \longrightarrow L_a^2(\mathbb{D}, dA)$ is an integral operator

$$Pf(z) = \langle f, K_z\rangle = \int_{\mathbb{D}} K(z,w) f(w) dA(w),$$

where $K(z,w) = K_w(z) = \frac{1}{(1-\bar{w}z)^2}$ is the unique reproducing kernel of $L_a^2(\mathbb{D}, dA)$. The space $L^{\infty}(\mathbb{D}, dA)$ is the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ such that $\|f\|_{\infty} = esssup\{|f(z)| : z \in \mathbb{D}\} < \infty$ and $H_{\infty}(\mathbb{D}, dA)$ denotes the set of all analytic functions of the space $L^{\infty}(\mathbb{D}, dA)$.

Mathematics subject classification (2010): 47B35, 47A10, 46E22.
Keywords and phrases: Bergman space, little Hankel operator, k^{th}-order slant little Hankel operator, spectrum.

Paper OaM-13-14

The theory of Hankel operators on the Hardy spaces is an important area of mathematical analysis and lots of applications in different domains of mathematics have been found such as interpolation problems, rational approximation, stationary processes or pertubation theory [10, 11]. In the year 1996, M. C. Ho [8] investigated the basic properties of slant Toeplitz operators on Hardy spaces. After that in the year 2006, Arora [2] introduced the class of slant Hankel operators on Hardy spaces and discussed its characterizations.

In the present paper, we study spectral and commutative properties of k^{th}-order slant little Hankel operators on the Bergman space with essentially bounded harmonic symbols. More precisely, we describe the conditions under which k^{th}-order slant little Hankel operators commute and we prove that spectrum and approximate point spectrum of S^k_ϕ are same where $\phi(z) = \sum_{i=0}^{N} \bar{z}^i$ and $N \in \{0,1,\cdots 2k-1\}$. Basic properties of the Hardy space and Bergman spaces can be found in [5, 6]. We refer [2, 3, 8, 12] for the applications and extensions of study to Hankel operators, slant Toeplitz operators, slant Hankel operators and its generalization on Hardy spaces. Let T be a bounded linear operator on a complex Banach space X then spectrum of T is defined as the set of all complex number λ such that $\lambda I - T$ is not invertible in the algebra $\mathbb{B}(X)$, where I denotes the identity operator on X and $\mathbb{B}(X)$ denotes the set of all bounded linear operators on X. The spectrum of T is classified into three categories: point spectrum, $\sigma_P(T)$; residual spectrum, $\sigma_R(T)$ and continuous spectrum, $\sigma_C(T)$ where

$$\begin{aligned}
\sigma_P(T) &= \{\lambda \in \mathbb{C} : ker(\lambda I - T) \neq (0)\},\\
\sigma_R(T) &= \{\lambda \in \mathbb{C} : ker(\lambda I - T) = (0) \text{ and } Range(\lambda I - T)^- \neq X\},\\
\sigma_C(T) &= \{\lambda \in \mathbb{C} : ker(\lambda I - T) = (0) \text{ and } Range(\lambda I - T) \neq Range(\lambda I - T)^- = X\}.
\end{aligned}$$

There are some overlapping divisions of spectrum also, namely approximate point spectrum and compression spectrum where approximate point spectrum of T is the set of all complex number λ such that $\lambda I - T$ is not bounded below and compression spectrum of T is the set of all complex number λ such that $Range(\lambda I - T)^- \neq X$.

2. The k^{th}-order slant little Hankel operators on $L^2_a(\mathbb{D}, dA)$

Let $\phi \in L^\infty(\mathbb{D}, dA)$ then for any $f \in L^2_a(\mathbb{D}, dA)$, the Toeplitz operator $T_\phi : L^2_a(\mathbb{D}, dA) \longrightarrow L^2_a(\mathbb{D}, dA)$ is defined as $T_\phi(f) = P(\phi f)$ and the little Hankel operator $H_\phi : L^2_a(\mathbb{D}, dA) \longrightarrow L^2_a(\mathbb{D}, dA)$ is defined as $H_\phi(f) = PJM_\phi(f)$ where P is the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D}, dA)$, $J : L^2(\mathbb{D}, dA) \longrightarrow L^2(\mathbb{D}, dA)$ is defined by $J(f(z)) = f(\bar{z})$ and M_ϕ is the multiplication operator on $L^2_a(\mathbb{D}, dA)$ defined as $M_\phi(f) = \phi f$. It is well known that H_ϕ is bounded with $\|H_\phi\| \leqslant \|\phi\|_\infty$. For $\phi = \sum_{j=0}^{\infty} a_j \bar{z}^j + \sum_{j=1}^{\infty} b_j z^j \in L^\infty(\mathbb{D}, dA)$, the $(m,n)^{th}$ entry of matrix representation of little Hankel operator on $L^2_a(\mathbb{D}, dA)$ with respect to orthonormal basis $\{\sqrt{n+1} z^n\}_{n=0}^{\infty}$ is

$$\begin{aligned}\left\langle H_{\phi}\sqrt{n+1}z^{n},\sqrt{m+1}z^{m}\right\rangle &= \sqrt{n+1}\sqrt{m+1}\,\langle PJ(\phi z^{n}),z^{m}\rangle \\ &= \sqrt{n+1}\sqrt{m+1}\left\langle (\sum_{j=0}^{\infty}a_{j}\bar{z}^{j}+\sum_{j=1}^{\infty}b_{j}z^{j})z^{n},\bar{z}^{m}\right\rangle \\ &= \sqrt{n+1}\sqrt{m+1}\sum_{j=0}^{\infty}a_{j}\left\langle \bar{z}^{j}z^{n},\bar{z}^{m}\right\rangle \\ &= \frac{\sqrt{n+1}\sqrt{m+1}}{(m+n+1)}a_{m+n}\end{aligned}$$

