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SPECTRAL MAPPING THEOREMS FOR WEYL SPECTRUM AND ISOLATED SPECTRAL POINTS

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Abstract. Spectral mapping theorems for Weyl spectrum and isolated spectral points were discussed by Gramsch, Lay and Oberai, etc. In this paper, $\mathscr{L}(\mathscr{X})$ means the space of all bounded linear operator on an infinite-dimensional complex Banach space $\mathscr{X}$, $f\in\mathscr{H}(\sigma(T))$ means f is holomorphic on an open set $\mathscr{U}$ containing the spectrum $\sigma(T)$, and $f\in\mathscr{H}_{lnc}(\sigma(T))$ means f is holomorphic and locally nonconstant. Firstly, it is shown that, if $T\in\mathscr{L}(\mathscr{X})$ and $f\in\mathscr{H}(\sigma(T))$, then (1) $\sigma_{uw}(f(T))\subseteq f(\sigma_{uw}(T))$ where $\sigma_{uw}(T)$ means the upper semi-Weyl spectrum; (2) $\sigma_{uw}(f(T))\supseteq f(\sigma_{uw}(T))$ is equivalent to the assertion that T is of stable sign index on $\rho_{uf}(T)$ where $\rho_{uf}(T)$ means the upper semi-Fredholm resolvent. Secondly, let $T\in\mathscr{L}(\mathscr{X})$, (1) if $f\in\mathscr{H}_{lnc}(\sigma(T))$ or T is polaroid, then $\sigma(f(T))\backslash\pi_{00}(f(T))\subseteq f(\sigma(T)\backslash\pi_{00}(T))$; (2) if T is isoloid, then $\sigma(f(T))\backslash\pi_{00}(f(T))\supseteq f(\sigma(T)\backslash\pi_{00}(T))$. Some two-out-of-three results on spectral mapping theorems and Weyl type theorems are also given. At the end, an example is provided which implies that the conditions "$f\in\mathscr{H}_{lnc}(\sigma(T))$", "$T$ is polaroid" and "T is isoloid" are crucial and inevitable.

1. Introduction

In this paper, $\mathscr{L}(\mathscr{X})$ means the space of all bounded linear operator on an infinite-dimensional complex Banach space $\mathscr{X}$, $f\in\mathscr{H}(\sigma(T))$ means f is holomorphic on an open set $\mathscr{U}$ containing the spectrum $\sigma(T)$, and $f\in\mathscr{H}_{lnc}(\sigma(T))$ means f is holomorphic and locally nonconstant on an open set $\mathscr{U}$ containing $\sigma(T)$.

Let $\sigma_p(T)$, $\sigma_f(T)$, $\sigma_w(T)$ and $\pi_{00}(T)$ mean the point spectrum, Fredholm spectrum, Weyl spectrum and the set of all isolated eigenvalues of finite multiplicity of an operator T respectively.

In 1971, Gramsch and Lay [13, Theorem 2] discussed the spectral mapping theorem for Weyl spectrum via F-semigroup.

THEOREM 1.1. ([13]) *Let $T\in\mathscr{L}(\mathscr{X})$ and $f\in\mathscr{H}(\sigma(T))$, then*

$$\sigma_w(f(T))\subseteq f(\sigma_w(T)).$$

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In general, equality does not hold in Theorem 1.1 [13, page 23].

Let iso $\sigma(T)$ be the set of all isolated point of $\sigma(T)$. An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be isoloid if iso $\sigma(T) \subseteq \sigma_p(T)$. In 1977, Oberai [15] proved some results on spectral mapping theorems for isolated spectral points and Weyl theorem.

THEOREM 1.2. ([15]) *Let $T \in \mathscr{L}(\mathscr{X})$ and $p(t)$ a polynomial. Then*

(1) $\sigma(p(T))\backslash\pi_{00}(T) \subseteq p(\sigma(T)\backslash\pi_{00}(T))$;

(2) *If T is isoloid, then* $\sigma(p(T))\backslash\pi_{00}(T) \supseteq p(\sigma(T)\backslash\pi_{00}(T))$.

In general, Theorem 1.2 (2) may fail if T is not assumed to be isoloid [15, Example 1]. An operator $T \in (W)$ means Weyl theorem holds for T, that is,

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T).$$

THEOREM 1.3. ([15]) *Let $T \in (W)$ and $p(t)$ a polynomial. If T is isoloid, then $\sigma_w(p(T)) = p(\sigma_w(T))$ if and only if $p(T) \in (W)$.*

Let $\sigma_a(T)$, $\sigma_{uf}(T)$, $\sigma_{bf}(T)$, $\sigma_{ubf}(T)$, $\sigma_{uw}(T)$, $\sigma_{bw}(T)$ and $\sigma_{ubw}(T)$ mean the approximate point spectrum, upper semi-Fredholm spectrum, B-Fredholm spectrum, upper semi-B-Fredholm spectrum, upper semi-Weyl spectrum, B-Weyl spectrum and upper semi-B-Weyl spectrum of an operator T respectively (see [4]).

