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IMAGES OF MULTILINEAR POLYNOMIALS OF DEGREE UP TO FOUR ON UPPER TRIANGULAR MATRICES

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Abstract. We describe the images of multilinear polynomials of degree up to four on the upper triangular matrix algebra.

1. Introduction

A famous open problem known as Lvov-Kaplansky conjecture asserts: the image of a multilinear polynomial in noncommutative variables over a field $\mathbb{K}$ on the matrix algebra $M_n(\mathbb{K})$ is always a vector space [4].

Recently, Kanel-Belov, Malev and Rowen [7] made a major breakthrough and solved the problem for $n = 2$.

A special case on polynomials of degree two has been known for long time ([9] and [1]). Recently, Mesyan [8] and Buzinski and Winstanley [3] extended this result for nonzero multilinear polynomials of degree three and four, respectively.

We will study the following variation of the Lvov-Kaplansky conjecture:

CONJECTURE 1. *The image of a multilinear polynomial on the upper triangular matrix algebra is a vector space.*

In this paper, we will answer Conjecture 1 for polynomials of degree up to four. We point out that whereas in [3] and [8] the results describe conditions under which the image of a multilinear polynomial p, $Im(p)$, contains a certain subset of $M_n(\mathbb{K})$, our results give the explicit forms of $Im(p)$ on the upper triangular matrix algebra in each case.

Throughout the paper UT_n will denote the set of upper triangular matrices. The set of all strictly upper triangular matrices will be denoted by $UT_n^{(0)}$. More generally, if $k \geqslant 0$, the set of all matrices in UT_n whose entries (i,j) are zero, for $j-i \leqslant k$, will be denoted by $UT_n^{(k)}$. Also if $i,j \in \{1,\dots,n\}$, we denote by e_{ij} the $n\times n$ matrix with 1 in the entry (i,j), and 0 elsewhere. These will be called matrix units. In particular, $UT_n^{(k)}$ is the vector space spanned by the e_{ij} with $j-i > k$.

Our main goal in this paper is to prove the following:

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THEOREM 2. *Let $n \geqslant 2$ be an integer.*

(1) *If $\mathbb{K}$ is an any field and p is a multilinear polynomial over $\mathbb{K}$ of degree two, then $Im(p)$ over UT_n is $\{0\}$, $UT_n^{(0)}$ or UT_n;*

(2) *If $\mathbb{K}$ is a field with at least n elements and p is a multilinear polynomial over $\mathbb{K}$ of degree three, then $Im(p)$ over UT_n is $\{0\}$, $UT_n^{(0)}$ or UT_n;*

(3) *If $\mathbb{K}$ is a zero characteristic field and p is a multilinear polynomial over $\mathbb{K}$ of degree four, then $Im(p)$ over UT_n is $\{0\}$, $UT_n^{(1)}$, $UT_n^{(0)}$ or UT_n.*

To prove the statement (1) we use some ideas of Shoda [9] and Albert and Muckenhoupt [1], and for statements (2) and (3) we use the polynomial reductions of Mesyan [8], Špenko [10] and Buzinski and Winstanley [3].

2. The linear span of a multilinear polynomial on UT_n

Throughout this section we will denote by $\mathbb{K}$ an arbitrary field and by $p(x_1,\ldots,x_m)$ a multilinear polynomial in $\mathbb{K}\langle X\rangle$. We will also denote by $\langle Im(p)\rangle$ the linear span of $Im(p)$ on UT_n.

We start with an analogous result to Lemma 4 from [7], where we analyse the image of a multilinear polynomial $p(x_1,\ldots,x_m)\in\mathbb{K}\langle X\rangle$ on upper triangular matrix units.

Let $e_{i_1,j_1},\ldots,e_{i_m,j_m}$ be upper triangular matrix units. Then $i_q \leqslant j_q$ for all q. We know that

$$e_{i_1,j_1}\cdots e_{i_m,j_m} \tag{1}$$

is nonzero (and equal to e_{i_1,j_m}) if and only if $j_q = i_{q+1}$, for all q.

