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FUGLEDE–PUTNAM THEOREM AND QUASISIMILARITY OF CLASS p–$wA(s,t)$ OPERATORS

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Abstract. We show that p-$wA(s,t)$ operators S,T^* $(s+t\leqslant 1,\ 0<p\leqslant 1)$ with $\ker(S)\subseteq\ker(S^*)$ and $\ker(T^*)\subseteq\ker(T)$ satisfy Fuglede-Putnam theorem, i.e., $SX=XT$ for some X implies $S^*X=XT^*$. Also, we show that two quasisimilar p-$wA(s,t)$ operators S,T $(s+t\leqslant 1,\ 0<p\leqslant 1)$ with $\ker(S)\subseteq\ker(S^*)$ and $\ker(T)\subseteq\ker(T^*)$ have equal spectra and essential spectra.

1. Introduction

Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space and $B(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Every operator $T\in B(\mathscr{H})$ can be decomposed into $T=U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined by the kernel condition $\ker U=\ker|T|$, then this decomposition is called the polar decomposition. In this paper, $T=U|T|$ denotes the polar decomposition with kernel condition $\ker U=\ker|T|$. Aluthge transformation introduced by Aluthge [1] as $\tilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Recall that an operator $T\in B(\mathscr{H})$ is said to be p-hyponormal if $(T^*T)^p\geqslant(TT^*)^p$ ([1]), w-hyponormal if $|\tilde{T}|\geqslant|T|\geqslant|\tilde{T}^*|$ ([2]), class A if $|T^2|\geqslant|T|^2$ ([7]), class $A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}}\geqslant|T^*|^{2t}$ ([6]) , and class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}}\geqslant|T^*|^{2t}$ and $|T|^{2s}\geqslant(|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([8, 20]). As an extension of class $wA(s,t)$, Prasad and Tanahashi [12] introduced class p-$wA(s,t)$ operators as follows;

DEFINITION 1. Let $T=U|T|$ be the polar decomposition of $T\in B(\mathscr{H})$ and let $s,t>0$ and $0<p\leqslant 1$. T is called class p-$wA(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}}\geqslant|T^*|^{2tp} \tag{1.1}$$

and

$$(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}\leqslant|T|^{2sp}. \tag{1.2}$$

Many interesting properties of class p-$wA(s,t)$ operators have been studied in [3, 12, 14, 16, 18]. In this paper, we study Fuglede-Putnam theorem and quasisimilarity of p-$wA(s,t)$ operators.

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2. Fuglede-Putnam theorem of class p-$wA(s,t)$ operators

The following Proposition is called Fuglede-Putnam theorem.

PROPOSITION 1. [Fuglede-Putnam] *Let* $S \in B(\mathscr{H})$ *and* $T^* \in B(\mathscr{K})$ *be normal operators and* $SX = XT$ *for some operator* $X \in B(\mathscr{H}, \mathscr{K})$. *Then* $S^*X = XT^*$, $[\text{ran } X]$ *reduces* S, $(\ker X)^{\perp}$ *reduces* T *and* $S|_{[\text{ran } X]}$, $T|_{(\ker X)^{\perp}}$ *are unitarily equivalent normal operators.*

Various extensions of the Fuglede-Putnam Theorem can be found in the literature. (See [13], [19]). In the present section, we extend the above theorem for class p-$wA(s,t)$ operators with reducing kernel.

