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MATRICES WHOSE POWERS EVENTUALLY HAVE CERTAIN PROPERTIES

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Abstract. The matrices whose powers eventually have some special properties is an interesting object of study, such as eventually positive matrices. This paper investigates the matrices whose powers eventually have certain structural properties. We completely characterize those complex square matrices whose powers become and remain diagonal, Toeplitz, normal, respectively.

1. Introduction

A complex square matrix A is called *nilpotent* if there exists a positive integer k such that A^k is a zero matrix. Thus nilpotent matrices are the matrices whose powers become and remain zero.

Two matrices X and Y are said to be *permutation similar* if there exists a permutation matrix P such that $P^TXP = Y$. A square matrix A is called *reducible* if A is permutation similar to a matrix of the form

$$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square matrices of order at least 1, and 0 is a zero matrix. A square matrix that is not reducible is called *irreducible*.

Recall that every square matrix has its *Frobenius normal form* [1, p. 57]. That is, for a square matrix A of order n, there exists a permutation matrix P of order n and an integer $t \geqslant 1$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ 0 & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{tt} \end{bmatrix}, \tag{1}$$

where $A_{11}, A_{22}, \ldots, A_{tt}$ are irreducible matrices.

A matrix is *positive (nonnegative)* if all of its entries are positive (nonnegative) real numbers. Let A be an irreducible nonnegative square matrix. Denote by $\rho(A)$ the

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spectral radius of A. Suppose A has exactly t eigenvalues of modulus $\rho(A)$. If $t=1$, then A is said to be *primitive*; otherwise A is *imprimitive*. A theorem of Frobenius [10, p. 134] states that a nonnegative square matrix A of order at least 2 is primitive if and only if there exists a positive integer k such that A^k is a positive matrix. Thus the primitive matrices of order at least 2 are the nonnegative matrices whose powers become and remain positive.

Inspired by the above examples, it is natural and interesting to consider the following general problem.

PROBLEM. *Characterize the (nonnegative) square matrices A for which there exists a positive integer k such that A^i has a certain property for all integers $i \geqslant k$.*

The complex square matrices, whose powers become and remain positive, nonnegative, reducible and irreducible respectively, have been studied extensively. See [3, 4, 6–9] and the references therein. In this paper, we focus on another three basic kinds of matrices: diagonal matrices, Toeplitz matrices and normal matrices. Our aim is to describe those matrices whose powers become and remain diagonal, Toeplitz and normal respectively. We give a complete characterization of them in Sections 2-4.

2. Matrices whose powers become and remain diagonal

If A is a matrix, $A(i,j)$ denotes its entry in the i-th row and j-th column. The following fact is clear.

LEMMA 2.1. *Let A be a complex matrix of order n. If there exists a positive integer k such that A^i is diagonal for all integers $i \geqslant k$, then given an integer j with $1 \leqslant j \leqslant n$, either $A^i(j,j)=0$ for all integers $i \geqslant k$, or $A^i(j,j) \neq 0$ for all integers $i \geqslant k$.*

The following theorem characterizes the matrices whose powers become and remain diagonal.

THEOREM 2.2. *Let A be an $n \times n$ complex matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is diagonal for all integers $i \geqslant k$.

(ii) There exists a permutation matrix P such that $A = P\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix}P^T$, where N is a nilpotent matrix of order m and Λ is an invertible diagonal matrix of order $n-m$.

Proof. (i)$\Rightarrow$(ii). By Lemma 2.1, there exists a permutation matrix P such that $P^TA^iP = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_i \end{bmatrix}$ for all integers $i \geqslant k$, where Λ_i is an invertible diagonal matrix. Since A has exactly m eigenvalues equal to 0, so does A^i. Thus Λ_i is a matrix of order $n-m$ for all integers $i \geqslant k$.

Partition $P^TAP = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where A_1 is of order m and A_4 is of order $n-m$. Since $AA^i = A^iA$, $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_i \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ for all integers $i \geqslant k$. Then

$A_2\Lambda_i = 0, \Lambda_i A_3 = 0$. By the invertibility of Λ_i, $A_2 = A_3 = 0$. Thus $P^TAP = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$. Since $\begin{bmatrix} A_1^i & 0 \\ 0 & A_4^i \end{bmatrix} = (P^TAP)^i = P^TA^iP = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_i \end{bmatrix}$, A_1 is a nilpotent matrix and A_4^i is an invertible diagonal matrix for all integers $i \geqslant k$. Thus $A_4 = (A_4^k)^{-1}A_4^{k+1}$ is an invertible diagonal matrix. Let $N = A_1, \Lambda = A_4$. Then $A = P\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix}P^T$.