Hence, if we change the order of the product in (1) we will obtain either 0 or e_{i_1,j_m}. To verify this claim, we will assume that we get a nonzero matrix unit after changing the order of some terms in (1). It is enough analyse just when we change the first or the last term. So, if $e_{i_k,j_k}\cdots e_{i_1,j_1}\cdots e_{i_m,j_m}$ is nonzero then $i_k \leqslant i_1$, and by the product (1) we also have $i_1 \leqslant i_k$, which proves that $i_k = i_1$ and therefore $e_{i_k,j_k}\cdots e_{i_1,j_1}\cdots e_{i_m,j_m} = e_{i_1,j_m}$. Analogously we prove that if $e_{i_1,j_1}\cdots e_{i_m,j_m}\cdots e_{i_k,j_k}$ is nonzero then this product will be e_{i_1,j_m}.

In this way, p evaluated on upper triangular matrix units is equal to zero or to some multiple of an upper triangular matrix unit.

DEFINITION 3. Let $A = \sum_{i,j=1}^{n} a_{i,j}e_{i,j} \in UT_n$. For each $k\in\{1,\ldots,n\}$ the k-th diagonal of A is the one with entries in positions $(1,k),(2,k+1),\ldots,(n-k+1,n)$. We say that the k-th diagonal of A is nonzero if at least one entry in its k-th diagonal is nonzero.

The next lemma shows that if an upper triangular matrix unit can be obtained as an evaluation of a multilinear polynomial on matrix units, then all matrix units in the same diagonal can also be obtained by one such evaluation.

LEMMA 4. *Assume that a nonzero multiple of* $e_{i,i+k-1}$ *can be written as an evaluation of* p *on upper triangular matrix units, for some* i *and* k. *Then* $e_{1,k}, e_{2,k+1}, \dots, e_{n-k+1,n} \in Im(p)$.

Proof. We write $\alpha e_{i,i+k-1} = p(e_{i_1,j_1}, \dots, e_{i_m,j_m})$, for some nonzero $\alpha \in \mathbb{K}$. Hence,

$$\alpha e_{1,k} = p(e_{i_1-i+1,j_1-i+1}, \dots, e_{i_m-i+1,j_m-i+1}),$$

and since $Im(p)$ is closed under scalar multiplication, $e_{1,k} \in Im(p)$. Analogously, we prove that $e_{2,k+1}, \dots, e_{n-k+1,n} \in Im(p)$. □

LEMMA 5. *Assume that a nonzero multiple of* $e_{i,i+k-1}$ *can be written as an evaluation of* p *on upper triangular matrix units, for some* i *and* k. *Then* $e_{i,i+k} \in Im(p)$.

Proof. We write $\alpha e_{i,i+k-1} = p(e_{i_1,j_1}, \dots, e_{i_m,j_m})$ for some nonzero $\alpha \in \mathbb{K}$. Hence $i+k-1 = j_l$ for some indexes $l \in \{1, \dots, m\}$. Replacing for each l the corresponding j_l by $j_l + 1$ we get

$$\alpha e_{i,i+k} = p(e_{i_1,j_1}, \dots, e_{i_l,j_l+1}, \dots, e_{i_m,j_m})$$

which proves that $e_{i,i+k} \in Im(p)$. □

If we also denote UT_n by $UT_n^{(-1)}$, then we have the main result of this section.

PROPOSITION 6. *Let* p *be a multilinear polynomial over* $\mathbb{K}$. *Then* $\langle Im(p) \rangle$ *is either* $\{0\}$ *or* $UT_n^{(k)}$ *for some integer* $k \geqslant -1$.

Proof. Assume that $Im(p)$ is nonzero. Hence, if $A = \sum_{i,j=1}^{n} a_{ij} e_{ij} \in Im(p)$ is nonzero, writing A as a linear combination of evaluations of p on upper triangular matrix units, we get that a multiple of e_{ij} belongs to $Im(p)$, for each nonzero (i,j) entry of A.