Let $T = U|T|$ be the polar decomposition of T and $0 < s,t$. Then generalized Aluthge transformation $T(s,t)$ is defined by $T(s,t) = |T|^s U |T|^t$. The following results are due to Prasad, Tanahashi and Uchiyama [12, 18]

PROPOSITION 2. [12] *If* $T \in B(\mathscr{H})$ *is a class* p-$wA(s,t)$ *operator with* $0 < s$, t, $0 < p \leqslant 1$, *then* $T(s,t)$ *is* $\frac{\min\{sp,tp\}}{s+t}$*-hyponormal.*

PROPOSITION 3. [18] *If* T *is a class* p-$wA(s,t)$ *operator and* $0 < s \leqslant s_1$, $0 < t \leqslant t_1$, $0 < p_1 \leqslant p \leqslant 1$, *then* T *is a class* p_1-$wA(s_1,t_1)$ *operator.*

PROPOSITION 4. [18] *Let* $T = U|T|$ *be a class* p-$wA(s,t)$ *operator with* $0 < s+t \leqslant 1$, $0 < p \leqslant 1$. *If* $T(s,t) = |T|^s U |T|^t$ *is normal, then* T *is also normal.*

The following result is due to Conway (Proposition 2.1 of [4]).

PROPOSITION 5. [4] *If* $T \in B(\mathscr{H})$, *then there is a reducing subspace* $\mathscr{M}$ *for* T *such that*

$$T|_{\mathscr{M}} \text{ is a normal operator;} \tag{2.1}$$

$$T|_{\mathscr{M}^{\perp}} \text{has no reducing subspace on which it is normal.} \tag{2.2}$$

DEFINITION 2. We say $T|_{\mathscr{M}}$ is normal part and $T|_{\mathscr{M}^{\perp}}$ is pure part of T.

LEMMA 1. *Let* $T \in B(\mathscr{H})$ *be a class* p-$wA(s,t)$ *operator with* $0 < s$, t, $s+t \leqslant 1$, $0 < p \leqslant 1$ *such that* $\ker T \subseteq \ker T^*$ *and let* $S \in B(\mathscr{K})$ *be a normal operator. If there exists an operator* $X \in B(\mathscr{K}, \mathscr{H})$ *with dense range such that* $TX = XS$, *then* T *is normal.*

Proof. We may assume $s+t = 1$ by Proposition 3. Since $\ker T \subseteq \ker T^*$, $\ker T = \ker |T|$ reduces T. Let $T = 0 \oplus T_1$ on $\ker|T| \oplus \overline{\text{ran}|T|}$. Let $T_1 = U_1|T_1|$ be the polar decomposition of T_1. Then $T = U|T| = (0 \oplus U_1)(0 \oplus |T_1|)$ is the polar decomposition of T. Hence $T(s,t) = |T|^s U |T|^t = 0 \oplus |T_1|^s U_1 |T_1|^t = 0 \oplus T_1(s,t)$. Put $W = |T|^s X$ and $\mathscr{H}_1 = \overline{\text{ran}|T|}$. Then $W \in B(\mathscr{K}, \mathscr{H}_1)$ has dense range and $T_1(s,t)$ satisfies

$$T_1(s,t)W = WS.$$

Since $T_1(s,t)$ is $\frac{\min\{st,tp\}}{s+t}$-hyponormal, $T_1(s,t)$ is normal by by [9, Lemma 3]. Hence $T(s,t)$ is normal. Thus T is normal by Proposition 4. $\square$

The above result for p-hyponormal operators are due to Jeon and Duggal [9] and for class $wA(s,t)$ due to Rashid [15].

LEMMA 2. *Let $T = U|T| \in B(\mathscr{H})$ be a class p-$wA(s,t)$ operator with $0 < s$, t, $s+t=1$, $0 < p \leqslant 1$ and $\ker T \subset \ker T^*$. Let $T(s,t) = |T|^s U|T|^t$. Suppose $T(s,t)$ be of the form $N \oplus T'$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$ where N is a normal operator on $\mathscr{M}$. Then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$ where T_1 is a class p-$wA(s,t)$ operator with $\ker T_1 \subset \ker T_1^*$ and $N = U_{11}|N|$ is the polar decomposition of N.*