and its matrix representation is

$$[H_{\phi}] = \begin{bmatrix} a_0 & \frac{1}{\sqrt{2}}a_1 & \frac{1}{\sqrt{3}}a_2 & \frac{1}{\sqrt{4}}a_3 & \frac{1}{\sqrt{5}}a_4 & \dots \\ \frac{1}{\sqrt{2}}a_1 & \frac{2}{3}a_2 & \frac{\sqrt{6}}{4}a_3 & \frac{\sqrt{8}}{5}a_4 & \frac{\sqrt{10}}{6}a_5 & \dots \\ \frac{1}{\sqrt{3}}a_2 & \frac{\sqrt{6}}{4}a_3 & \frac{3}{5}a_4 & \frac{\sqrt{12}}{6}a_5 & \frac{\sqrt{15}}{7}a_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

whose adjoint is given by

$$[H_{\phi}]^{*} = \begin{bmatrix} \overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{\sqrt{4}}\overline{a_3} & \frac{1}{\sqrt{5}}\overline{a_4} & \dots \\ \frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{\sqrt{8}}{5}\overline{a_4} & \frac{\sqrt{10}}{6}\overline{a_5} & \dots \\ \frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_4} & \frac{\sqrt{12}}{6}\overline{a_5} & \frac{\sqrt{15}}{7}\overline{a_6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

From the above matrices, we conclude that $H_{\phi}^{*}=H_{\hat{\phi}}$ where $\hat{\phi}(z)=\sum_{j=0}^{\infty}\overline{a_j}\,\bar{z}^{j}+\sum_{j=1}^{\infty}\overline{b_j}z^{j}$ $\in L^{\infty}(\mathbb{D},dA)$.

For $k\geqslant 2$, define $W_k : L_a^2(\mathbb{D},dA)\longrightarrow L_a^2(\mathbb{D},dA)$ by $W_k(z^{kn})=z^{n}$, $W_k(z^{kn+p})=0$ for all $n\in\mathbb{N}\cup\{0\}$ and $p=1,2,\dots k-1$. Clearly, W_k is a bounded linear operator with $\|W_k\|=\sqrt{k}$ and the adjoint of W_k is given by $W_k^{*}(z^{m})=\frac{km+1}{m+1}z^{km}$ for $m\geqslant 0$ (see [9]).

In year 2008, Arora and Bhola [3] discussed about the k^{th}-order slant Hankel operators acting on H^2 space. We extend the definition of little Hankel operators on the Bergman space to k^{th}-order slant little Hankel operators in the following manner:

DEFINITION 1. For $k \geqslant 2$ and $\phi(z) = \sum_{j=0}^{\infty} a_j \bar{z}^j + \sum_{j=1}^{\infty} b_j z^j$ in $L^\infty(\mathbb{D}, dA)$, a linear operator $S_\phi^k : L_a^2(\mathbb{D}, dA) \longrightarrow L_a^2(\mathbb{D}, dA)$ is defined by

$$S_\phi^k(f) = W_k H_\phi(f) \text{ for all } f \in L_a^2(\mathbb{D}, dA).$$

We call S_ϕ^k, the k^{th}-order slant little Hankel operator on $L_a^2(\mathbb{D}, dA)$ with symbol ϕ and

$$\|S_\phi^k\| = \|W_k H_\phi\| \leqslant \|W_k\| \|H_\phi\| = \sqrt{k} \|H_\phi\| \leqslant \sqrt{k} \|\phi\|_\infty.$$

Thus, S_ϕ^k is bounded. We denote the set of all k^{th}-order slant little Hankel operators on $L_a^2(\mathbb{D}, dA)$ by SHO(L_a^2).

The $(m,n)^{th}$ entry of matrix representation of S_ϕ^k on $L_a^2(\mathbb{D}, dA)$ with respect to orthonormal basis $\{\sqrt{n+1} z^n\}_{n=0}^{\infty}$ is

$$\begin{aligned}
\left\langle S_\phi^k \sqrt{n+1} z^n, \sqrt{m+1} z^m \right\rangle &= \sqrt{n+1}\sqrt{m+1} \, \langle W_k P J(\phi z^n), z^m \rangle \\
&= \sqrt{n+1}\sqrt{m+1} \, \langle P J(\phi z^n), W_k^* z^m \rangle \\
&= \sqrt{n+1}\sqrt{m+1} \left\langle P J(\phi z^n), \frac{km+1}{m+1} z^{km} \right\rangle \\
&= \sqrt{n+1}\sqrt{m+1} \left\langle \Big(\sum_{j=0}^{\infty} a_j \bar{z}^j + \sum_{j=1}^{\infty} b_j z^j\Big) z^n, \frac{km+1}{m+1} \bar{z}^{km} \right\rangle \\
&= \frac{\sqrt{n+1}(km+1)}{\sqrt{m+1}} \sum_{k=0}^{\infty} a_j \left\langle \bar{z}^j z^n, \bar{z}^{km} \right\rangle \\
&= \frac{\sqrt{n+1}(km+1)}{\sqrt{m+1}(n+km+1)} a_{n+km}
\end{aligned}$$

and its matrix representation is given as

$$\begin{bmatrix}
a_0 & \frac{1}{\sqrt{2}} a_1 & \frac{1}{\sqrt{3}} a_2 & \frac{1}{\sqrt{4}} a_3 & \frac{1}{\sqrt{5}} a_4 & \cdots \\
\frac{1}{\sqrt{2}} a_k & \frac{(k+1)\sqrt{2}}{(k+2)\sqrt{2}} a_{k+1} & \frac{(k+1)\sqrt{3}}{(k+3)\sqrt{2}} a_{k+2} & \frac{(k+1)\sqrt{4}}{(k+4)\sqrt{2}} a_{k+3} & \frac{(k+1)\sqrt{5}}{(k+5)\sqrt{2}} a_{k+4} & \cdots \\
\frac{1}{\sqrt{3}} a_{2k} & \frac{(2k+1)\sqrt{2}}{(2k+2)\sqrt{3}} a_{2k+1} & \frac{(2k+1)\sqrt{3}}{(2k+3)\sqrt{3}} a_{2k+2} & \frac{(2k+1)\sqrt{4}}{(2k+4)\sqrt{3}} a_{2k+3} & \frac{(2k+1)\sqrt{5}}{(2k+5)\sqrt{3}} a_{2k+4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}. \tag{2.1}$$

For $k = 2$, S_ϕ^k is simply called slant little Hankel operator on $L_a^2(\mathbb{D}, dA)$. It is denoted by S_ϕ where W_2 is denoted by W.