Let k be the minimal integer such that the k-th diagonal of some matrix $A = \sum_{i,j=1}^{n} a_{ij} e_{ij} \in Im(p)$ is nonzero. Then there exists some $a_{i,i+k-1} \neq 0$ and therefore $\alpha e_{i,i+k-1} = p(e_{i_1,j_1}, \dots, e_{i_m,j_m})$ for some nonzero $\alpha \in \mathbb{K}$. By Lemma 4 all the matrix units $e_{1,k}, \dots, e_{n-k+1,n}$ belong to $Im(p)$. By Lemma 5 $e_{i,i+k} \in Im(p)$. Using the above lemmas alternatively, we get that $UT_n^{(k-2)} \subset \langle Im(p) \rangle$. By the minimality of k we have $\langle Im(p) \rangle = UT_n^{(k-2)}$. □

By the above proposition we can restate Conjecture 1 as

CONJECTURE 7. *The image of a multilinear polynomial on the upper triangular matrix algebra is either* $\{0\}$ *or* $UT_n^{(k)}$ *for some integer* $k \geqslant -1$.

3. A technical proposition

We start with a fact about the image of multilinear polynomials of any degree on UT_n. We will prove that no subset between $UT_n^{(0)}$ and UT_n can be the image of a multilinear polynomial over UT_n.

PROPOSITION 8. *Let $\mathbb{K}$ be any field, $m \geqslant 2$ an integer and*

$$p(x_1,\dots,x_m) = \sum_{\sigma\in S_m} \alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(m)}, \alpha_\sigma \in \mathbb{K},$$

a nonzero multilinear polynomial.

(1) *if* $\sum_{\sigma\in S_m} \alpha_\sigma \neq 0$, *then* $Im(p) = UT_n$;

(2) *if* $\sum_{\sigma\in S_m} \alpha_\sigma = 0$ *and* $UT_n^{(0)} \subset Im(p)$, *then* $Im(p) = UT_n^{(0)}$.

Proof. If $\sum_{\sigma\in S_m} \alpha_\sigma \neq 0$, then replacing $m-1$ variables by I_n (the identity matrix) and the last one by $(\sum_{\sigma\in S_m} \alpha_\sigma)^{-1}A$ where A is any matrix in UT_n, we get $Im(p) = UT_n$, from which (1) follows.

If $\sum_{\sigma\in S_m} \alpha_\sigma = 0$ and $UT_n^{(0)} \subset Im(p)$, then let $\tau \in S_n$ such that $\alpha_\tau \neq 0$ (there exists such a permutation because $p \neq 0$). Then, $\alpha_\tau = -\sum_{\sigma\in S_m\setminus\{\tau\}} \alpha_\sigma$.

So,

$$p(x_1,\dots,x_m) = \sum_{\sigma\in S_m\setminus\{\tau\}} \alpha_\sigma(x_{\sigma(1)}\cdots x_{\sigma(m)} - x_{\tau(1)}\cdots x_{\tau(m)}).$$

Therefore, replacing $x_1,\dots,x_m$ by upper triangular matrices we obtain in each term of the sum above a matrix with just zeros in the main diagonal. Indeed, the main diagonal of a product of upper triangular matrices is the same, regardless of the order.

With this, we conclude that $Im(p) \subset UT_n^{(0)}$ and by hypothesis, $Im(p) = UT_n^{(0)}$. □

4. The images of multilinear polynomials of degree two

We consider a multilinear polynomial of degree two, which has the following form: $p(x,y) = \alpha xy + \beta yx$ for some $\alpha,\beta \in \mathbb{K}$. We will divide the study of the image of p in two cases.

Case 1. $\alpha + \beta \neq 0$.

In this case we can use Proposition 8 (1) and get $Im(p) = UT_n$.

Case 2. $\alpha+\beta=0$.

If $\alpha=\beta=0$ then $Im(p)=\{0\}$. Otherwise, we may assume that $p(x,y)=xy-yx$. Let $A=(a_{ij})\in UT_n^{(0)}$. Take $B=\sum_{k=1}^{n-1} e_{k,k+1}$ and $C=(c_{ij})\in UT_n$. So,

$$BC-CB = (\sum_{k=1}^{n-1} e_{k,k+1})(\sum_{i,j=1}^{n} c_{ij}e_{ij}) - (\sum_{i,j=1}^{n} c_{ij}e_{ij})(\sum_{k=1}^{n-1} e_{k,k+1}) \tag{2}$$
$$= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(c_{i+1,j}-c_{i,j-1})e_{ij}$$

Using $c_{ij}=0$ for $i>j$, we note that the diagonal entries of the matrix $BC-CB$ above are all zero.