Proof. Since

$$|T(s,t)|^{2pr} \geqslant |T|^{2pr} \geqslant |T(s,t)^*|^{2pr}$$

for $r \in (0, \min\{s,t\}]$, we have

$$|N|^{2pr} \oplus |T'|^{2pr} \geqslant |T|^{2pr} \geqslant |N|^{2pr} \oplus |T'^*|^{2pr}$$

by assumption. Let

$$|T|^{2pr} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Then

$$S_{11} = |N|^{2pr}, \ |T'|^{2pr} \geqslant S_{22} \geqslant |T'^*|^{2pr}$$

and

$$\left\langle |T|^{2pr} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} |N|^{2pr} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$
$$= \langle |N|^{2pr} x, x \rangle + 2\mathrm{Re}\, \langle S_{12} x, y \rangle + \langle S_{22} y, y \rangle.$$

for all $x \in \mathscr{M}, y \in \mathscr{M}^{\perp}$. Hence $S_{12} = 0$ and $|T| = |N| \oplus L$ for some positive operator L. Let

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Then $T(s,t) = |T|^s U |T|^t$ means

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^s & 0 \\ 0 & L^s \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^t & 0 \\ 0 & L^t \end{pmatrix}$$
$$= \begin{pmatrix} |N|^s U_{11} |N|^t & |N|^s U_{12} L^t \\ L^s U_{21} |N|^t & L^s U_{22} L^t \end{pmatrix}.$$

Since $\ker T \subset \ker T^*$,

$$[\mathrm{ran}\ U] = [\mathrm{ran}\ T] = (\ker T^*)^{\perp} \subset (\ker T)^{\perp} = [\mathrm{ran}\ |T|]$$

where $[\mathrm{ran}\ X]$ is the norm closure of the range of X. Let $Nx = 0$ for $x \in \mathscr{M}$. Then $x \in \ker |T| = \ker U$, and

$$Ux = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} x \\ U_{21} x \end{pmatrix} = 0.$$

Hence

$$\ker N \subset \ker U_{11} \cap \ker U_{21}.$$

Let $x \in \mathscr{M}$. Then

$$U\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} \in [\text{ran } |T|] = [\text{ran } (|N| \oplus L)].$$

Hence

$$\text{ran } U_{11} \subset [\text{ran } |N|], \quad \text{ran } U_{21} \subset [\text{ran } L].$$

Similarly

$$\text{ran } U_{12} \subset [\text{ran } |N|], \quad \text{ran } U_{22} \subset [\text{ran } L].$$

Let $Lx = 0$ for $x \in \mathscr{M}^{\perp}$. Then $x \in \ker|T| = \ker U$ and

$$U\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} = 0.$$

Hence

$$\ker L \subset \ker U_{12} \cap \ker U_{22}.$$

Let $N = V|N|$ be the polar decomposition of N. Then

$$(V|N|^s - |N|^s U_{11})|N|^t = N - |N|^s U_{11}|N|^t = 0.$$

Hence $V|N|^s - |N|^s U_{11} = 0$ on $[\text{ran } |N|]$. Since $\ker N \subset \ker U_{11}$, this implies $0 = V|N|^s - |N|^s U_{11} = |N|^s(V - U_{11})$. Hence

$$\text{ran } (V - U_{11}) \subset \ker|N| \cap [\text{ran } |N|] = \{0\}.$$

Hence $V = U_{11}$ and $N = U_{11}|N|$ is the polar decomposition of N. Since $|N|^s U_{12} L^t = 0$,

$$\text{ran } U_{12} L^t \subset \ker|N| \cap [\text{ran } |N|] = \{0\}.$$

Hence $U_{12}L^t = 0$ and $U_{12} = 0$ because $\ker L \subset \ker U_{12}$. Similarly we have $U_{21} = 0$ by $L^s U_{21}|N|^t = 0$. Hence $U = U_{11} \oplus U_{22}$. So we obtain

