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A NOTE ON TRIANGULAR OPERATORS ON SMOOTH SEQUENCE SPACES

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Dedicated to the memory of Prof. Dr. Tosun Terzioğlu

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Abstract. For a scalar sequence $(\theta_n)_{n\in\mathbb{N}}$, let C be the matrix defined by $c_n^k = \theta_{n-k+1}$ if $n \geqslant k$, $c_n^k = 0$ if $n < k$. The map between Köthe spaces $\lambda(A)$ and $\lambda(B)$ is called a Cauchy Product map if it is determined by the triangular matrix C. In this note we introduced some necessary and sufficient conditions for a Cauchy Product map on a nuclear Köthe space $\lambda(A)$ to nuclear G_1-space $\lambda(B)$ to be linear and continuous. Its transpose is also considered.

1. Introduction

We refer the reader to [3], [4] and [5] for the terminology used but not defined here. Let $A = (a_n^k)_{n,k\in\mathbb{N}}$ be a matrix of real numbers such that $0 \leqslant a_n^k \leqslant a_n^{k+1}$ for all n,k and $\sup\limits_k a_n^k > 0$. The ℓ^1 - Köthe space $\lambda(A)$ defined by the matrix A is the space of all sequences of scalars $x = (x_n)$ such that

$$\|x\|_k = \sum_n |x_n| a_n^k < \infty, \quad \forall k \in \mathbb{N}.$$

With the topology generated by the system of seminorms $\{\|.\|_k, k \in \mathbb{N}\}$, it is a Fréchet space.

The topological dual of $\lambda(A)$ is isomorphic to the space of all sequences u for which $|u_n| \leqslant C a_n^k$ for some k and $C > 0$.

It is well known that a Köthe space $\lambda(A)$ associated with the matrix A is nuclear if and only if for each k there exists m such that

$$\sum_n \frac{a_n^k}{a_n^m} < +\infty$$

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and in this case the fundamental system of norms $\|x\|_k = \sum_n |x_n| a_n^k$ can be replaced by the equivalent system of norms

$$\|x\|_k = \sup_n |x_n| a_n^k, \quad k \in \mathbb{N}.$$

The infinite and finite type power series spaces are well known examples of Köthe spaces given by the matrices $(e^{k\alpha_n})$ respectively $(e^{-\frac{\alpha_n}{k}})$ where (α_n) is a monotonically increasing sequence going to infinity. The space $A(\mathbb{C})$ of all entire functions on $\mathbb{C}$ and the space $A(\mathbb{D})$ of all holomorphic functions on the unit disc can be represented as an infinite respectively finite type power series spaces.

Smooth sequence spaces were introduced in [6] as a generalization of power series spaces. A Köthe set $A = \{(a_n^k)\}$ is called a G_∞-set and the corresponding Köthe space $\lambda(A)$ a G_∞-space if A satisfies the followings:

(1) $a_n^1 = 1$, $a_n^k \leqslant a_{n+1}^k$ for each k and n;

(2) $\forall k\ \exists j$ with $(a_n^k)^2 = O(a_n^j)$.

A Köthe set $B = \{(b_n^k)\}$ is called a G_1-set and the corresponding Köthe space $\lambda(B)$ a G_1-space if B satisfies the followings:

(1) $0 < b_{n+1}^k \leqslant b_n^k < 1$ for each k and n;

(2) $\forall k\ \exists j$ with $b_n^k = O((b_n^j)^2)$.

We need the following result [1].

LEMMA 1. *Let $\lambda(A)$ and $\lambda(B)$ be Köthe spaces. A map $T : \lambda(A) \longrightarrow \lambda(B)$ is continuous linear map if and only if for each k there exists m such that*

$$\sup_n \frac{\|Te_n\|_k}{\|e_n\|_m} < +\infty.$$

If (a_n), (b_n) are two sequences of scalars, then the Cauchy product $(c_n) = (a_n) * (b_n)$ of (a_n) and (b_n) is defined by $c_n = \sum_{k=1}^{n} a_{n+1-k} b_k$.

Now let $\theta = (\theta_n)$ be a fixed sequence of scalars and let $\lambda(A)$, $\lambda(B)$ be two nuclear ℓ^1-Köthe spaces. We define the Cauchy Product mapping T_θ from $\lambda(A)$ into $\lambda(B)$ by $T_\theta x = \theta * x, \quad x = (x_n) \in \lambda(A)$. So, $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ can be determined by the lower triangular matrix

$$C = \begin{pmatrix} \theta_1 & 0 & 0 & 0 & \cdots \\ \theta_2 & \theta_1 & 0 & 0 & \cdots \\ \theta_3 & \theta_2 & \theta_1 & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}.$$

2. Cauchy product map on Köthe spaces

In this section we introduce some necessary and sufficient conditions for the map T_θ to be linear and continuous.

THEOREM 1. *Let $\lambda(A)$ be a nuclear Köthe space, $\lambda(B)$ be a nuclear G_1-space. Then the Cauchy product map $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ is linear continuous operator if and only if the following hold:*

i) $\theta \in \lambda(B)$;

ii) $\lambda(A) \subset \lambda(B)$.

Proof. Let $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ be a continuous linear operator.
Note that $\|T_\theta e_n\|_k = \|(0,0,...,0,\theta_1,\theta_2,\cdots)\|_k = \sup_{j\geqslant n} |\theta_{j-n+1}| b_j^k$, for $n \in \mathbb{N}$. Clearly $\|e_n\|_m = a_n^m$. So, by Lemma 1 $\forall k$, $\exists m$, $\exists \rho > 0$ such that

$$\sup_{j\geqslant n} |\theta_{j-n+1}| b_j^k \leqslant \rho a_n^m, \quad \forall n \in \mathbb{N}.$$

Choose $j = n$. Then $\forall k$, $\exists m$, $\exists C > 0$ such that

$$b_n^k \leqslant C a_n^m,$$

i.e. $\lambda(A) \subset \lambda(B)$. Since $T_\theta e_1 \in \lambda(B)$, it follows that $\theta \in \lambda(B)$.

