Operators
and
Matrices
Volume 13, Number 1 (2019), 221–229
doi:10.7153/oam-2019-13-15

ON THE SUM OF POWERS OF SQUARE MATRICES

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(*Communicated by R.-C. Li*)

Abstract. Given a 2×2 matrix A, we obtain the formula for sum of A^n, $(n\in\mathbb{Z})$, using its trace and determinant only; this includes the negative powers in the case of a nonsingular matrix too. Here we mean by sum, the sum of all the entries of the matrix. Various special cases arising out of values of trace and determinant are discussed and as an application we also derive Marcus-Newman inequality proved by D. London. $2\,\mathrm{su}(A^3)\geqslant \mathrm{su}(A)\,\mathrm{su}(A^2)$, for all $A\in\mathscr{F}_2\cap\mathscr{M}_2^+$.

1. Introduction

The study of sum of powers of matrices was initiated by Marcus and Newman [4] in 1962. For a nonnegative symmetric $n\times n$ matrix A, they conjectured the inequality $n\,\mathrm{su}(A^3)\geqslant \mathrm{su}(A)\,\mathrm{su}(A^2)$. Later London, in 1966, [2], [3], disproved it for $n=4$ and proved the same for $n\leqslant 3$. The inequality $n\,\mathrm{su}(A^2)\geqslant \mathrm{su}(A)\,\mathrm{su}(A)$ is proved with the help of simple counting argument in graph theory when A is a nonnegative symmetric matrix by Seymour [6]. In 1985, Kankaanpää and Merikoski [1] generalized the Marcus-Newman inequality. Merikoski [5] surveyed extensively the results related to the trace and sums of a matrix and its powers. However, the formula of sum of the powers of a matrix in terms of trace and determinant is not found in the literature. We initiate the investigation of this aspect to obtain a formula for sum of a power of a 2×2 matrix using determinant and trace of a matrix. In what follows, Δ will denote the determinant of the matrix under consideration and $M_n,\mathscr{F}_n$ and M_n^+ will denote the spaces of real, real symmetric and (element-wise) nonnegative $n\times n$ matrices respectively, where $n\in\mathbb{Z}_+$.

The first section is devoted to the main result observing the pattern in the sums of higher orders of a 2×2 matrix. As a corollary to the main result, we record many particular cases of importance and also give, as an application, an inequality proved by London [3]. In second section, we obtain the formula for the sum of a negative power of A where A is a nonsingular 2×2 matrix. Again the special cases are recorded and an analogue of the inequality for negative powers is obtained.

Mathematics subject classification (2010): Primary 15A24, Secondary 15A45.
Keywords and phrases: Sum of elements of a matrix, power of a matrix, nonsingular matrix, symmetric matrix, trace, determinant.
The first author gratefully acknowledges the UGC support under UGC-SAP-DRS programme F-510/5/DRS/2004 (SAP-II) as well as F-510/3/DRS/2009 (SAP-III) to the Department of Mathematics, Sardar Patel University.

2. Sum of a positive power of A

For a matrix $A=[a_{ij}]$, $\mathrm{su}(A)$ denotes the sum of all the entries of A, that is $\mathrm{su}(A)=\sum_i\sum_j a_{ij}$.

While going through the rigorous computations of the sum of all entries of higher order of a 2×2 matrix, we found the characteristic equation playing a major role. To exhibit the power of the Cayley-Hamilton theorem and the motivation behind the proof, we state the formulae of $\mathrm{su}(A^k)$, in their final form, for some values of k. We prove the general formula by induction in Theorem 2.1.

Let A be a 2×2 matrix.