(ii)$\Rightarrow$(i). This is clear by direct verification and it suffice to choose k equal to m. $\square$

LEMMA 2.3. *Let A be a nilpotent nonnegative matrix. Then A is permutation similar to a strictly upper triangular matrix.*

Proof. Consider the Frobenius normal form (1) of A. Since A is nilpotent, each A_{ii} is nilpotent, $i = 1,2,\dots,t$. By the Perron-Frobenius theorem [10, p. 123], each $A_{ii} = 0$ is of order 1, $i = 1,2,\dots,t$. Thus A is permutation similar to a strictly upper triangular matrix. $\square$

The following theorem characterizes the nonnegative matrices whose powers become and remain diagonal.

THEOREM 2.4. *Let A be an $n \times n$ nonnegative matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is diagonal for all integers $i \geqslant k$.

(ii) There exists a permutation matrix P such that $A = P\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix}P^T$, where N is a strictly upper triangular matrix of order m and Λ is a positive diagonal matrix of order $n-m$.

Proof. (i)$\Rightarrow$(ii). By Theorem 2.2, there exists a permutation matrix P_1 such that $A = P_1\begin{bmatrix} N_1 & 0 \\ 0 & \Lambda \end{bmatrix}P_1^T$, where N_1 is a nilpotent nonnegative matrix of order m and Λ is a positive diagonal matrix of order $n-m$. By Lemma 2.3, there exists a permutation matrix P_2 of order m such that $P_2^TN_1P_2$ is a strictly upper triangular matrix. Denote by I_n the identity matrix of order n. Let $P = P_1\begin{bmatrix} P_2 & 0 \\ 0 & I_{n-m} \end{bmatrix}$, $N = P_2^TN_1P_2$. Then P is a permutation matrix and $A = P\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix}P^T$.

(ii)$\Rightarrow$(i). This is clear by direct verification and it suffice to choose k equal to m. $\square$

It is predictable that a nonnegative matrix whose powers become and remain diagonal cannot have too many positive entries. By Theorem 2.4, we can determine the maximum number of positive entries in a nonnegative matrix whose powers become and remain diagonal as well as characterize those matrices that attain the maximum number.

COROLLARY 2.5. *Let A be a nonnegative matrix of order $n \geqslant 2$. If there exists a positive integer k such that A^i is diagonal for all integers $i \geqslant k$, then we have the following conclusions:*

(i) If $n = 2$, then A has at most 2 positive entries, and A has exactly 2 positive entries if and only if A is a positive diagonal matrix.

(ii) If $n = 3$, then A has at most 3 positive entries, and A has exactly 3 positive entries if and only if A is either a positive diagonal matrix or permutation similar to a strictly upper triangular matrix with all the entries above the main diagonal being positive.

(iii) If $n \geqslant 4$, then A has at most $n(n-1)/2$ positive entries, and A has exactly $n(n-1)/2$ positive entries if and only if A is permutation similar to a strictly upper triangular matrix with all the entries above the main diagonal being positive.

3. Matrices whose powers become and remain Toeplitz

A matrix of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \cdots & a_0 \end{bmatrix}$$

is called a *Toeplitz matrix*. A matrix of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ ra_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ ra_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ra_1 & ra_2 & ra_3 & \cdots & a_0 \end{bmatrix}$$

is called an *r-circulant matrix*. In particular, a 1-circulant matrix is a circulant matrix, and a 0-circulant matrix is an upper triangular Toeplitz matrix. Lower triangular Toeplitz matrices together with r-circulant matrices are called *generalized circulant matrices*. Clearly, generalized circulant matrices are special Toeplitz matrices.