NOTE 1. Since b_j does not appear in the matrix (2.1) for any natural number j therefore we have $S_{\sum_{j=0}^{\infty} a_j \bar{z}^j + \sum_{j=1}^{\infty} b_j z^j} = S_{\sum_{j=0}^{\infty} a_j \bar{z}^j}$ on $L_a^2(\mathbb{D}, dA)$.

PROPOSITION 1. For ϕ_1, ϕ_2 in $L^\infty(\mathbb{D}, dA)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ then
i) $S^k_{\lambda_1\phi_1+\lambda_2\phi_2} = \lambda_1 S^k_{\phi_1} + \lambda_2 S^k_{\phi_2}$.
ii) J is a self adjoint unitary operator on $L^2(\mathbb{D}, dA)$.

The following proposition follows directly from the above matrix with respect to the orthonormal basis.

PROPOSITION 2. The mapping $\Gamma : L^\infty(\mathbb{D}, dA) \to \mathrm{SHO}(L^2_a)$ defined by $\Gamma(\phi) = S^k_\phi$ is linear but not one- one. For, if $\phi_1 - \phi_2 \in zH_\infty(\mathbb{D}, dA)$ for some $\phi_1, \phi_2 \in L^\infty(\mathbb{D}, dA)$ then $S^k_{\phi_1} = S^k_{\phi_2}$. This gives that the operator S^k_ϕ does not have unique symbol ϕ.

In [4], Arora and Bhola obtained the point spectrum of k^{th}-order slant Hankel operators on the space H^2 with symbol function $\bar{z}^i$ for $i \geqslant 0$ and related its spectrum and approximate point spectrum. Similar to their work, we obtain the following results.

THEOREM 1. *If $\phi(z) = \sum_{i=0}^N \bar{z}^i \in L^\infty(\mathbb{D}, dA)$ where $N \in \{0, 1, \cdots 2k-1\}$ then the point spectrum of S^k_ϕ is*

$$\sigma_p(S^k_\phi) = \begin{cases} \{0,1\} & \text{if } 0 \leqslant N < k \\ \{0, \lambda_1, \lambda_2\} & \text{if } k \leqslant N \leqslant 2k-1, \end{cases}$$

where $\lambda_1 = \frac{2k+3+\sqrt{2k^2+8k+9}}{2(k+2)}$ and $\lambda_2 = \frac{2k+3-\sqrt{2k^2+8k+9}}{2(k+2)}$.

Proof. Let $\lambda \in \sigma_p(S^k_\phi)$ then there exists $f \neq 0$ in $L^2_a(\mathbb{D}, dA)$ such that $S^k_\phi f = \lambda f$. Consider $f = \sum_{n=0}^\infty a_n z^n$ in $L^2_a(\mathbb{D}, dA)$ then $W_k PJM_\phi(f) = \lambda f$, giving

$$W_k PJ\left(\sum_{i=0}^N \bar{z}^i \sum_{n=0}^\infty a_n z^n\right) = \lambda \sum_{n=0}^\infty a_n z^n$$

then $W_k P\left(\sum_{i=0}^N \sum_{n=0}^\infty a_n z^i \bar{z}^n\right) = \lambda \sum_{n=0}^\infty a_n z^n$. Thus,

$$W_k\left(\sum_{i=0}^N \sum_{n=0}^i \frac{i-n+1}{(i+1)} a_n z^{i-n}\right) = \lambda \sum_{n=0}^\infty a_n z^n. \tag{2.2}$$

Case 1. If $0 \leqslant N < k$, then equation (2.2) gives $\sum_{i=0}^N \frac{1}{i+1} a_i = \lambda \sum_{n=0}^\infty a_n z^n$. This yields, $\lambda a_0 = \sum_{i=0}^N \frac{1}{i+1} a_i$ and $\lambda a_n = 0$ for all $n \geqslant 1$.

If $\lambda \neq 0$ and $\lambda \neq 1$ then $a_n = 0$ for all $n \geqslant 1$. This yields $a_0 = \lambda a_0$ which gives $a_0 = 0$ leads to $f = 0$, a contradiction. Hence, 0 is the eigen value of S^k_ϕ corresponding to the eigen vector $f(z) = \sum_{n=0}^\infty a_n z^n$ with $\sum_{i=0}^N \frac{1}{i+1} a_i = 0$ and 1 is the eigen value of S^k_ϕ corresponding to the eigen vector $f(z) = a_0$.

Case 2. If $k \leqslant N \leqslant 2k-1$, then equation (2.2) gives $\sum_{i=0}^N \frac{1}{i+1} a_i + \sum_{i=k}^N \frac{k+1}{i+1} a_{i-k} z = \lambda \sum_{n=0}^\infty a_n z^n$. This yields

$$\sum_{i=k}^N \frac{k+1}{i+1} a_{i-k} = \lambda a_1, \quad \sum_{i=0}^N \frac{1}{i+1} a_i = \lambda a_0 \text{ and } \lambda a_n = 0 \text{ for all } n \geqslant 2. \tag{2.3}$$

If $\lambda \neq 0$ then equation (2.3) gives $a_n = 0$ for all $n \geqslant 2$,

$$a_0 + \frac{1}{2}a_1 = \lambda a_0 \tag{2.4}$$

and

$$a_0 + \frac{k+1}{k+2}a_1 = \lambda a_1. \tag{2.5}$$

On solving equations (2.4) and (2.5), it follows that $a_1 = 2(\lambda-1)a_0$ then substituting the value of a_1 in equation (2.5), it becomes $a_0(\lambda-\lambda_1)(\lambda-\lambda_2) = 0$ where $\lambda_1 = \frac{2k+3+\sqrt{2k^2+8k+9}}{2(k+2)}$ and $\lambda_2 = \frac{2k+3-\sqrt{2k^2+8k+9}}{2(k+2)}$.