Conversely, since B is a G_1-set and by $ii)$ and $i)$ we have for a given k, there are $m_1(k)$ and $m_2(m_1)$ such that

$$\begin{aligned}\|T_\theta e_n\|_k &= \sup_{j\geqslant n} |\theta_{j-n+1}| b_j^k \leqslant C_1 \sup_{j\geqslant n} |\theta_{j-n+1}| (b_j^{m_1})^2 \leqslant C_1 \sup_{j\geqslant n} (|\theta_{j-n+1}| b_j^{m_1})(b_n^{m_1}) \\ &\leqslant C_2 \sup_{j\geqslant n} (|\theta_{j-n+1}| b_j^{m_1})(a_n^{m_2}) \leqslant C_2 \sup_{j\geqslant n} (|\theta_{j-n+1}| b_{j-n+1}^{m_1})(a_n^{m_2}) \leqslant C a_n^{m_2}.\end{aligned}$$

Therefore, $\forall k$, $\exists m_2$ such that

$$\sup_n \frac{\|T_\theta e_n\|_k}{\|e_n\|_{m_2}} < \infty,$$

that is, T_θ is continuous. $\square$

We consider the map $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ which is determined by the matrix C^t (the transpose of C) and try to find necessary and sufficient conditions for the continuity of $T_\theta{}'$.

THEOREM 2. *Let $\lambda(A)$ be a nuclear G_∞-space, $\lambda(B)$ be a nuclear Köthe space. Then, $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ which is given above is linear continuous operator if and only if the following hold:*

i) $\theta \in \lambda(A)'$;

ii) $\lambda(A) \subset \lambda(B)$.

Proof. The matrix C^t of the operator $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ is the following upper triangular matrix:

$$C^t = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \cdots \\ 0 & \theta_1 & \theta_2 & \theta_3 & \cdots \\ 0 & 0 & \theta_1 & \theta_2 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}.$$

Let $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ be a continuous linear operator.
Note that $\|T_\theta{}' e_n\|_k = \|(\theta_n, \theta_{n-1}, \cdots, \theta_1, 0, 0, \cdots)\|_k = \sup_{1 \leqslant i \leqslant n} |\theta_{n+1-i}| b_i^k$, for $n \in \mathbb{N}$. So, by Lemma 1 $\forall k$, $\exists m$, $\exists \mu > 0$ such that

$$\sup_{1 \leqslant i \leqslant n} |\theta_{n+1-i}| b_i^k \leqslant \mu a_n^m, \quad \forall n \in \mathbb{N}.$$

Let $i = 1$. Hence $\exists m$, $\exists C = \frac{\mu}{b_1^k} > 0$ such that

$$|\theta_n| \leqslant C a_n^m, \quad \forall n,$$

i.e. $\theta \in \lambda(A)'$.
Let $i = n$. Then $\forall k$, $\exists m$ such that

$$b_n^k \leqslant \frac{\mu}{|\theta_1|} a_n^m,$$

i.e.

$$\lambda(A) \subset \lambda(B).$$

On the other hand, since A is a G_∞-set and by $i)$ and $ii)$ for a given k, there are m_1 and $m_2(k)$ and $m = max\{m_1, m_2\}$ such that

$$\begin{aligned} \|T_\theta{}' e_n\|_k &= \sup_{1 \leqslant i \leqslant n} |\theta_{n-i+1}| b_i^k \leqslant C_1 \sup_{1 \leqslant i \leqslant n} a_{n-i+1}^{m_1} b_i^k \leqslant C_1 \sup_{1 \leqslant i \leqslant n} a_{n-i+1}^{m_1} a_i^{m_2} \leqslant C_1 a_n^{m_1} a_n^{m_2} \\ &\leqslant C_2 (a_n^m)^2. \end{aligned}$$

Since $\lambda(A)$ is G_∞ - space, for this m, $\exists j$ such that

$$\sup_n \frac{(a_n^m)^2}{a_n^j} < \infty.$$

Therefore, $\forall k$, $\exists j$ such that

$$\sup_n \frac{\|T_\theta{}' e_n\|_k}{\|e_n\|_j} < \infty,$$

that is, $T_\theta{}'$ is continuous. □

It is known that $\mathscr{S}$ is a normal sequence space if whenever $|x_i| < |y_i|$ and $y = (y_i) \in \mathscr{S}$, then $x = (x_i) \in \mathscr{S}$ [2].

REMARK 1. Now we write $\theta \in \mathscr{S}$ when the Cauchy product map $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ above is continuous. If $\theta, \eta \in \mathscr{S}$, $\lambda \in \mathscr{K}$, then clearly $T_{\theta+\eta}$ and $T_{\lambda\theta}$ will be continuous since T_θ and T_η are continuous. Hence $\mathscr{S}$ is a vector space.

Now, let $|\theta_i| < |\eta_i|, \forall i,\ \eta \in \mathscr{S}$. Since T_η is continuous, for all k we find m so that

$$\sup_n \left\{ \sup_{j \geqslant n} \left|\theta_{j-n+1}\right| \frac{b_j^k}{a_n^m} \right\} \leqslant \sup_n \left\{ \sup_{j \geqslant n} \left|\eta_{j-n+1}\right| \frac{b_j^k}{a_n^m} \right\} < \infty,$$

i.e. T_θ is continuous.

Therefore $\theta \in \mathscr{S}$. Hence we obtain that $\mathscr{S}$ is a normal sequence space.

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