LEMMA 3.1. *Let A be a Toeplitz matrix. If A^2 is still a Toeplitz matrix, then A is a generalized circulant matrix.*

Proof. Suppose A has order n. Since A is a Toeplitz matrix, we can denote $A(i,j) = a_{j-i}$. Then $A^2(i,j) = \sum_{k=1}^{n} A(i,k)A(k,j) = \sum_{k=1}^{n} a_{k-i}a_{j-k}$. Since A^2 is a Toeplitz matrix, $A^2(i,j) = A^2(i+1,j+1)$ for all $i,j = 1,2,\ldots,n-1$; i.e., $\sum_{k=1}^{n} a_{k-i}a_{j-k} = \sum_{k=1}^{n} a_{k-i-1}a_{j+1-k}$. Thus $a_{n-i}a_{j-n} = a_{-i}a_j$ for all $i,j = 1,2,\ldots,n-1$. It follows that the $(n-1)\times 2$ matrix

$$B=\begin{bmatrix} a_{-1} & a_{n-1}\\ a_{-2} & a_{n-2}\\ \vdots & \vdots\\ a_{1-n} & a_1 \end{bmatrix}$$

has all its minors of order 2 equal to 0; i.e., $\operatorname{rank} B=0$ or 1. This implies that A is a generalized circulant matrix. $\square$

The *Fourier transform matrix* of order n is defined to be

$$F=\frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1}\\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix},$$

where $\omega=e^{\frac{2\pi i}{n}}$ $(i=\sqrt{-1})$. Note that $\det F\neq 0$.

LEMMA 3.2. ([2]) *Let $r\neq 0$. Then A is an r-circulant matrix of order n if and only if there exist $\lambda_1,\lambda_2,\ldots,\lambda_n$ such that $F^{-1}D^{-1}ADF=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, where $D=\operatorname{diag}(1,r^{\frac{1}{n}},r^{\frac{2}{n}},\ldots,r^{\frac{n-1}{n}})$ and F is the Fourier transform matrix of order n.*

LEMMA 3.3. *Any power of a generalized circulant matrix is still a generalized circulant matrix.*

Proof. Let A be a generalized circulant matrix. We distinguish two cases.

(i) Suppose A is a lower triangular Toeplitz matrix. Direct verification shows that any power of a lower triangular Toeplitz matrix is still a lower triangular Toeplitz matrix.

(ii) Suppose A is an r-circulant matrix. If $r\neq 0$, by Lemma 3.2, any power of A is still an r-circulant matrix. If $r=0$, then A is an upper triangular Toeplitz matrix. Direct verification shows that any power of an upper triangular Toeplitz matrix is still an upper triangular Toeplitz matrix. $\square$

LEMMA 3.4. *The inverse of an invertible generalized circulant matrix is still a generalized circulant matrix.*

Proof. Let A be an invertible generalized circulant matrix of order n. We distinguish two cases.

(i) Suppose A is a lower triangular Toeplitz matrix. Assume that the characteristic polynomial of A is $\lambda^n+c_1\lambda^{n-1}+\cdots+c_{n-1}\lambda+c_n$. Since A is invertible, $c_n\neq 0$. By the Cayley-Hamilton Theorem, $A^n+c_1A^{n-1}+\cdots+c_{n-1}A+c_nI_n=0$. Thus $A^{-1}=-\frac{1}{c_n}(A^{n-1}+c_1A^{n-2}+\cdots+c_{n-1}I_{n-1})$ is a lower triangular Toeplitz matrix.

(ii) Suppose A is an r-circulant matrix. If $r\neq 0$, by Lemma 3.2, A^{-1} is still an r-circulant matrix. If $r=0$, then A is an upper triangular Toeplitz matrix. A similar argument as in (i) shows that the inverse of an invertible upper triangular Toeplitz matrix is still an upper triangular Toeplitz matrix. $\square$

THEOREM 3.5. *Let A be a Toeplitz matrix. Then the following statements are equivalent:*

(i) Any power of A is a Toeplitz matrix.
(ii) A^2 is a Toeplitz matrix.
(iii) A is a generalized circulant matrix.
(iv) Any power of A is a generalized circulant matrix.

Proof. (i)$\Rightarrow$(ii). Obvious.
(ii)$\Rightarrow$(iii). Lemma 3.1.
(iii)$\Rightarrow$(iv). Lemma 3.3.
(iv)$\Rightarrow$(i). Obvious. $\square$

LEMMA 3.6. *Let A be an r-circulant matrix of order n and let B be a lower triangular Toeplitz matrix of order n. If $AB = BA$, then either A or B is a scalar matrix.*

Proof. Suppose A and B have the forms

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ ra_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ ra_1 & ra_2 & \cdots & a_0 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_{n-1} & \cdots & b_1 & b_0 \end{bmatrix}.$$

If $AB = BA$, it suffices to prove $A = a_0 I_n$ or $B = b_0 I_n$.