If $\lambda \neq 0$, $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$ then $a_0 = 0$ and $a_1 = 0$ which gives $f = 0$, a contadiction. Hence, 0 is the eigen value of S_ϕ^k corresponding to the eigen vector $f(z) = \sum_{n=2}^{\infty} a_n z^n$ and λ_1 and λ_2 are the eigen values of S_ϕ^k corresponding to the eigen vector $f(z) = a_0 + a_1 z$ with $a_1 = 2(\lambda_1 - 1)a_0$ and $a_1 = 2(\lambda_2 - 1)a_0$ respectively. □

REMARK 1. From the proof of theorem (1), for $\phi(z) = \sum_{i=0}^{N} \bar{z}^i \in L^\infty(\mathbb{D}, dA)$ we have the following observations:

Case 1. If $0 \leqslant N < k$ then for any complex number $\lambda \neq 0, 1$ we have $Range(S_\phi^k - \lambda I) = \{(\sum_{i=0}^{N} \frac{1}{i+1} a_i - \lambda a_0) - \lambda \sum_{n=1}^{\infty} a_n z^n : \sum_{n=0}^{\infty} a_n z^n \in L_a^2(\mathbb{D}, dA)\}$ which is dense in $L_a^2(\mathbb{D}, dA)$. Therefore residual spectrum of $S_\phi^k - \lambda I$, $\sigma_R(S_\phi^k - \lambda I)$ is empty.

Case 2. If $k \leqslant N < 2k-1$ then for any complex number λ except $\lambda \in \{0, \lambda_1, \lambda_2\}$ we obtain that $Range(S_\phi^k - \lambda I) = \{(\sum_{i=0}^{N} \frac{1}{i+1} a_i - \lambda a_0) + (\sum_{i=k}^{N} \frac{k+1}{i+1} a_{i-k} - \lambda a_1) z - \lambda \sum_{n=2}^{\infty} a_n z^n : \sum_{n=0}^{\infty} a_n z^n \in L_a^2(\mathbb{D}, dA)\}$ is dense in $L_a^2(\mathbb{D}, dA)$. Thus, $\sigma_R(S_\phi^k - \lambda I) = \Phi$.

As a consequence of the above theorem, we obtain the following result:

THEOREM 2. *Let $\sigma_{AP}(S_\phi^k)$ and $\sigma(S_\phi^k)$ denote the approximate point spectrum and spectrum of S_ϕ^k respectively, where $\phi(z) = \sum_{i=0}^{N} \bar{z}^i \in L^\infty(\mathbb{D}, dA)$ and $N \in \{0, 1, \cdots 2k-1\}$ then $\sigma_{AP}(S_\phi^k) = \sigma(S_\phi^k)$.*

Proof. It is well known that [7, 1], $\sigma(T) = \sigma_{AP}(T) \bigcup \sigma_{CP}(T)$ for any bounded linear operator T on Hilbert space H, where $\sigma(T), \sigma_{AP}(T)$ and $\sigma_{CP}(T)$ denotes spectrum, approximate point spectrum and compression spectrum of T, respectively. Therefore

$$\sigma(S_\phi^k) = \sigma_{AP}(S_\phi^k) \bigcup \sigma_{CP}(S_\phi^k), \tag{2.6}$$

where $\phi(z) = \sum_{i=0}^{N} \bar{z}^i \in L^\infty(\mathbb{D}, dA)$ and $N \in \{0, 1, \cdots 2k-1\}$. Also from [7], we have $\sigma_R(S_\phi^k) = \sigma_{CP}(S_\phi^k) \backslash \sigma_P(S_\phi^k)$. By remark (1), it is evident that $\sigma_{CP}(S_\phi^k) \subseteq \sigma_P(S_\phi^k)$, but $\sigma_P(S_\phi^k) \subseteq \sigma_{AP}(S_\phi^k)$. Thus, $\sigma_{CP}(S_\phi^k) \subseteq \sigma_{AP}(S_\phi^k)$. Hence from equation (2.6), it follows that $\sigma_{AP}(S_\phi^k) = \sigma(S_\phi^k)$. □

Similarly we conclude the following result:

THEOREM 3. *For $i \geqslant 0$, the point spectrum of $S^k_{\bar{z}^i}$ is the following:*

$$\sigma_p(S^k_{\bar{z}^i}) = \begin{cases} \{0\} & \text{if } i \text{ is not a multiple of } (k+1) \\ \{0, (\frac{k+1}{k+2})\} & \text{if } i \text{ is a multiple of } (k+1). \end{cases}$$

and $\sigma_{AP}(S^k_{\bar{z}^i}) = \sigma(S^k_{\bar{z}^i})$.

3. Commutativity of k^{th}-order slant little Hankel operators

In this section, we are dealing with the commutative properties of k^{th}-order slant little Hankel operators and we show that under some assumptions k^{th}-order slant little Hankel operators on $L^2_a(\mathbb{D}, dA)$ commute if and only if the symbol functions are linearly dependent.