Recall the characteristic equation of A

$$\det(A-\lambda I)=\lambda^2-\mathrm{Tr}(A)\lambda+\Delta=0.$$

By the Cayley Hamilton theorem, we have $A^2-\mathrm{Tr}(A)A+\Delta I=0$. Multiplying both sides by A^n, we get,

$$A^{n+2}-\mathrm{Tr}(A)A^{n+1}+\Delta A^n=0. \tag{1}$$

The sum being the linear operator, gives (2), a recurring relation, which is central to this note.

$$\mathrm{su}(A^{n+2})-\mathrm{Tr}(A)\,\mathrm{su}(A^{n+1})+\Delta\,\mathrm{su}(A^n)=0.$$

Rewriting the same we get the following,

$$\mathrm{su}(A^{n+2})=\mathrm{Tr}(A)\,\mathrm{su}(A^{n+1})-\Delta\,\mathrm{su}(A^n). \tag{2}$$

Putting a particular value of n and simplifying, we have the following.

$$\mathrm{su}(A^2)=\mathrm{Tr}(A)\,\mathrm{su}(A)-2\Delta, \tag{3}$$

$$\mathrm{su}(A^3)=\mathrm{su}(A)(\mathrm{Tr}(A)^2-\Delta)-2\Delta\,\mathrm{Tr}(A), \tag{4}$$

$$\begin{aligned}
\mathrm{su}(A^4)&=\mathrm{su}(A)\left(\mathrm{Tr}(A)^3+\frac{(-1)^1}{1!}2\,\mathrm{Tr}(A)^{3-2(1)}\Delta\right)\\
&\quad-2\Delta\left(\mathrm{Tr}(A)^2+\frac{(-1)^1}{1!}\mathrm{Tr}(A)^{2-2(1)}\Delta\right),\\
\mathrm{su}(A^5)&=\mathrm{su}(A)\left(\mathrm{Tr}(A)^4+\frac{(-1)^1}{1!}3\,\mathrm{Tr}(A)^{4-2(1)}\Delta+\frac{(-1)^2}{2!}2\cdot 1\,\mathrm{Tr}(A)^{4-2(2)}\Delta^2\right)\\
&\quad-2\Delta\left(\mathrm{Tr}(A)^3+\frac{(-1)^1}{1!}2\,\mathrm{Tr}(A)^{3-2(1)}\Delta\right),\\
\mathrm{su}(A^6)&=\mathrm{su}(A)\left(\mathrm{Tr}(A)^5+\frac{(-1)^1}{1!}4\,\mathrm{Tr}(A)^{5-2(1)}\Delta+\frac{(-1)^2}{2!}3\cdot 2\,\mathrm{Tr}(A)^{5-2(2)}\Delta^2\right)\\
&\quad-2\Delta\left(\mathrm{Tr}(A)^4+\frac{(-1)^1}{1!}3\,\mathrm{Tr}(A)^{4-2(1)}\Delta+\frac{(-1)^2}{2!}2\cdot 1\,\mathrm{Tr}(A)^{4-2(2)}\Delta^2\right).
\end{aligned}$$

It is quite apparent that the complexity of the formula increases as the power increases. Well within the ninth power, the formula really becomes highly involved.

$$\begin{aligned}
\mathrm{su}(A^9) = \mathrm{su}(A)\Bigg(&\mathrm{Tr}(A)^8 + \frac{(-1)^1}{1!}7\,\mathrm{Tr}(A)^{8-2(1)}\Delta + \frac{(-1)^2}{2!}6\cdot 5\,\mathrm{Tr}(A)^{8-2(2)}\Delta^2 \\
&+\frac{(-1)^3}{3!}5\cdot 4\cdot 3\,\mathrm{Tr}(A)^{8-2(3)}\Delta^3 + \frac{(-1)^4}{4!}4\cdot 3\cdot 2\cdot 1\,\mathrm{Tr}(A)^{8-2(4)}\Delta^4\Bigg) \\
&-2\Delta\Bigg(\mathrm{Tr}(A)^7 + \frac{(-1)^1}{1!}6\,\mathrm{Tr}(A)^{7-2(1)}\Delta + \frac{(-1)^2}{2!}5\cdot 4\,\mathrm{Tr}(A)^{7-2(2)}\Delta^2 \\
&+\frac{(-1)^3}{3!}4\cdot 3\cdot 2\,\mathrm{Tr}(A)^{7-2(3)}\Delta^3\Bigg).
\end{aligned}$$

Empirically, we reach the following formulation, which we prove by the mathematical induction.