DEFINITION 1.1. Let $T \in \mathscr{L}(\mathscr{X})$.

(1) T is said to be of stable sign index on $\rho_f(T) := C\backslash\sigma_f(T)$ if for each λ, $\mu \in \rho_f(T)$, $\mathrm{ind}(T-\lambda)$ and $\mathrm{ind}(T-\mu)$ have the same sign.

(2) T is said to be of stable sign index on $\rho_{uf}(T) := C\backslash\sigma_{uf}(T)$ if for each λ, $\mu \in \rho_{uf}(T)$, $\mathrm{ind}(T-\lambda)$ and $\mathrm{ind}(T-\mu)$ have the same sign.

(3) T is said to be of stable sign index on $\rho_{bf}(T) := C\backslash\sigma_{bf}(T)$ if for each λ, $\mu \in \rho_{bf}(T)$, $\mathrm{ind}(T-\lambda)$ and $\mathrm{ind}(T-\mu)$ have the same sign.

(4) T is said to be of stable sign index on $\rho_{ubf}(T) := C\backslash\sigma_{ubf}(T)$ if for each λ, $\mu \in \rho_{ubf}(T)$, $\mathrm{ind}(T-\lambda)$ and $\mathrm{ind}(T-\mu)$ have the same sign.

Let $\sigma_b(T)$, $\sigma_{ub}(T)$, $\sigma_{bb}(T)$ and $\sigma_{ubb}(T)$ mean the Browder spectrum, upper semi-Browder spectrum, B-Browder spectrum and upper semi-B-Browder spectrum of an operator T respectively (see [4]). Denote $P(T) := \sigma(T)\backslash\sigma_{bb}(T)$ the poles of the resolvent of T, $P_0(T) := \sigma(T)\backslash\sigma_b(T)$ the poles of the resolvent of T with finite rank, acc $\sigma(T) := \sigma(T)\backslash$iso $\sigma(T)$, and $\pi_0(T) := \sigma_p(T) \cap$ iso $\sigma(T)$. An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be polaroid if iso $\sigma(T) \subseteq P(T)$.

Theorem 1.1-1.3 are extended to Theorem 1.4-1.6 respectively.

THEOREM 1.4. ([16]) *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$, the following assertions are equivalent:*

(1) T *is of stable sign index on* $\rho_f(T)$.

(2) $\sigma_w(f(T)) = f(\sigma_w(T))$.

(3) $\sigma_w(p(T)) = p(\sigma_w(T))$ *for each polynomial* p.

THEOREM 1.5. ([14, 16]) *Let* $T \in \mathscr{L}(\mathscr{X})$ *be isoloid. If* $f \in \mathscr{H}(\sigma(T))$, *then*

$$\sigma(f(T)) \backslash \pi_{00}(f(T)) = f(\sigma(T) \backslash \pi_{00}(T)).$$

It should be pointed out that Theorem 1.5 may fail when $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$. See Example 5.1 (3) for details.

THEOREM 1.6. ([12]) *Let* T *be polaroid and* $f \in \mathscr{H}(\sigma(T))$. *If* $T \in (W)$, *then* T *is of stable sign index on* $\rho_f(T)$ *(i.e.,* $\sigma_w(f(T)) = f(\sigma_w(T))$ *) if and only if* $f(T) \in (W)$.

In this work, the authors will give extensions of Theorem 1.4-1.6. In Section 2, the spectral mapping theorems for Weyl type spectrums, such as upper semi-Weyl spectrum, B-Weyl spectrum and upper semi-B-Weyl spectrum, are considered (see Theorem 2.1, Theorem 2.2, Theorem 2.3). Moreover, the spectral mapping theorems for B-Weyl spectrum and upper semi-B-Weyl spectrum may fail if $f \notin \mathscr{H}_{lnc}(\sigma(T))$ (see Example 5.1 (1)-(2)).

In Section 3, the spectral mapping theorems for isolated spectral points are discussed (see Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4). Especially, Example 5.1 (3)-(10) are provided which illustrate the results may fail without the condition "$f \in \mathscr{H}_{lnc}(\sigma(T))$" or "$T$ is polaroid".