THEOREM 4. *Let $\phi(z) = \sum_{i=0}^{n} a_i \bar{z}^i, \zeta(z) = \sum_{j=0}^{n} b_j \bar{z}^j$ be such that $\phi, \zeta \in L^\infty(\mathbb{D}, dA)$, where n is any non negative integer and $a_n \neq 0$, $b_n \neq 0$ then S^k_ϕ and S^k_ζ commute if and only if ϕ and ζ are linearly dependent.*

Proof. Let ϕ and ζ are linearly dependent then it is obvious that S^k_ϕ and S^k_ζ commute. Conversely, suppose that S^k_ϕ and S^k_ζ commute. If $n = 0$ then result is trivially true. For $n > 0$ let $n = kp + r$ where $p \geqslant 0$, $0 \leqslant r \leqslant k-1$ be integers. Since S^k_ϕ and S^k_ζ commute therefore,

$$S^{k\,*}_\phi S^{k\,*}_\zeta (z^p) = S^{k\,*}_\zeta S^{k\,*}_\phi (z^p). \tag{3.1}$$

Consider

$$\begin{aligned} S^{k\,*}_\phi S^{k\,*}_\zeta (z^p) &= PJM_{\hat{\phi}} W_k^* PJM_{\hat{\zeta}} W_k^* (z^p) = PJM_{\hat{\phi}} W_k^* PJ \left(\sum_{j=0}^{n} \overline{b_j}\, \bar{z}^j \frac{kp+1}{p+1} z^{kp} \right) \\ &= \frac{kp+1}{p+1} PJM_{\hat{\phi}} W_k^* P \left(\sum_{j=0}^{n} \overline{b_j} z^j\, \bar{z}^{kp} \right) \\ &= \frac{kp+1}{p+1} PJM_{\hat{\phi}} W_k^* \left(\sum_{j=kp}^{n} \frac{(j-kp+1)}{(j+1)} \overline{b_j} z^{j-kp} \right). \end{aligned} \tag{3.2}$$

The following two cases arise:

Case 1. If $r = 0$ then equation (3.2) becomes

$$S^{k\,*}_\phi S^{k\,*}_\zeta (z^p) = \frac{1}{p+1} PJM_{\hat{\phi}} W_k^* \overline{b_n} = \frac{1}{p+1} \overline{b_n} PJ \left(\sum_{i=0}^{n} \overline{a_i}\, \bar{z}^i \right) = \frac{1}{p+1} \overline{b_n} \left(\sum_{i=0}^{n} \overline{a_i} z^i \right). \tag{3.3}$$

Similar calculation gives

$$S^{k\,*}_\zeta S^{k\,*}_\phi (z^p) = \frac{1}{p+1} \overline{a_n} \left(\sum_{j=0}^{n} \overline{b_j} z^j \right). \tag{3.4}$$

From equations (3.1), (3.3) and (3.4), it follows that

$$\frac{1}{p+1}\overline{b_n}\left(\sum_{i=0}^{n}\overline{a_i}z^i\right)=\frac{1}{p+1}\overline{a_n}\left(\sum_{j=0}^{n}\overline{b_j}z^j\right). \tag{3.5}$$

Since $\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ forms an orthonormal basis for the Bergman space, so equation (3.5) gives $\overline{b_n}\overline{a_i}=\overline{a_n}\overline{b_i}$ for all $0\leqslant i\leqslant n$. This yields $b_i=\lambda a_i$ for all $0\leqslant i\leqslant n$ where $\lambda=\frac{b_n}{a_n}$. Hence, $\zeta(z)=\lambda\phi(z)$.

Case 2. If $r>0$ then it follows from equation (3.2) that

$$\begin{aligned}
S_\phi^{k*}S_\zeta^{k*}(z^p)&=\frac{kp+1}{p+1}PJM_{\hat{\phi}}\left(\sum_{j=kp}^{n}\frac{(k(j-kp)+1)}{(j+1)}\overline{b_j}z^{k(j-kp)}\right)\\
&=\frac{kp+1}{p+1}PJ\left(\sum_{i=0}^{n}\overline{a_i}\bar{z}^i\sum_{j=kp}^{n}\frac{(k(j-kp)+1)}{(j+1)}\overline{b_j}z^{k(j-kp)}\right)\\
&=\frac{kp+1}{p+1}P\left(\sum_{i=0}^{n}\overline{a_i}z^i\sum_{j=kp}^{n}\frac{(k(j-kp)+1)}{(j+1)}\overline{b_j}\bar{z}^{k(j-kp)}\right)\\
&=\frac{kp+1}{p+1}P\left(\sum_{i=0}^{n}\overline{a_i}z^i\sum_{q=0}^{r}\frac{(kq+1)}{(q+kp+1)}\overline{b_{q+kp}}\bar{z}^{kq}\right)\\
&=\frac{kp+1}{p+1}\left(\sum_{q=0}^{min(r,p)}\sum_{i=kq}^{n}\frac{(kq+1)(i-kq+1)}{(q+kp+1)(i+1)}\overline{a_i}\overline{b_{q+kp}}z^{i-kq}\right).
\end{aligned} \tag{3.6}$$

Similarly, we can obtain

$$S_\zeta^{k*}S_\phi^{k*}(z^p)=\frac{kp+1}{p+1}\left(\sum_{s=0}^{min(r,p)}\sum_{j=ks}^{n}\frac{(ks+1)(j-ks+1)}{(s+kp+1)(j+1)}\overline{b_j}\overline{a_{s+kp}}z^{j-ks}\right). \tag{3.7}$$

Equations (3.1), (3.6) and (3.7) yield

$$\begin{aligned}
&\left(\sum_{q=0}^{min(r,p)}\sum_{i=kq}^{n}\frac{(kq+1)(i-kq+1)}{(q+kp+1)(i+1)}\overline{a_i}\overline{b_{q+kp}}z^{i-kq}\right)\\
&\qquad=\left(\sum_{s=0}^{min(r,p)}\sum_{j=ks}^{n}\frac{(ks+1)(j-ks+1)}{(s+kp+1)(j+1)}\overline{b_j}\overline{a_{s+kp}}z^{j-ks}\right).
\end{aligned} \tag{3.8}$$

Therefore for every integer m such that $n-k<m\leqslant n$, we have $\overline{a_m}\overline{b_{kp}}=\overline{b_m}\overline{a_{kp}}$. It gives $b_m=\lambda a_m$ where $\lambda=\frac{b_{kp}}{a_{kp}}$. Similarly from equation (3.8) it follows that for every integer m such that $n-2k<m\leqslant n-k$, we have $\frac{1}{kp+1}\overline{a_m}\overline{b_{kp}}+\frac{(k+1)(m+1)}{(kp+2)(m+k+1)}\overline{a_{m+k}}\overline{b_{kp+1}}$ $=\frac{1}{kp+1}\overline{b_m}\overline{a_{kp}}+\frac{(k+1)(m+1)}{(kp+2)(m+k+1)}\overline{b_{m+k}}\overline{a_{kp+1}}$. Since $n-k<m+k\leqslant n$ and $r<k$ so for all