THEOREM 2.1. *If $A = [a_{ij}] \in M_2$, then*

$$\begin{aligned}
\mathrm{su}(A^{n+2}) = \mathrm{su}(A) &\sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^r \binom{n+1-r}{r} \mathrm{Tr}(A)^{n+1-2r}\Delta^r \\
&-2\Delta \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} \mathrm{Tr}(A)^{n-2r}\Delta^r,
\end{aligned} \tag{5}$$

where $\lfloor\ \rfloor$ is the floor value function.

Proof. As already observed, (5) holds for $n = 0$. That is, $\mathrm{su}(A^2) = \mathrm{Tr}(A)\,\mathrm{su}(A) - 2\Delta$. Now we assume that (5) holds for all positive integers less than n so that

$$\begin{aligned}
\mathrm{su}(A^{n}) = \mathrm{su}(A) &\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^r \binom{n-1-r}{r} \mathrm{Tr}(A)^{n-1-2r}\Delta^r \\
&-2\Delta \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^r \binom{n-2-r}{r} \mathrm{Tr}(A)^{n-2-2r}\Delta^r,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathrm{su}(A^{n+1}) = \mathrm{su}(A) &\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} \mathrm{Tr}(A)^{n-2r}\Delta^r \\
&-2\Delta \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^r \binom{n-1-r}{r} \mathrm{Tr}(A)^{n-1-2r}\Delta^r.
\end{aligned} \tag{7}$$

Now multiplying (6) by $-\triangle$, and (7) by $\mathrm{Tr}(A)$, and substituting in (2), we have,

$$
\begin{aligned}
&\mathrm{su}(A^{n+2})\\
&= \mathrm{Tr}(A)\,\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\binom{n-r}{r}\mathrm{Tr}(A)^{n-2r}\Delta^r - 2\Delta\,\mathrm{Tr}(A)\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-1-2r}\Delta^r\\
&\quad -\triangle\,\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-1-2r}\Delta^r + 2\Delta^2\sum_{r=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^r\binom{n-2-r}{r}\mathrm{Tr}(A)^{n-2-2r}\Delta^r\\
&= \mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\binom{n-r}{r}\mathrm{Tr}(A)^{n+1-2r}\Delta^r - 2\Delta\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-2r}\Delta^r\\
&\quad -\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-1-2r}\Delta^{r+1} + 2\Delta\sum_{r=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^r\binom{n-2-r}{r}\mathrm{Tr}(A)^{n-2-2r}\Delta^{r+1}\\
&= \mathrm{su}(A)\left(\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\binom{n-r}{r}\mathrm{Tr}(A)^{n+1-2r}\Delta^r + \sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^{r+1}\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-1-2r}\Delta^{r+1}\right)\\
&\quad -2\Delta\left(\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-2r}\Delta^r + \sum_{r=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^{r+1}\binom{n-2-r}{r}\mathrm{Tr}(A)^{n-2-2r}\Delta^{r+1}\right)\\
&= \mathrm{su}(A)\left(\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\binom{n-r}{r}\mathrm{Tr}(A)^{n+1-2r}\Delta^r + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor+1}(-1)^r\binom{n-r}{r-1}\mathrm{Tr}(A)^{n+1-2r}\Delta^r\right)\\
&\quad -2\Delta\left(\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\binom{n-1-r}{r}\mathrm{Tr}(A)^{n-2r}\Delta^r + \sum_{r=1}^{\lfloor\frac{n-2}{2}\rfloor+1}(-1)^r\binom{n-1-r}{r-1}\mathrm{Tr}(A)^{n-2r}\Delta^r\right)\\
&= \mathrm{su}(A)\,\mathrm{Tr}(A)^{n+1} + \mathrm{su}(A)\left(\sum_{r=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^r\,\mathrm{Tr}(A)^{n+1-2r}\Delta^r\left(\binom{n-r}{r}+\binom{n-r}{r-1}\right)\right)\\
&\quad -2\Delta\,\mathrm{Tr}(A)^n - 2\Delta\left(\sum_{r=1}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\,\mathrm{Tr}(A)^{n-2r}\Delta^r\left(\binom{n-1-r}{r}+\binom{n-1-r}{r-1}\right)\right)\\
&= \mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^r\binom{n+1-r}{r}\mathrm{Tr}(A)^{n+1-2r}\Delta^r - 2\Delta\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\binom{n-r}{r}\mathrm{Tr}(A)^{n-2r}\Delta^r. \quad \square
\end{aligned}
$$