Weyl type theorems have been studied extensively in the last two decades (see [1, 5, 17]). Theorems 1.3 and 1.6 say that there is a close relation between spectral mapping theorems and Weyl type theorems.

In Section 4, we prove some two-out-of-three results on spectral mapping theorems for Weyl type spectrums, isolated spectral points and Weyl type theorems.

Lastly, we show an example which implies that the conditions "T is isoloid", "T is polaroid" or "$f \in \mathscr{H}_{lnc}(\sigma(T))$" are crucial and inevitable.

2. Spectral mapping theorems for Weyl type spectrums

For every $n \in \mathscr{Z}$, let us define $\Omega_n := \{\mu \in \sigma(T) : \mathrm{ind}(\mu - T) = n\}$.

THEOREM 2.1. *Let* $T \in \mathscr{L}(\mathscr{X})$ *and* $f \in \mathscr{H}(\sigma(T))$.

(1) $\sigma_{uw}(f(T)) \subseteq f(\sigma_{uw}(T))$.

(2) *The following assertions are equivalent:*

(a) T *is of stable sign index on* $\rho_{uf}(T)$.

(b) $\sigma_{uw}(f(T)) \supseteq f(\sigma_{uw}(T))$.

(c) $\sigma_{uw}(p(T)) \supseteq p(\sigma_{uw}(T))$ *for each polynomial* p.

Theorem 2.1 says that Theorem 1.4 holds for upper semi-Weyl spectrum. Since the assertion "T or T^* has SVEP" ensures "T is of stable sign index on $\rho_{uf}(T)$" (see [1, Theorem 3.36]), Theorem 2.1 is an extension of [2, Corollary 2.6].

Proof. (1) Suppose that $f \in \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \in \sigma_{uw}(f(T))$. Then

$$f(T) - \lambda = \Pi_{i=1}^{n}(T-\mu_i)^{k_i}h(T) \tag{2.1}$$

where $\mu_1, \cdots, \mu_n$ are different spectral points of T and $h(T)$ is invertible. Thus, there exists $\mu_0 \in \{\mu_i, i = 1, \cdots, n\}$ with $\mu_0 \in \sigma_{uw}(T)$ ([1, Remark 1.54]). So $\lambda = f(\mu_0) \in f(\sigma_{uw}(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \in \sigma_{uw}(f(T))$. Let $g(z) = f(z) - \lambda$, then g is defined on an open set $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ with $\mathscr{U}_1$, $\mathscr{U}_2$ open, $\mathscr{U}_1 \cap \mathscr{U}_2 = \phi$, $\sigma_1 := \sigma(T) \cap \mathscr{U}_1 \neq \phi$, $\sigma_2 := \sigma(T) \cap \mathscr{U}_2 \neq \phi$, $g|_{\mathscr{U}_1} \equiv 0$ and $g \in \mathscr{H}_{lnc}(\sigma_2)$. Let $E = E(\sigma_2)$ be the Riesz idempotent corresponding to σ_2, $T_1 = T|_{\ker(E)}$, $T_2 = T|_{E(\mathscr{X})}$. Then $\mathscr{X} = \ker(E) \oplus E(\mathscr{X})$, $\sigma(T_i) = \sigma_i$ $(i = 1, 2)$.

Assume to the contrary that $\lambda \notin f(\sigma_{uw}(T)) \supseteq f(\sigma_{uf}(T)) = \sigma_{uf}(f(T))$, thus $\lambda \in \rho_{uf}(f(T))$. By [1, Lemma 3.62] or [13, Theorem 1],

$$\mathrm{ind}(g(T)) = \Sigma_{n\neq 0} n\alpha_n$$

where α_n is the number of zeros of g on Ω_n. Since $\sigma_{uw}(T) = \sigma_{uf}(T) \cup (\cup_{n>0}\Omega_n)$ and $\lambda \notin f(\sigma_{uw}(T))$, we have

$$\mathrm{ind}(g(T)) = \Sigma_{n<0} n\alpha_n \leqslant 0.$$

So $\lambda \notin \sigma_{uw}(f(T))$. This is a contradiction.