$y \geqslant 0$, $kp+y > n-k = kp+r-k$, therefore, $a_m b_{kp} = b_m a_{kp}$ implies $b_m = \lambda a_m$. Proceeding like this, by using equation (3.8) it follows that $b_m = \lambda a_m$ for $0 \leqslant m \leqslant n$ where $\lambda = \frac{b_{kp}}{a_{kp}}$. Hence, $\zeta(z) = \lambda \phi(z)$. □

LEMMA 1. *Let* $\phi(z) = \sum_{i=0}^{n} a_i \bar{z}^i$ *and* $\zeta(z) = \sum_{j=0}^{m} b_j \bar{z}^j$ *be such that* $\phi, \zeta \in L^\infty(\mathbb{D}, dA)$ *where* n *and* m *are non negative integers with* $n > m$. *Let* $n = kp_1 + r_1$, $m = kp_2 + r_2$ *where* p_1, p_2, r_1, r_2 *are integers such that* p_1, $p_2 \geqslant 0$, $0 \leqslant r_1, r_2 < k$ *and also let* $a_{kp_1}^2 + b_{kp_2}^2 \neq 0$ *and* $b_m \neq 0$. *If* S_ϕ^k *and* S_ζ^k *commute then* $a_j = 0$ *for each integer* j *such that* $m < j \leqslant n$.

Proof. Since S_ϕ^k and S_ζ^k commute, therefore,

$$S_\phi^{k\,*} S_\zeta^{k\,*}(f) = S_\zeta^{k\,*} S_\phi^{k\,*}(f) \text{ for all } f \in L_a^2(\mathbb{D}, dA). \tag{3.9}$$

The following three cases arise:

Case 1. If $m = 0$ and $n = kp_1 + r_1$. Since $n > m$, so either $p_1 = 0$ and $0 < r_1$ or $p_1 > 0$ and $0 \leqslant r_1 < k$ then

$$\begin{aligned} S_\phi^{k\,*} S_\zeta^{k\,*}(1) &= PJM_{\hat{\phi}} W_k^* PJM_{\hat{\zeta}} W_k^*(1) = PJM_{\hat{\phi}} W_k^*(\overline{b_m}) = PJM_{\hat{\phi}}(\overline{b_m}) \\ &= \overline{b_m} PJ\Big(\sum_{i=0}^{n} \overline{a_i}\, \bar{z}^i\Big) = \overline{b_m} \sum_{i=0}^{n} \overline{a_i} z^i, \end{aligned} \tag{3.10}$$

$$\begin{aligned} S_\zeta^{k\,*} S_\phi^{k\,*}(1) &= PJM_{\hat{\zeta}} W_k^* PJM_{\hat{\phi}} W_k^*(1) = PJM_{\hat{\phi}} W_k^* PJ\Big(\sum_{i=0}^{n} \overline{a_i}\, \bar{z}^i\Big) \\ &= PJM_{\hat{\phi}}\Big(\sum_{i=0}^{n} \frac{ki+1}{i+1} \overline{a_i} z^{ki}\Big) = \overline{b_m} P\Big(\sum_{i=0}^{n} \frac{ki+1}{i+1} \overline{a_i}\, \bar{z}^{ki}\Big) = \overline{b_m}\,\overline{a_0}. \end{aligned} \tag{3.11}$$

So, from equations (3.9), (3.10) and (3.11) it follows that $\overline{b_m}\,\overline{a_i} = 0$ for $0 < i \leqslant n$. Since $b_m \neq 0$, therefore, $a_i = 0$ for all i such that $m < i \leqslant n$.

Case 2. If $m = r_2$ where $0 < r_2 < k$.

If $n = r_1$ then since $n > m$, so $0 < r_2 < r_1 < k$.

$$\begin{aligned} S_\phi^{k\,*} S_\zeta^{k\,*}(1) &= PJM_{\hat{\phi}} W_k^* PJM_{\hat{\zeta}} W_k^*(1) = PJM_{\hat{\phi}} W_k^*\Big(\sum_{j=0}^{m} \overline{b_j} z^j\Big) = PJM_{\hat{\phi}}\Big(\sum_{j=0}^{m} \frac{kj+1}{j+1} \overline{b_j} z^{kj}\Big) \\ &= PJ\Big(\sum_{i=0}^{n} \overline{a_i}\, \bar{z}^i \sum_{j=0}^{m} \frac{kj+1}{j+1} \overline{b_j} z^{kj}\Big) = P\Big(\sum_{i=0}^{n} \overline{a_i} z^i \sum_{j=0}^{m} \frac{kj+1}{j+1} \overline{b_j}\, \bar{z}^{kj}\Big) = \overline{b_0} \sum_{i=0}^{n} \overline{a_i} z^i. \end{aligned} \tag{3.12}$$

$$\begin{aligned} S_\zeta^{k\,*} S_\phi^{k\,*}(1) &= PJM_{\hat{\zeta}} W_k^* PJM_{\hat{\phi}} W_k^*(1) = PJM_{\hat{\zeta}} W_k^*\Big(\sum_{i=0}^{n} \overline{a_i} z^i\Big) = PJM_{\hat{\zeta}}\Big(\sum_{i=0}^{n} \frac{ki+1}{i+1} \overline{a_i} z^{ki}\Big) \\ &= PJ\Big(\sum_{j=0}^{m} \overline{b_j}\, \bar{z}^j \sum_{i=0}^{n} \frac{ki+1}{i+1} \overline{a_i} z^{ki}\Big) = P\Big(\sum_{j=0}^{m} \overline{b_j} z^j \sum_{i=0}^{n} \frac{ki+1}{i+1} \overline{a_i}\, \bar{z}^{ki}\Big) = \overline{a_0} \sum_{j=0}^{m} \overline{b_j} z^j. \end{aligned} \tag{3.13}$$

From equations (3.9), (3.12) and (3.13) it follows that $b_0a_i = 0$ for $m < i \leqslant n$ and $b_0a_j = b_ja_0$ for $0 \leqslant j \leqslant m$. In particular, for $j = m$, $b_0a_m = a_0b_m$. If $b_0 = 0$ then $a_0 = 0$, a contradiction as $a_{kp_1}^2 + b_{kp_2}^2 = a_0^2 + b_0^2 \neq 0$. Therefore, $b_0 \neq 0$. This yields $a_i = 0$ for all i such that $m < i \leqslant n$.