Assume that $b_i \neq 0$ for some i with $1 \leqslant i \leqslant n-1$. First consider the $(i, n-i)$ entry of $AB = BA$, we have $a_{n-i} = 0$. Then successively consider the $(i-1, n-i), (i-2, n-i), \ldots, (1, n-i)$ entries of $AB = BA$, we have $a_{n-i+1} = 0, a_{n-i+2} = 0, \ldots, a_{n-1} = 0$. Finally successively consider the $(i, n-i-1), (i, n-i-2), \ldots, (i, 1)$ entries of $AB = BA$, we have $a_{n-i-1} = 0, a_{n-i-2} = 0, \ldots, a_1 = 0$. Thus $a_1 = a_2 = \cdots = a_{n-1} = 0$; i.e., $A = a_0 I_n$. This completes the proof. $\square$

LEMMA 3.7. *Let A be a complex square matrix. If there exists a positive integer k such that A^i is a generalized circulant matrix for all integers $i \geqslant k$, then either A^i is an r-circulant matrix for all integers $i \geqslant k$, or A^i is a lower triangular Toeplitz matrix for all integers $i \geqslant k$.*

Proof. First note that a matrix is an r-circulant matrix as well as a lower triangular Toeplitz matrix if and only if it is a scalar matrix. Assume that there exists an integer $j \geqslant k$ such that A^j is a lower triangular Toeplitz matrix but not a scalar matrix. By Lemma 3.6 and $A^i A^j = A^j A^i$, it follows that A^i is a lower triangular Toeplitz matrix for all integers $i \geqslant k$. Otherwise there exist $r_k, r_{k+1}, \ldots$ such that A^i is an r_i-circulant matrix for all integers $i \geqslant k$. In this case, on the one hand $(A^k)^i$ is still an r_k-circulant matrix, on the other hand $(A^i)^k$ is still an r_i-circulant matrix. Since $(A^k)^i = (A^i)^k$, $r_i = r_k \triangleq r$. Thus A^i is an r-circulant matrix for all integers $i \geqslant k$. $\square$

The following theorem characterizes the matrices whose powers become and remain Toeplitz.

THEOREM 3.8. *Let A be an $n\times n$ complex matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is a Toeplitz matrix for all integers $i \geqslant k$.

(ii) There exists a positive integer k such that A^i is a generalized circulant matrix for all integers $i \geqslant k$.

(iii) Either A is a triangular Toeplitz matrix, or there exists $r\neq 0$ and a permutation matrix P such that $A = DFP\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} P^{-1}F^{-1}D^{-1}$, where $D = \mathrm{diag}(1, r^{\frac{1}{n}}, r^{\frac{2}{n}}, \ldots, r^{\frac{n-1}{n}})$, F is the Fourier transform matrix of order n, N is a nilpotent matrix of order m and Λ is an invertible diagonal matrix of order $n-m$.

Proof. (i)$\Rightarrow$(ii). Since $A^k, A^{k+1}, A^{k+2}, \ldots$ are all Toeplitz matrices, so is $(A^i)^2$ for all integers $i \geqslant k$. By Lemma 3.1, A^i is a generalized circulant matrix for all integers $i \geqslant k$.

(ii)$\Rightarrow$(iii). Since A^i is a generalized circulant matrix for all integers $i \geqslant k$, by Lemma 3.7, either A^i is an r-circulant matrix for all integers $i \geqslant k$, or A^i is a lower triangular Toeplitz matrix for all integers $i \geqslant k$.

Suppose A^i is an r-circulant matrix for all integers $i \geqslant k$. We distinguish two cases.