The following corollaries are important cases of Theorem 2.1.

COROLLARY 2.2. *If* $\det(A) = 0$, *then* $\mathrm{su}(A^n) = \mathrm{su}(A)\,[\mathrm{Tr}(A)]^{n-1}$.

COROLLARY 2.3. *If* $\mathrm{Tr}(A) = 0$, *then*

$$\mathrm{su}(A^n) = \begin{cases} 2(-1)^{\frac{n}{2}}\Delta^{\frac{n}{2}} & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}}\,\mathrm{su}(A)\Delta^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

COROLLARY 2.4. *If* $\mathrm{su}(A) = 0$, *then*

$$\mathrm{su}(A^{n+2}) = -2\Delta \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} \mathrm{Tr}(A)^{n-r}\Delta^r.$$

COROLLARY 2.5. *If* $\mathrm{su}(A) = 0 = \mathrm{Tr}(A)$, *then*

$$\mathrm{su}(A^n) = \begin{cases} 2(-1)^{\frac{n}{2}}\Delta^{\frac{n}{2}} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

COROLLARY 2.6. *If* $\mathrm{su}(A) = 0 = \det(A)$, *then* $\mathrm{su}(A^n) = 0$ *for all* $n \geqslant 2$.

COROLLARY 2.7. *If* $\mathrm{Tr}(A) = det(A) = 0$, *then* $\mathrm{su}(A^n) = 0$ *for all* $n \geqslant 2$.

COROLLARY 2.8. *If* $\mathrm{su}(A) = \mathrm{Tr}(A) = det(A) = 0$, *then* $\mathrm{su}(A^n) = 0$ *for all* $n \geqslant 2$.

COROLLARY 2.9. *If* $\mathrm{su}(A) \neq 0$ *and* $\det(A) = 0$, *then* $\frac{\mathrm{su}(A^n)}{\mathrm{su}(A^{n-1})} = \mathrm{Tr}(A)$.

Proof. From Corollary 2.2

$$\frac{\mathrm{su}(A^n)}{\mathrm{su}(A^{n-1})} = \frac{\mathrm{su}(A)[\mathrm{Tr}(A)]^{n-1}}{\mathrm{su}(A)[\mathrm{Tr}(A)]^{n-2}} = \mathrm{Tr}(A). \quad \square$$

London [3], Kankaanpää and Merikoski [1] obtained the following as the main theorem, we obtain the same as the application to theorem 1 for 2×2 matrix.