Now we consider the system defined by the equations $c_{i+1,j}-c_{i,j-1}=a_{ij}$. A solution of this system is $c_{1k}=0$, $k=1,\dots,n$ and $c_{i+1,j}=a_{ij}+a_{i-1,j-1}+\cdots+a_{1,j-(i-1)}$ where $i<j$ and $i=2,\dots,n-1$, $j=2,\dots,n$.

So, $Im(p)\supset UT_n^{(0)}$ and by Proposition 8 (2), we have $Im(p)=UT_n^{(0)}$.

In resume, we have proved the following

PROPOSITION 9. *Let $p(x,y)\in\mathbb{K}\langle X\rangle$ be a multilinear polynomial where $\mathbb{K}$ is any field. Then $Im(p)$ on UT_n is $\{0\}$, $UT_n^{(0)}$ or UT_n.*

5. The images of multilinear polynomials of degree three

To start this section we prove the following lemma, which is an analogous of Lemma 1.2 of [2].

LEMMA 10. *Let $\mathbb{K}$ be a field with at least n elements and let $d_{11},\dots,d_{nn}\in\mathbb{K}$ be distinct elements. Then for $D=diag(d_{11},\dots,d_{nn})$ and $k\geqslant 0$, we have*

$$[UT_n^{(k)},D]=UT_n^{(k)} \text{ and } [UT_n,D]=UT_n^{(0)}.$$

Proof. Clearly, $[UT_n^{(k)},D]\subset UT_n^{(k)}$.

Now, let $A=\sum_{j-i>k} a_{ij}e_{ij}$ be an arbitrary element of $UT_n^{(k)}$. Then,

$$[A,D] = AD-DA = (\sum_{i,j=1}^{n} a_{ij}e_{ij})(\sum_{l=1}^{n} d_{ll}e_{ll}) - (\sum_{l=1}^{n} d_{ll}e_{ll})(\sum_{i,j=1}^{n} a_{ij}e_{ij})$$
$$= \sum_{j-i>k}^{n} a_{ij}(d_{jj}-d_{ii})e_{ij}$$

Hence, if $B=\sum_{j-i>k}^{n} b_{ij}e_{ij}\in UT_n^{(k)}$, we choose $a_{ij}=b_{ij}(d_{jj}-d_{ii})^{-1}$, for $j-i>k$. This proves that $[UT_n^{(k)},D]\supset UT_n^{(k)}$, and the first equality is proved.

Now we prove the second equality. It is immediate that $[UT_n, D] \subset UT_n^{(0)}$. Also, since $UT_n^{(0)} \subset UT_n$, we have $[UT_n^{(0)}, D] \subset [UT_n, D]$. Hence, from the first equation for $k = 0$, we have $UT_n^{(0)} = [UT_n^{(0)}, D] \subset [UT_n, D]$. And the second equality is proved. □

Following the proof of Theorem 13 of [8], we obtain the next theorem, where we determine the image of multilinear polynomials of degree 3 on UT_n.

THEOREM 11. *Let $\mathbb{K}$ be a field with at least n elements and let $p(x,y,z) \in \mathbb{K}\langle X\rangle$ be a multilinear polynomial. Then $Im(p)$ is $\{0\}$, $UT_n^{(0)}$ or UT_n.*

Proof. Let $p(x,y,z) \in \mathbb{K}\langle X\rangle$ be a nonzero multilinear polynomial. So,

$$p(x,y,z) = \alpha_1 xyz + \alpha_2 xzy + \alpha_3 yxz + \alpha_4 yzx + \alpha_5 zxy + \alpha_6 zyx, \alpha_l \in \mathbb{K}.$$

If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \neq 0$ then using Proposition 8 (1) we have $Im(p) = UT_n$.

Hence, we may assume that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0$. So, we write p as

$$p(x,y,z) = \alpha_1(xyz - zyx) + \alpha_2(xzy - zyx) + \alpha_3(yxz - zyx) + \alpha_4(yzx - zyx) + \alpha_5(zxy - zyx).$$

If any of $p(1,y,z), p(x,1,z)$ or $p(x,y,1)$ are non-zero, then we have by Proposition 9 that $Im(p)$ contains all upper triangular matrices with zero main diagonal. Then by Proposition 8 we have that $Im(p)$ is $UT_n^{(0)}$ or UT_n.