Case 1. $r \neq 0$. By Lemma 3.2, $F^{-1}D^{-1}A^iDF = \Lambda_i$, where Λ_i is a diagonal matrix for all integers $i \geqslant k$, $D = \mathrm{diag}(1, r^{\frac{1}{n}}, r^{\frac{2}{n}}, \ldots, r^{\frac{n-1}{n}})$ and F is the Fourier transform matrix of order n. Let $B = F^{-1}D^{-1}ADF$. Then $B^i = \Lambda_i, i = k, k+1, k+2, \ldots$. Since A has exactly m eigenvalues equal to 0, so does B. By Theorem 2.2, there exists a permutation matrix P such that $B = P\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} P^T$, where N is a nilpotent matrix of order m and Λ is an invertible diagonal matrix of order $n-m$. Thus $A = DFP\begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} P^{-1}F^{-1}D^{-1}$.

Case 2. $r = 0$. Then $A^k, A^{k+1}, A^{k+2}, \ldots$ are all upper triangular Toeplitz matrices.

If A is invertible, then $(A^k)^{-1}$ exists. By Lemma 3.4, $(A^k)^{-1}$ is an upper triangular Toeplitz matrix. It is not difficult to verify that the product of upper triangular Toeplitz matrices is still an upper triangular Toeplitz matrix. Thus $A = (A^k)^{-1}A^{k+1}$ is an upper triangular Toeplitz matrix.

If A is not invertible, then each diagonal entry of A^k is 0. Thus $(A^k)^n = 0$; i.e., A is nilpotent. It follows that $m = n$. Let $D = P = I_n$. Then $A = DFPNP^{-1}F^{-1}D^{-1}$, where $N = F^{-1}AF$ is nilpotent.

Suppose A^i is a lower triangular Toeplitz matrix for all integers $i \geqslant k$. This is similar to Case 2 above.

(iii)$\Rightarrow$(i). This is clear by direct verification and it suffice to choose k equal to m. $\square$

COROLLARY 3.9. *Let A be an $n\times n$ complex matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is a circulant matrix for all integers $i \geqslant k$.

(ii) There exists a permutation matrix P such that $A = FP \begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} P^{-1}F^{-1}$, where F is the Fourier transform matrix of order n, N is a nilpotent matrix of order m and Λ is an invertible diagonal matrix of order $n-m$.

4. Matrices whose powers become and remain normal

A *family* of matrices is a nonempty finite or infinite set of matrices, and a *commuting family* is a family of matrices in which every pair of matrices commutes under multiplication.

LEMMA 4.1. ([5, p. 103]) *Let Ω be a commuting family of matrices. Then there exists a unitary matrix U such that for every $A \in \Omega$, U^*AU is an upper triangular matrix.*

The following theorem characterizes the matrices whose powers become and remain normal.

THEOREM 4.2. *Let A be an $n \times n$ complex matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is normal for all integers $i \geqslant k$.

(ii) There exists a unitary matrix U such that $A = U \begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} U^*$, where N is a strictly upper triangular matrix of order m and Λ is an invertible diagonal matrix of order $n-m$.

Proof. (i)$\Rightarrow$(ii). First note that $\{A^i\}_{i \geqslant k}$ is a commuting family. By Lemma 4.1, there exists a unitary matrix V such that V^*A^iV is an upper triangular matrix for all integers $i \geqslant k$. Since V^*A^iV is normal, $V^*A^iV = (V^*AV)^i$ is diagonal for all integers $i \geqslant k$. By Theorem 2.2, there exists a permutation matrix P such that $V^*AV = P \begin{bmatrix} N_1 & 0 \\ 0 & \Lambda \end{bmatrix} P^T$, where N_1 is a nilpotent matrix of order m and Λ is an invertible diagonal matrix of order $n-m$. Then there exists a unitary matrix W of order m such that W^*N_1W is a strictly upper triangular matrix. Let $N = W^*N_1W$, $U = VP \begin{bmatrix} W & 0 \\ 0 & I_{n-m} \end{bmatrix}$. Then U is unitary and $A = U \begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} U^*$.

(ii)$\Rightarrow$(i). This is clear by direct verification and it suffice to choose k equal to m. $\square$

COROLLARY 4.3. *Let A be an $n \times n$ complex matrix with exactly m eigenvalues equal to 0. Then the following statements are equivalent:*

(i) There exists a positive integer k such that A^i is Hermitian for all integers $i \geqslant k$.

(ii) There exists a unitary matrix U such that $A = U \begin{bmatrix} N & 0 \\ 0 & \Lambda \end{bmatrix} U^*$, where N is a strictly upper triangular matrix of order m and Λ is an invertible real diagonal matrix of order $n-m$.

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