THEOREM 2.10. *If* $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_2^+$, *then*

$$2\,\mathrm{su}(A^3) \geqslant \mathrm{su}(A)\,\mathrm{su}(A^2). \tag{8}$$

Proof. From equations (3) and (4)

$$\begin{aligned} &2\,\mathrm{su}(A^3) - \mathrm{su}(A)\,\mathrm{su}(A^2) \\ &\quad = 2\left[\mathrm{su}(A)(\mathrm{Tr}(A)^2 - \Delta) - 2\Delta\,\mathrm{Tr}(A)\right] - \mathrm{su}(A)[\mathrm{Tr}(A)\,\mathrm{su}(A) - 2\Delta] \\ &\quad = \mathrm{su}(A)\,\mathrm{Tr}(A)\,[2\,\mathrm{Tr}(A) - \mathrm{su}(A)] - 4\Delta\,\mathrm{Tr}(A) \\ &\quad = \mathrm{su}(A)\,\mathrm{Tr}(A)[a + c - 2b] - 4[ac - b^2]\,\mathrm{Tr}(A) \end{aligned}$$

$$\begin{aligned}
&= \operatorname{Tr}(A)[\operatorname{su}(A)(a+c-2b)-4ac+4b^2] \\
&= \operatorname{Tr}(A)[(a+2b+c)(a+c-2b)-4ac+4b^2] \\
&= \operatorname{Tr}(A)(a-c)^2 \geqslant 0.
\end{aligned}$$

This completes the proof. □

REMARK 2.11.

1. If $a=c$, then equality holds in (8).
2. The proof of Theorem 2.1 reveals that the the inequality (8) holds even when $A \in M_2^+$ is replaced by a weaker condition $\operatorname{Tr}(A) \geqslant 0$.

3. Sum of a negative power of a nonsingular matrix A

The analogue of the formula (5) for the sum of the nonnegative powers also holds for the negative powers. We deal with the same in this section. The proof is on the same line by means of the mathematical induction.

THEOREM 3.1. *If $A=[a_{ij}] \in M_2$, is a nonsingular matrix, then*

$$\operatorname{su}(A^{-n}) = \frac{1}{\Delta^n}\left[2\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^r\binom{n-r}{r}\operatorname{Tr}(A)^{n-2r}\Delta^r - \operatorname{su}(A)\sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor -1}(-1)^r\binom{n-r-1}{r}\operatorname{Tr}(A)^{n-1-2r}\Delta^r\right],$$

where $\lfloor\ \rfloor$ is the floor value function.

Proof. Let us note that (1) and all equations based on it hold even for negative powers of A provided A is invertible. Henceforth we shall be exploiting the same with the assumption that A is invertible. Rewriting (2), we have for $n \in \mathbb{N}$,

$$\operatorname{su}(A^{-n}) = \frac{1}{\Delta}[\operatorname{Tr}(A)\operatorname{su}(A^{-n+1}) - \operatorname{su}(A^{-n+2})]. \tag{9}$$

For $n=1$, in (9), we have $\operatorname{su}(A^{-1}) = \frac{1}{\Delta}[\operatorname{Tr}(A)\operatorname{su}(A^0) - \operatorname{su}(A^1)]$. That is,

$$\operatorname{su}(A^{-1}) = \frac{1}{\Delta}[2\operatorname{Tr}(A) - \operatorname{su}(A^1)],$$

because $A^0 = I$. We assume that the result holds for $n \leqslant k$. That is,

$$\operatorname{su}(A^{-k}) = \frac{1}{\Delta^k}\left[2\sum_{r=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^r\binom{k-r}{r}\operatorname{Tr}(A)^{k-2r}\Delta^r - \operatorname{su}(A)\sum_{r=0}^{\lfloor \frac{k+1}{2} \rfloor -1}(-1)^r\binom{k-r-1}{r}\operatorname{Tr}(A)^{k-1-2r}\Delta^r\right] \tag{10}$$