Otherwise, the equations $p(1,y,z) = p(x,1,z) = p(x,y,1) = 0$ imply that $\alpha_3 = \alpha_5, \alpha_2 = \alpha_4$ and $\alpha_1 = -\alpha_2 - \alpha_3$. Therefore,

$$\begin{aligned} p(x,y,z) &= (-\alpha_2 - \alpha_3)(xyz - zyx) + \alpha_2(xzy - zyx + yzx - zyx) \\ &\quad + \alpha_3(yxz - zyx + yzx - xyz) \\ &= \alpha_2(xyz - zyx + yzx - xyz) + \alpha_3(yxz - zyx + zxy - xyz) \\ &= \alpha_2[x,[z,y]] + \alpha_3[z,[x,y]] \end{aligned}$$

Since $p \neq 0$, renaming the variables if necessary, we may assume that $\alpha_2 \neq 0$ and therefore assume

$$p(x,y,z) = [x,[z,y]] + \alpha[z,[x,y]],$$

for some $\alpha \in \mathbb{K}$.

By Lemma 10, $UT_n^{(0)} = [D, UT_n^{(0)}] = [D,[UT_n, D]]$. So, taking $x = y = D$ and $z = A$ any matrix in UT_n we get all of $UT_n^{(0)}$. So, $Im(p) = UT_n^{(0)}$. □

6. The images of multilinear polynomials of degree four

In this section we will determinate the image of multilinear polynomials of degree four over a field $\mathbb{K}$ of zero characteristic.

We start with the following lemma.

LEMMA 12. *Let $\mathbb{K}$ be any field. Then $[UT_n^{(0)}, UT_n^{(0)}] = UT_n^{(1)}$.*

Proof. Clearly, $[UT_n^{(0)}, UT_n^{(0)}] \subset UT_n^{(1)}$.

Now, let $A = \sum_{k=1}^{n} e_{k,k+1} \in UT_n^{(0)}$ and $B = \sum_{i,j=1}^{n} b_{ij}e_{ij} \in UT_n^{(0)}$. The same computations as in equation (2) yields

$$[A,B] = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(b_{i+1,j} - b_{i,j-1})e_{ij}.$$

So, for $C = (c_{ij}) \in UT_n^{(1)}$, the system below has solution

$$\begin{cases} b_{23} - b_{12} &= c_{13} \\ &\vdots \\ b_{2n} - b_{1,n-1} &= c_{1n} \\ &\vdots \\ b_{n-1,n} - b_{n-2,n-1} &= c_{n-2,n} \end{cases}.$$

Indeed, we may choose $b_{1k} = 0, k = 2, \ldots, n-1$ and $b_{i+1,j} = c_{i,j} + \cdots + c_{1,j-(i-1)}$, $i = 1, \ldots, n-2$, $j = 3, \ldots, n$. Therefore, $C \in [UT_n^{(0)}, UT_n^{(0)}]$. □

Now we prove the main result for polynomials of degree 4. Our proof is based on the proof of Theorem 1 of [3].

THEOREM 13. *Let $\mathbb{K}$ be a field of zero characteristic and let $p(x_1,x_2,x_3,x_4) \in \mathbb{K}\langle X\rangle$ be a multilinear polynomial. Then the image of p on UT_n is $\{0\}$, $UT_n^{(1)}$, $UT_n^{(0)}$ or UT_n.*

Proof. We may assume that $p \neq 0$. If any of $p(1,x_2,x_3,x_4)$, $p(x_1,1,x_3,x_4)$, $p(x_1,x_2,1,x_4)$ or $p(x_1,x_2,x_3,1)$ are nonzero, then by Proposition 8 and Theorem 11, we have $Im(p) = UT_n^{(0)}$ or UT_n. So, we may assume that

$$p(1,x_2,x_3,x_4) = p(x_1,1,x_3,x_4) = p(x_1,x_2,1,x_4) = p(x_1,x_2,x_3,1) = 0.$$

Then by Falk's theorem [5] we have

$$\begin{aligned} p(x_1,x_2,x_3,x_4) &= L(x_1,x_2,x_3,x_4) + \alpha_{1234}[x_1,x_2][x_3,x_4] + \alpha_{1324}[x_1,x_3][x_2,x_4] \\ &\quad + \alpha_{1423}[x_1,x_4][x_2,x_3] + \alpha_{2314}[x_2,x_3][x_1,x_4] + \alpha_{2413}[x_2,x_4][x_1,x_3] \\ &\quad + \alpha_{3412}[x_3,x_4][x_1,x_2] \end{aligned}$$

where $L(x_1,x_2,x_3,x_4)$ is a Lie polynomial and α_{1234}, α_{1324}, α_{1423}, α_{2314}, α_{2413}, $\alpha_{3412} \in \mathbb{K}$.