(2) (a)$\Rightarrow$(b) Suppose that $f \in \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \notin \sigma_{uw}(f(T)) \supseteq \sigma_{uf}(f(T))$, thus $\lambda \in \rho_{uf}(f(T))$. By (2.1),

$$0 \geqslant \mathrm{ind}(f(T) - \lambda) = \Sigma_{i=1}^{n} k_i \mathrm{ind}(T - \mu_i).$$

Hence $\mathrm{ind}(T-\mu_i) \leqslant 0$ and $\mu_i \notin \sigma_{uw}(T)$ for $i = 1, \cdots, n$. So $\lambda \notin f(\sigma_{uw}(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \notin \sigma_{uw}(f(T))$. Let $g(z) = f(z) - \lambda$ as in the proof of (1), then $g(T) = g(T_1) \oplus g(T_2) = 0 \oplus g(T_2)$. Since $\lambda \notin \sigma_{uf}(f(T))$, we have $0 \notin \sigma_{uf}(g(T_1))$ with $\mathrm{ind}(g(T_1)) = 0$ and $0 \notin \sigma_{uf}(g(T_2))$. Hence $\dim(\mathscr{X}_1) < \infty$ and $\sigma(T_1) = \sigma_1 \subseteq P_0(T)$. On the other hand, $0 \geqslant \mathrm{ind}(g(T)) = \mathrm{ind}(g(T_2))$ and $g \in \mathscr{H}_{lnc}(\sigma_2)$ deduce that the zeros of g on $\sigma(T_2)$ do not belong to $\sigma_{uw}(T_2)$. Since $\sigma_1 \cap \sigma_2 = \phi$, the zeros of g on $\sigma(T_2)$ do not belong to $\sigma_{uw}(T)$. So that $\lambda \notin f(\sigma_{uw}(T))$.

(b)$\Rightarrow$(c) Clear.

Proof of (c)$\Rightarrow$(a) is similar to [16, Theorem 2]: Assume to the contrary that T is not of stable sign index on $\rho_{uf}(T)$. Then there are λ_1, $\lambda_2 \in \rho_{uf}(T)$ with $\mathrm{ind}(T - \lambda_1) > 0$ and $\mathrm{ind}(T-\lambda_2) < 0$. Let $k = \mathrm{ind}(T-\lambda_1)$, $m = -\mathrm{ind}(T-\lambda_2)$, $p(z) = (z-\lambda_1)^m(z-\lambda_2)^k$. Then $p(T)$ is an upper semi-Fredholm operator ([1, Remark 1.54]) and $\mathrm{ind}(p(T)) = km + k(-m) = 0$, that is, $0 \notin \sigma_{uw}(p(T))$. Meanwhile, $\lambda_1 \in \sigma_{uw}(T)$ and $0 = p(\lambda_1) \in p(\sigma_{uw}(T))$. This is a contradiction. $\square$

The following Theorem 2.2 says that Theorem 1.4 holds for B-Weyl spectrum and $f \in \mathscr{H}_{lnc}(\sigma(T))$.

THEOREM 2.2. *Let* $T \in \mathscr{L}(\mathscr{X})$ *and* $f \in \mathscr{H}(\sigma(T))$.

(1) $\sigma_{bw}(f(T)) \subseteq f(\sigma_{bw}(T))$.

(2) *The following assertions are equivalent:*

(a) T *is of stable sign index on* $\rho_{bf}(T)$.

(b) $\sigma_{bw}(f(T)) \supseteq f(\sigma_{bw}(T))$ *for each* $f \in \mathscr{H}_{lnc}(\sigma(T))$.

(c) $\sigma_{bw}(p(T)) \supseteq p(\sigma_{bw}(T))$ *for each nonconstant polynomial* p.

Theorem 2.2 is a generalization of [9, Theorem 2.4], [18, Theorem 2.1] and [11, Corollary 2.8]. Theorem 2.2 (2) may fail without the condition "$f \in \mathscr{H}_{lnc}(\sigma(T))$", see (1) of Example 5.1.

Proof. (1) The case that f is constant is obvious, and it is sufficient to prove the case $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ since [9, Theorem 2.4] proved the case $f \in \mathscr{H}_{lnc}(\sigma(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \in \sigma_{bw}(f(T))$. Let $g(z) = f(z) - \lambda$ as in the proof of Theorem 2.1 (1). Since $\lambda \in \sigma_{bw}(f(T))$, $g(T) = g(T_1) \oplus g(T_2)$ and $g|_{\mathscr{U}_1} \equiv 0$, then $g(T_2)$ is not a B-Weyl operator. By $g \in \mathscr{H}_{lnc}(\sigma_2) = \mathscr{H}_{lnc}(\sigma(T_2))$, there exists $\mu \in \sigma_{bw}(T_2) \subseteq \sigma_{bw}(T)$ such that $\lambda = f(\mu)$.