is true and we prove it for $n=k+1$. Using (9) and (10),

$$
\begin{aligned}
&\mathrm{su}(A^{-(k+1)})\\
&=\frac{1}{\Delta}[\mathrm{Tr}(A)\,\mathrm{su}(A^{-k})-\mathrm{su}(A^{-(k-1)})]\\
&=\frac{1}{\Delta^{k+1}}\left[2\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^r\binom{k-r}{r}\mathrm{Tr}(A)^{k+1-2r}\Delta^r-\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{k+1}{2}\rfloor-1}(-1)^r\binom{k-r-1}{r}\mathrm{Tr}(A)^{k-2r}\Delta^r\right]\\
&\quad-\frac{1}{\Delta^{k}}\left[2\sum_{r=0}^{\lfloor\frac{k-1}{2}\rfloor}(-1)^r\binom{k-1-r}{r}\mathrm{Tr}(A)^{k-1-2r}\Delta^r-\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor-1}(-1)^r\binom{k-r-2}{r}\mathrm{Tr}(A)^{k-2-2r}\Delta^r\right]\\
&=\frac{1}{\Delta^{k+1}}\left[2\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^r\binom{k-r}{r}\mathrm{Tr}(A)^{k+1-2r}\Delta^r+2\sum_{r=0}^{\lfloor\frac{k-1}{2}\rfloor}(-1)^{r+1}\binom{k-1-r}{r}\mathrm{Tr}(A)^{k-1-2r}\Delta^{r+1}\right]\\
&\quad-\frac{\mathrm{su}(A)}{\Delta^{k+1}}\left[\sum_{r=0}^{\lfloor\frac{k+1}{2}\rfloor-1}(-1)^r\binom{k-r-1}{r}\mathrm{Tr}(A)^{k-2r}\Delta^r+\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor-1}(-1)^{r+1}\binom{k-r-2}{r}\mathrm{Tr}(A)^{k-2-2r}\Delta^{r+1}\right]\\
&=\frac{1}{\Delta^{k+1}}\left[2\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^r\binom{k-r}{r}\mathrm{Tr}(A)^{k+1-2r}\Delta^r+2\sum_{r=1}^{\lfloor\frac{k-1}{2}\rfloor+1}(-1)^r\binom{k-r}{r-1}\mathrm{Tr}(A)^{k+1-2r}\Delta^r\right]\\
&\quad-\frac{\mathrm{su}(A)}{\Delta^{k+1}}\left[\sum_{r=0}^{\lfloor\frac{k+1}{2}\rfloor-1}(-1)^r\binom{k-r-1}{r}\mathrm{Tr}(A)^{k-2r}\Delta^r+\sum_{r=1}^{\lfloor\frac{k}{2}\rfloor}(-1)^r\binom{k-r-1}{r-1}\mathrm{Tr}(A)^{k-2r}\Delta^r\right]\\
&=\frac{1}{\Delta^{k+1}}\left[2\,\mathrm{Tr}(A)^{k+1}+2\sum_{r=1}^{\lfloor\frac{k+1}{2}\rfloor}(-1)^r\left(\binom{k-r}{r}+\binom{k-r}{r-1}\right)\mathrm{Tr}(A)^{k+1-2r}\Delta^r\right]\\
&\quad-\frac{\mathrm{su}(A)}{\Delta^{k+1}}\left[\mathrm{Tr}(A)^k+\sum_{r=1}^{\lfloor\frac{k+2}{2}\rfloor-1}(-1)^r\left(\binom{k-r-1}{r}+\binom{k-r-1}{r-1}\right)\mathrm{Tr}(A)^{k-2r}\Delta^r\right]\\
&=\frac{1}{\Delta^{k+1}}\left[2\sum_{r=0}^{\lfloor\frac{k+1}{2}\rfloor}(-1)^r\binom{k+1-r}{r}\mathrm{Tr}(A)^{k+1-2r}\Delta^r-\mathrm{su}(A)\sum_{r=0}^{\lfloor\frac{k+2}{2}\rfloor-1}(-1)^r\binom{k-r}{r}\mathrm{Tr}(A)^{k-2r}\Delta^r\right].
\end{aligned}
$$

□

As the special cases, we deduce the following results from Theorem 3.1.