Using Hall basis (see [6]) we can write

$$\begin{aligned} L(x_1,x_2,x_3,x_4) &= \alpha_1[[[x_2,x_1],x_3],x_4] + \alpha_2[[[x_3,x_1],x_2],x_4] + \alpha_3[[[x_4,x_1],x_2],x_3] \\ &\quad + \alpha_4[[x_4,x_1],[x_3,x_2]] + \alpha_5[[x_4,x_2],[x_3,x_1]] + \alpha_6[[x_4,x_3],[x_2,x_1]], \end{aligned}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\in\mathbb{K}$.

Opening the brackets for the three last terms we can assume p as

$$\begin{aligned}p(x_1,x_2,x_3,x_4) = {}& \alpha_1[[[x_2,x_1],x_3],x_4]+\alpha_2[[[x_3,x_1],x_2],x_4]+\alpha_3[[[x_4,x_1],x_2],x_3]\\ &+\alpha_{1234}[x_1,x_2][x_3,x_4]+\alpha_{1324}[x_1,x_3][x_2,x_4]+\alpha_{1423}[x_1,x_4][x_2,x_3]\\ &+\alpha_{2314}[x_2,x_3][x_1,x_4]+\alpha_{2413}[x_2,x_4][x_1,x_3]+\alpha_{3412}[x_3,x_4][x_1,x_2].\end{aligned}$$

Now suppose that for some $i=1,2,3$ we have $\alpha_i\neq 0$. Without loss of generality, we may assume that $\alpha_1\neq 0$. So, replacing x_1,x_3 and x_4 by $D=diag(d_{11},\ldots,d_{nn})$ with $d_{11},\ldots,d_{nn}$ distinct elements in $\mathbb{K}$, we get $p(D,x_2,D,D)=\alpha_1[[[x_2,D],D],D]$. Now, using Lemma 10 we have $Im(p)=UT_n^{(0)}$. So, we may assume that $\alpha_1=\alpha_2=\alpha_3=0$ and then

$$\begin{aligned}p(x_1,x_2,x_3,x_4) = {}& \alpha_{1234}[x_1,x_2][x_3,x_4]+\alpha_{1324}[x_1,x_3][x_2,x_4]+\alpha_{1423}[x_1,x_4][x_2,x_3]\\ &+\alpha_{2314}[x_2,x_3][x_1,x_4]+\alpha_{2413}[x_2,x_4][x_1,x_3]+\alpha_{3412}[x_3,x_4][x_1,x_2].\end{aligned}$$

We may assume $n\geqslant 3$ since for $n=2$ the polynomial p above is a polynomial identity for UT_2. Clearly, $Im(p)\subset UT_n^{(1)}$.

We will consider two cases.

Case 1. Assume $\alpha_{1234}=\alpha_{2314}=\alpha_{3412}=\alpha_{1423}=-\alpha_{1324}=-\alpha_{2413}$. Then we may assume that

$$\begin{aligned}p(x_1,x_2,x_3,x_4) = {}& [x_1,x_2][x_3,x_4]+[x_3,x_4][x_1,x_2]+[x_2,x_3][x_1,x_4]+[x_1,x_4][x_2,x_3]\\ &-[x_1,x_3][x_2,x_4]-[x_2,x_4][x_1,x_3].\end{aligned}$$