$$T = U|T| = U_{11}|N| \oplus U_{22}L = N \oplus T_1,$$

where $T_1 = U_{22}L$. □

THEOREM 6. *Let $S \in B(\mathscr{H})$ and $T^* \in B(\mathscr{K})$ are class p-$wA(s,t)$ operators with $0 < s,\ t,\ s+t \leqslant 1$, $0 < p \leqslant 1$ and $\ker S \subset \ker S^*$, $\ker T^* \subset \ker T$. Let $SX = XT$ for some operator $X \in B(\mathscr{K}, \mathscr{H})$. Then $S^*X = XT^*$, $[\text{ran } X]$ reduces S, $(\ker X)^{\perp}$ reduces T, and $S|_{[\text{ran } X]}, T|_{(\ker X)^{\perp}}$ are unitarily equivalent normal operators.*

Proof. We may assume $s + t = 1$ by Proposition 3. Decompose S, T^* into normal parts and pure parts as in Proposition 5, $S = S_1 \oplus S_2$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $T^* = T_1^* \oplus T_2^*$ on $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$ where S_1, T_1^* are normal and S_2, T_2^* are pure. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : \mathscr{K}_1 \oplus \mathscr{K}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_2.$$

Then $SX = XT$ implies

$$\begin{pmatrix} S_1X_{11} & S_1X_{12} \\ S_2X_{21} & S_2X_{22} \end{pmatrix} = \begin{pmatrix} X_{11}T_1 & X_{12}T_2 \\ X_{21}T_1 & X_{22}T_2 \end{pmatrix}.$$

Let $S_2 = U_2|S_2|, T_2^* = V_2^*|T_2^*|$ be the polar decompositions and $W = |S_2|^s X_{22}|T_2^*|^s$. Then

$$S_2(s,t)W = |S_2|^s S_2 X_{22}|T_2^*|^s = |S_2|^s X_{22}T_2|T_2^*|^s = W(T_2^*(s,t))^*.$$

Since S_2, T_2^* are class p-$wA(s,t)$ operators, $S_2(s,t), T_2^*(s,t)$ are $\min\{sp,tp\}$-hyponormal. Hence $[\text{ran } W]$ reduces $S_2(s,t)$, $(\ker W)^\perp$ reduces $T_2^*(s,t)$ and

$$S_2(s,t)|_{[\text{ran } W]} \simeq (T_2^*(s,t))^*|_{(\ker W)^\perp}$$

are unitarily equivalent normal operators by Theorem 7 of [5]. Since S_2, T_2^* are pure, we have $W = 0$ by Lemma 2. Then $X_{22} = 0$ as S_2, T_2^* are injective by assumption $\ker S \subset \ker S^*, \ker T^* \subset \ker T$. Since $S_2X_{21} = X_{21}T_1$ and $S_1X_{12} = X_{12}T_2$, we have $X_{21}T_1 = 0$ and $S_1X_{12} = 0$ by similar arguments. Then $X_{12} = 0, X_{21} = 0$ as S_2, T_2^* are injective. Since $S_1X_{11} = X_{11}T_1$, we have $S_1^*X_{11} = X_{11}T_1^*$, $[\text{ran } X_{11}]$ reduces S_1, $(\ker X_{11})^\perp$ reduces T_1, and $S_1|_{[\text{ran } X_{11}]}$, $T_1|_{(\ker X_{11})^\perp}$ are unitarily equivalent normal operators by Proposition 1. This implies that $S^*X = XT^*, [\text{ran } X]$ reduces $S, (\ker X)^\perp$ reduces T and $S|_{[\text{ran } X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators. □

COROLLARY 1. *For each $j = 1,2$, let $T_j \in B(\mathscr{H}_j)$ be a class p-$wA(s,t)$ operator with kernel condition $\ker T_j \subset \ker T_j^*$ and let $T_j = N_j \oplus P_j$ on $\mathscr{H}_j = \mathscr{H}_{j1} \oplus \mathscr{H}_{j2}$, where N_j is normal part and P_j is pure part of T_j. If T_1 and T_2 are quasisimilar, then N_1 and N_2 are unitarily equivalent and there exist $V \in B(\mathscr{H}_{22}, \mathscr{H}_1)$ and $W \in B(\mathscr{H}_{12}, \mathscr{H}_2)$ having dense range such that $P_1V = VP_2$ and $WP_1 = P_2W$.*