If $n = kp_1 + r_1$ where $p_1 > 0$ and $0 \leqslant r_1 < k$. Similar calculations gives $S_\phi^{k\,*}S_\zeta^{k\,*}(z^{p_1}) = 0$ and $S_\zeta^{k\,*}S_\phi^{k\,*}(z^{p_1}) = \frac{1}{p_1+1}\sum_{j=0}^{m}\overline{b_j}\overline{a_{kp_1}}z^j$. By using equation (3.9), it follows that $a_{kp_1}b_j = 0$ for $0 \leqslant j \leqslant m$. In particular, for $j = m$ we have $a_{kp_1}b_m = 0$. Since $b_m \neq 0$, therefore, $a_{kp_1} = 0$. Similarly, we obtain

$$S_\phi^{k\,*}S_\zeta^{k\,*}(1) = \sum_{j=0}^{min(r_2,p_1)}\sum_{i=kj}^{n}\frac{(kj+1)(i-kj+1)}{(j+1)(i+1)}\overline{a_i}\overline{b_j}z^{i-kj}$$

and

$$S_\zeta^{k\,*}S_\phi^{k\,*}(1) = \sum_{j=0}^{m}\overline{b_j}\overline{a_0}z^j.$$

By using equation (3.9), it follows that $a_ib_0 = 0$ for $n-k < i \leqslant n$. Since $n-k < kp_1 \leqslant n$, so $a_{kp_1}b_0 = 0$ but $a_{kp_1} = 0$ and $a_{kp_1}^2 + b_{kp_2}^2 = a_{kp_1}^2 + b_0^2 \neq 0$. Therefore, $b_0 \neq 0$. This yields $a_i = 0$ for $n-k < i \leqslant n$. Also, from equation (3.9) we have $a_ib_0 + \frac{(k+1)(i+1)}{2(i+k+1)}b_1a_{k+i} = 0$ for all i such that $n-2k < i \leqslant n-k$ but $a_{k+i} = 0$ for $n-k < i+k \leqslant n$. Hence, $a_i = 0$ for $n-2k < i \leqslant n-k$. Continuing in this way, we conclude that $a_i = 0$ for all i such that $m < i \leqslant n$.

Case 3. If $m = kp_2$ where $p_2 > 0$ and $n = kp_1 + r_1$ then since $n > m$ so, either $p_2 = p_1$ and $0 < r_1 < k$ or $p_1 > p_2$ and $0 \leqslant r_1 < k$. By the simple calculations, we obtain

$$S_\phi^{k\,*}S_\zeta^{k\,*}(z^{p_2}) = \frac{(kp_2+1)}{(p_2+1)(m+1)}\overline{b_m}\sum_{i=0}^{n}\overline{a_i}z^i$$

and

$$S_\zeta^{k\,*}S_\phi^{k\,*}(z^{p_2}) = \frac{(kp_2+1)}{(p_2+1)}\sum_{q=0}^{min(n-kp_2,p_2)}\sum_{j=kq}^{m}\frac{(kq+1)(j-kq+1)}{(q+kp_2+1)(j+1)}\overline{b_j}\overline{a_{q+kp_2}}z^{j-kq}.$$

By using equation (3.9), it follows that $b_ma_i = 0$ for $m < i \leqslant n$ but $b_m \neq 0$ which leads to $a_i = 0$ for all i such that $m < i \leqslant n$.

Case 4. If $m = kp_2 + r_2$ where $p_2 > 0$ and $0 < r_2 < k$.

If $n = kp_1 + r_1$ where $p_1 = p_2 = p$ (say) then since $n > m$, therefore, $0 < r_2 < r_1 < k$. Then,

$$S_\phi^{k\,*}S_\zeta^{k\,*}(z^p) = \frac{(kp+1)}{(p+1)}\sum_{q=0}^{min(r_2,p)}\sum_{i=kq}^{n}\frac{(kq+1)(i-kq+1)}{(q+kp+1)(i+1)}\overline{a_i}\overline{b_{q+kp}}z^{i-kq}$$

and

$$S_\zeta^{k\,*}S_\phi^{k\,*}(z^p) = \frac{(kp+1)}{(p+1)}\sum_{s=0}^{min(r_1,p)}\sum_{j=ks}^{m}\frac{(ks+1)(j-ks+1)}{(s+kp+1)(j+1)}\overline{b_j}\overline{a_{s+kp}}z^{j-ks}.$$

From equation (3.9), it follows that $a_i b_{kp} = 0$ for $m < i \leqslant n$. Also, $n-k < m$ as $r_1 < k < r_2 + k$, therefore, $a_m b_{kp} = b_m a_{kp}$. Since $b_m \neq 0$, so if $b_{kp} = 0$ then $a_{kp} = 0$, a contradiction. Hence $b_{kp} \neq 0$, therefore, $a_i = 0$ for each i such that $m < i \leqslant n$.