Consider $A\in UT_n^{(1)}$. Let $D=diag(d_{11},\ldots,d_{nn})$ where $d_{11},\ldots,d_{nn}$ are all distinct elements of $\mathbb{K}$. Then, by Lemma 10 there exists $G\in UT_n^{(1)}$ with $A=[D,G]$. By Lemma 12 there are $E,F\in UT_n^{(0)}$ such that $G=[E,F]$. Again by Lemma 10 we have $B,C\in UT_n$ such that $E=[D,B]$ and $F=[D,C]$. So, $A=[D,[[D,B],[D,C]]]$. Observing that

$$\begin{aligned}p(D,D^2,B,C) &= [D,D^2][B,C]+[B,C][D,D^2]+[D^2,B][D,C]+[D,C][D^2,B]\\ &\quad -[D,B][D^2,C]-[D^2,C][D,B]\\ &= [D^2,B][D,C]+[D,C][D^2,B]-[D,B][D^2,C]-[D^2,C][D,B]\\ &= [D,[[D,B],[D,C]]],\end{aligned}$$

we have $A\in Im(p)$, proving in this way that $Im(p)=UT_n^{(1)}$.

Case 2. Assume that at least one of following $\alpha_{1234}=\alpha_{2314}=\alpha_{3412}=\alpha_{1423}=-\alpha_{1324}=-\alpha_{2413}$ does not hold. So, there are $A,B,C\in UT_n$ such that at least one of the following expressions is not zero:

$$\begin{aligned}
p(A,A,B,C) &= (\alpha_{1324}+\alpha_{2314})[A,B][A,C]+(\alpha_{1423}+\alpha_{2413})[A,C][A,B],\\
p(A,B,A,C) &= (\alpha_{1234}-\alpha_{2314})[A,B][A,C]+(\alpha_{3412}-\alpha_{1423})[A,C][A,B],\\
p(A,B,C,A) &= (-\alpha_{1234}-\alpha_{2413})[A,B][A,C]+(-\alpha_{1324}-\alpha_{3412})[A,C][A,B],\\
p(B,A,A,C) &= (-\alpha_{1234}-\alpha_{1324})[A,B][A,C]+(-\alpha_{2413}-\alpha_{3412})[A,C][A,B],\\
p(B,A,C,A) &= (-\alpha_{1423}+\alpha_{1234})[A,B][A,C]+(\alpha_{3412}+\alpha_{2314})[A,C][A,B],\\
p(B,C,A,A) &= (\alpha_{1324}+\alpha_{1423})[A,B][A,C]+(\alpha_{2314}+\alpha_{2413})[A,C][A,B].
\end{aligned}$$

Therefore, we may reduce the problem to prove that with the expression

$$[A,B][A,C]+\lambda[A,C][A,B],\ \ \lambda\in\mathbb{K},$$

we get all elements in $UT_n^{(1)}$. Using Lemma 10 and taking $A = diag(a_{11},\dots,a_{nn})$ where all $a_{11},\dots,a_{nn}$ are distinct elements of $\mathbb{K}$, there exist $B \in UT_n$ such that $\sum_{k=1}^{n-1} e_{k,k+1}$ $= [A,B]$. Writing $[A,C] = \sum_{i,j=1}^{n} b_{ij}e_{ij}$ we have

$$\begin{aligned}
[A,B][A,C]+\lambda[A,C][A,B] &= (\sum_{k=1}^{n-1} e_{k,k+1})(\sum_{i,j=1}^{n} b_{ij}e_{ij})+\lambda(\sum_{i,j=1}^{n} b_{ij}e_{ij})(\sum_{k=1}^{n-1} e_{k,k+1})\\
&= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(b_{i+1,j}+\lambda b_{i,j-1})e_{ij}
\end{aligned}$$

So, for an arbitrary $M = (c_{ij}) \in UT_n^{(1)}$, the system below has solution.

$$\begin{cases}
b_{23}+\lambda b_{12} &= c_{13}\\
&\vdots\\
b_{2n}+\lambda b_{1,n-1} &= c_{1n}\\
&\vdots\\
b_{n-1,n}+\lambda b_{n-2,n-1} &= c_{n-2,n}
\end{cases}.$$

Indeed, we may choose $b_{1k} = 0,\ k = 2,\dots,n-1$ and

$$b_{i+1,j} = c_{i,j}-\lambda c_{i-1,j-1}+\cdots+(-\lambda)^{i-1}c_{1,j-(i-1)},\ \ i = 1,\dots,n-2,\ \ j = 3,\dots,n.$$

Therefore, $M \in Im(p)$, proving that $Im(p) = UT_n^{(1)}$. □

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