Proof. By hypothesis there exist $X \in B(\mathscr{H}_2, \mathscr{H}_1)$, $Y \in B(\mathscr{H}_1, \mathscr{H}_2)$ with injective and dense range such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} : \mathscr{H}_{21} \oplus \mathscr{H}_{22} \to \mathscr{H}_{11} \oplus \mathscr{H}_{12}.$$

Then

$$\begin{pmatrix} N_1X_1 & N_1X_2 \\ P_1X_3 & P_1X_4 \end{pmatrix} = \begin{pmatrix} X_1N_2 & X_2N_2 \\ X_3P_2 & X_4P_2 \end{pmatrix}.$$

Since $P_1X_3 = X_3N_2$, we have $[\text{ran} X_3]$ reduces P_1, $(\ker X_3)^\perp$ reduces N_2 and $P_1|_{[\text{ran} X_3]} = N_2|_{(\ker X_3)^\perp}$ are unitarily equivalent normal operators by Theorem 6. Since P_1 is pure, we have $X_3 = 0$. Hence X_1 is injective and $N_1X_1 = X_2N_2$. Then N_1 and N_2 are unitarily equivalent by Lemma 1.1 of [21]. Also, X_4 has dense range and $P_1X_4 = X_4P_2$. The rest of the proof is similar. □

3. Essential spectra of quasisimiar class p-$wA(s,t)$ operators

Two operators $T \in B(\mathscr{H})$ and $S \in B(\mathscr{K})$ is called quasisimilar if there exist injective operators $X \in B(\mathscr{H},\mathscr{K}), Y \in B(\mathscr{K},\mathscr{H})$ with dense rages such that $XT = SX$ and $YS = TY$. This equivalence relation of quasisimilarity was introduced by Sz.-Nagy and Foias and has received considerable attention. In general, quasisimilarity need not preserve the spectrum and essential spectrum. However, quasisimilarity preserves spectra in special classes of operators. For instance, if T and S are quasisimilar hyponormal operators then $\sigma(T) = \sigma(S)$ by Corollary 3 of [17] and $\sigma_e(T) = \sigma_e(S)$ by Theorem 2.4 of [21] (see [5, 10, 22]).

In this section, we show quasisimilar class p-$wA(s,t)$ operators have equal spectra and essential spectra.

Recall that an operator T is said to be subscalar if it is the restriction of a scalar operator to an invariant subspace. It is well known that subscalar operators satisfy Bishop's property (β).

PROPOSITION 7. ([18]) *If T is class p-$wA(s,t)$ with $0 < s,\ t$ and $0 < p \leqslant 1$, then T satisfies Bishop's property (β) and T is subscalar.*

THEOREM 8. *Let $S \in B(\mathscr{H})$ and $T \in B(\mathscr{K})$ be quasisimilar class p-$wA(s,t)$ operators with $0 < s,\ t,\ s+t \leqslant 1,\ 0 < p \leqslant 1$. If there exist $X \in B(\mathscr{K},\mathscr{H})$, $Y \in B(\mathscr{H},\mathscr{K})$ with dense ranges such that $SX = XT$, $YS = TY$. Then $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$.*

Proof. Since S and T satisfies Bishop's property (β) by Proposition 7, we have $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$ by Theorem 3.7.15 of [11]. $\square$

COROLLARY 2. *Let $S \in B(\mathscr{H})$ and $T \in B(\mathscr{K})$ be quasisimilar class p-$wA(s,t)$ operators with $0 < s,\ t,\ s+t \leqslant 1,\ 0 < p \leqslant 1$. Then $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$.*

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