COROLLARY 3.2. *If A is a nonsingular matrix with* $\mathrm{Tr}(A)=0$, *then*

$$
\mathrm{su}(A^{-n})=\begin{cases}\dfrac{2(-1)^{\frac{n}{2}}}{(\Delta)^{\frac{n}{2}}}, & \text{if } n\equiv 0 \mod 2;\\[2ex] \dfrac{(-1)^{\frac{n+1}{2}}\mathrm{su}(A)}{(\Delta)^{\frac{n+1}{2}}}, & \text{if } n\equiv 1 \mod 2.\end{cases}
$$

COROLLARY 3.3. *If A is a nonsingular matrix with* $\mathrm{su}(A)=0$*, then*

$$\mathrm{su}(A^{-n}) = \frac{1}{\Delta^n}[2\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^r \binom{n-r}{r} \mathrm{Tr}(A)^{n-2r}\Delta^r].$$

COROLLARY 3.4. *If A is a nonsingular matrix with* $\mathrm{Tr}(A)=0=\mathrm{su}(A)$*, then*

$$\mathrm{su}(A^{-n}) = \begin{cases} \frac{2(-1)^{\frac{n}{2}}}{(\Delta)^{\frac{n}{2}}}, & \text{if } n \equiv 0 \mod 2; \\ 0, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

As an application of Theorem 3.1, we derive the following inequalities analogous to Theorem 2.10.

THEOREM 3.5. *If* $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_2^+$ *is a nonsingular matrix, then*

$$2\,\mathrm{su}(A^{-3}) \geqslant \mathrm{su}(A^{-1})\,\mathrm{su}(A^{-2}) \text{ if } \Delta > 0; \tag{11}$$

$$2\,\mathrm{su}(A^{-3}) \leqslant \mathrm{su}(A^{-1})\,\mathrm{su}(A^{-2}) \text{ if } \Delta < 0. \tag{12}$$

Proof. From Theorem 3.1,

$$\begin{aligned} 2\,\mathrm{su}(A^{-3}) - \mathrm{su}(A^{-1})\,\mathrm{su}(A^{-2}) &= \frac{2}{\Delta^3}[2(\mathrm{Tr}(A)^3 - 2\Delta\,\mathrm{Tr}(A)) - \mathrm{su}(A)(\mathrm{Tr}(A)^2-\Delta)] \\ &\quad - \frac{1}{\Delta^3}[2\,\mathrm{Tr}(A) - \mathrm{su}(A^1)][2(\mathrm{Tr}(A)^2-\Delta) - \mathrm{Tr}(A)\,\mathrm{su}(A)] \\ &= \frac{1}{\Delta^3}[-4\Delta\,\mathrm{Tr}(A) + 2\,\mathrm{Tr}(A)^2\,\mathrm{su}(A) - \mathrm{Tr}(A)[\mathrm{su}(A)]^2] \\ &= \frac{\mathrm{Tr}(A)}{\Delta^3}[-4(ac-b^2) + \mathrm{su}(A)(2\,\mathrm{Tr}(A) - \mathrm{su}(A))] \\ &= \frac{\mathrm{Tr}(A)}{\Delta^3}[-4ac + 4b^2 + (a+c+2b)(a+c-2b)] \\ &= \frac{\mathrm{Tr}(A)}{\Delta^3}(a-c)^2. \end{aligned}$$

Hence the inequalities (11) and (12) follow. □

REMARK 3.6.

1. If $a = c$, then equality holds in the above theorem.
2. The condition $A \in M_2^+$ can be made even weaker by just considering $\mathrm{Tr}(A) \geqslant 0$.

Acknowledgement. Authors gratefully acknowledge a very careful reading of the manuscript by the referee annihilating even punctuation mistakes, in particular, inadvertent typos affecting the mathematical validity could be corrected only due to micro-level verification of mathematics by the referee.

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(Received July 13, 2017)

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