If $n = kp_1 + r_1$ where $p_1 > p_2$ and $0 \leqslant r_1, r_2 < k$. Then $S_\phi^{k\,*} S_\zeta^{k\,*}(z^{p_1}) = 0$ and

$$S_\zeta^{k\,*} S_\phi^{k\,*}(z^{p_1}) = \frac{(kp_1+1)}{(p_1+1)} \sum_{q=0}^{min(r_1,p_2)} \sum_{j=kq}^{m} \frac{(kq+1)(j-kq+1)}{(q+kp_1+1)(j+1)} \overline{b_j} \overline{a_{q+kp_1}} z^{j-kq}.$$

From equation (3.9) it follows that $b_j a_{kp_1} = 0$ for $m-k < j \leqslant m$. In particular, for $j = m$ we have $b_m a_{kp_1} = 0$. Since $b_m \neq 0$, therefore, $a_{kp_1} = 0$. Thus $b_{kp_2} \neq 0$ as $a_{kp_1}^2 + b_{kp_2}^2 \neq 0$. Again calculating

$$S_\phi^{k\,*} S_\zeta^{k\,*}(z^{p_2}) = \frac{(kp_2+1)}{(p_2+1)} \sum_{q=0}^{min(r_2,p_1)} \sum_{i=kq}^{n} \frac{(kq+1)(i-kq+1)}{(q+kp_2+1)(i+1)} \overline{a_i} \overline{b_{q+kp_2}} z^{i-kq}$$

and

$$S_\zeta^{k\,*} S_\phi^{k\,*}(z^{p_2}) = \frac{(kp_2+1)}{(p_2+1)} \sum_{s=0}^{min(n-kp_2,p_2)} \sum_{j=ks}^{m} \frac{(ks+1)(j-ks+1)}{(s+kp_2+1)(j+1)} \overline{b_j} \overline{a_{q+kp_2}} z^{j-kq}$$

gives $a_i b_{kp_2} = 0$ for $n-k < i \leqslant n$ (using equation (3.9)). Since $b_{kp_2} \neq 0$ so it gives $a_i = 0$ for $n-k < i \leqslant n$. Also, $\frac{1}{kp_2+1} a_i b_{kp_2} + \frac{(k+1)(i+1)}{(kp_2+2)(i+k+1)} a_{i+k} b_{kp_2+1} = 0$ for all i such that $n-2k < i \leqslant n-k$. Since $a_{i+k} = 0$ for $n-k < i+k \leqslant n$ and $b_{kp_2} \neq 0$ leads to $a_i = 0$ for $n-2k < i \leqslant n-k$. Continuing like this, we conclude that $a_i = 0$ for all i such that $m < i \leqslant n$. □

THEOREM 5. *Let $\phi(z) = \sum_{i=0}^{n} a_i \overline{z}^i$ and $\zeta(z) = \sum_{j=0}^{m} b_j \overline{z}^j$ be such that $\phi, \zeta \in L^\infty(\mathbb{D}, dA)$ where n and m are non negative integers such that $n > m$. Let $n = kp_1 + r_1$, $m = kp_2 + r_2$ where p_1, p_2, r_1, r_2 are integers such that p_1, $p_2 \geqslant 0$, $0 \leqslant r_1, r_2 < k$ and also let $b_{kp_2} \neq 0$ and $b_m \neq 0$ then S_ϕ^k and S_ζ^k commute if and only if ϕ and ζ are linearly dependent.*

Proof. Let ϕ and ζ are linearly dependent then it is obvious that S_ϕ^k and S_ζ^k commute. Conversely, suppose that S_ϕ^k and S_ζ^k commute. Since $b_{kp_2} \neq 0$, therefore, $a_{kp_1}^2 + b_{kp_2}^2 \neq 0$. Hence, by previous lemma, $a_j = 0$ for all integer j such that $m < j \leqslant n$. Let if possible, there exists a non negative integer t with $t \leqslant m$ such that $a_i = 0$ for each $t < i \leqslant m$ and $a_t \neq 0$. Then again by previous lemma, $b_j = 0$ for all integer j such that $t < j \leqslant m$ but $b_m \neq 0$ so, it gives $a_m \neq 0$. Hence, by theorem (4), it follows that ϕ and ζ are linearly dependent. □

Acknowledgement. Support of CSIR-UGC Research Grant(UGC) [Ref. No.: 21/12/2014(ii) EU-V, Sr. No. 2121440601] to second author for carrying out the research work is gratefully acknowledged.

REFERENCES

[1] P. AIENA, *Fredholm and local spectral theory, with applications to multipliers*, Kluwer Academic Publishers, Dordrecht, 2004.

[2] S. C. ARORA, R. BATRA AND M. P. SINGH, *Slant Hankel operators*, Arch. Math. (Brno) **42**, 2 (2006), 125–133.

[3] S. C. ARORA AND J. BHOLA, *The compression of a kth-order slant Hankel operator*, Ganita **59**, 1 (2008), 1–11.

[4] S. C. ARORA AND J. BHOLA, *Weyl's theorem for a class of operators*, Int. J. Contemp. Math. Sci. **6**, 25 (2011), 1213–1220.

[5] R. G. DOUGLAS, *Banach algebra techniques in operator theory*, Academic Press **49**, New York-London, 1972.

[6] P. DUREN AND A. SCHUSTER, *Bergman spaces, Mathematical Surveys and Monographs*, American Mathematical Society **100**, 2004.

[7] P. R. HALMOS, *A Hilbert space problem book*, D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto, Ont.-London, 1967.

[8] M. C. HO, *Properties of slant Toeplitz operators*, Indiana Univ. Math. J. **45**, 3 (1996), 843–862.

[9] Y. LU, C. LIU AND J. YANG, *Commutativity of kth-order slant Toeplitz operators*, Math. Nachr. **283**, 9 (2010), 1304–1313.

[10] V. V. PELLER, *An excursion into the theory of Hankel operators*, Math. Sci. Res. Inst. Publ., Holomorphic spaces (Berkeley, CA, 1995) **33**, Cambridge Univ. Press, Cambridge, 65–120, 1998.

[11] V. V. PELLER, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[12] S. C. POWER, *Hankel operators on Hilbert space*, Bull. London Math. Soc. **12**, 6 (1980), 422–442.

(Received June 7, 2017)

Anuradha Gupta
Associate professor, Delhi college of arts and commerce
Department of mathematics, Netaji Nagar, University of Delhi
Delhi, India
e-mail: dishna2@yahoo.in

Bhawna Gupta
Department of mathematics
University of Delhi
New Delhi, India
e-mail: swastik.bhawna26@gmail.com