(2) (a)$\Rightarrow$(b) See [9, Theorem 2.4]. (b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(a) Assume to the contrary that T is not of stable sign index on $\rho_{bf}(T)$. Then there are λ_1, $\lambda_2 \in \rho_{bf}(T)$ with $\mathrm{ind}(T-\lambda_1) > 0$ and $\mathrm{ind}(T-\lambda_2) < 0$. Let $k = \mathrm{ind}(T-\lambda_1)$, $m = -\mathrm{ind}(T-\lambda_2)$, $p(z) = (z-\lambda_1)^m (z-\lambda_2)^k$. Then $p(T)$ is a B-Fredholm operator ([7, Theorem 3.6], [6, Corollary 3.3]) and $\mathrm{ind}(p(T)) = km + k(-m) = 0$ ([8, Theorem 3.2]), that is, $0 \notin \sigma_{bw}(p(T))$. Meanwhile, λ_1, $\lambda_2 \in \sigma_{bw}(T)$ and $0 = p(\lambda_1) = p(\lambda_2) \in p(\sigma_{bw}(T))$. This is a contradiction. □

THEOREM 2.3. *Let* $T \in \mathscr{L}(\mathscr{X})$ *and* $f \in \mathscr{H}(\sigma(T))$.

(1) $\sigma_{ubw}(f(T)) \subseteq f(\sigma_{ubw}(T))$.

(2) *The following assertions are equivalent:*

(a) T *is of stable sign index on* $\rho_{ubf}(T)$.

(b) $\sigma_{ubw}(f(T)) \supseteq f(\sigma_{ubw}(T))$ *for each* $f \in \mathscr{H}_{lnc}(\sigma(T))$.

(c) $\sigma_{ubw}(p(T)) \supseteq p(\sigma_{ubw}(T))$ *for each nonconstant polynomial* p.

Theorem 2.3 is an extension of [18, Theorem 2.3], and Example 5.1 (2) below illustrates the condition "$f \in \mathscr{H}_{lnc}(\sigma(T))$" is crucial.

Proof. (1) The case that f is constant is obvious, and it is sufficient to prove the case $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ since [18, Theorem 2.3] proved the case $f \in \mathscr{H}_{lnc}(\sigma(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T)) \backslash \mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \in \sigma_{ubw}(f(T))$. Let $g(z) = f(z) - \lambda$ as in the proof of Theorem 2.2 (1). Since $\lambda \in \sigma_{ubw}(f(T))$ and $g(T) = g(T_1) \oplus$

$g(T_2)=0\oplus g(T_2)$, then $g(T_2)$ is not an upper semi-B-Weyl operator. By $g\in\mathscr{H}_{lnc}(\sigma(T_2))$ and $\sigma_1\cap\sigma_2=\phi$, there exists $\mu\in\sigma_{ubw}(T_2)\subseteq\sigma_{ubw}(T)$ such that $\lambda=f(\mu)$.

(2) (a)$\Rightarrow$(b) See [18, Theorem 2.3]. (b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(a) Assume to the contrary that T is not of stable sign index on $\rho_{ubf}(T)$. Then there are λ_1, $\lambda_2\in\rho_{ubf}(T)$ with $\mathrm{ind}(T-\lambda_1)>0$ and $\mathrm{ind}(T-\lambda_2)<0$. Let $k=\mathrm{ind}(T-\lambda_1)$, $m=-\mathrm{ind}(T-\lambda_2)$, $p(z)=(z-\lambda_1)^m(z-\lambda_2)^k$. Then $p(T)$ is an upper semi-B-Fredholm operator ([10, Corollary 4.4] or [7, Theorem 3.6]) and $\mathrm{ind}(p(T))=km+k(-m)=0$ ([8, Theorem 3.2]), that is, $0\notin\sigma_{ubw}(p(T))$. Meanwhile, $\lambda_1\in\sigma_{ubw}(T)$ and $0=p(\lambda_1)\in p(\sigma_{ubw}(T))$. This is a contradiction. $\square$

3. Spectral mapping theorems for isolated spectral points

THEOREM 3.1. *Let $T\in\mathscr{L}(\mathscr{X})$ and $f\in\mathscr{H}(\sigma(T))$.*

(1) *If $f\in\mathscr{H}_{lnc}(\sigma(T))$, then $\sigma(f(T))\backslash\pi_{00}(f(T))\subseteq f(\sigma(T)\backslash\pi_{00}(T))$.*

(2) *If T is polaroid, then $\sigma(f(T))\backslash\pi_{00}(f(T))\subseteq f(\sigma(T)\backslash\pi_{00}(T))$.*

(3) *If T is isoloid, then $\sigma(f(T))\backslash\pi_{00}(f(T))\supseteq f(\sigma(T)\backslash\pi_{00}(T))$.*

Theorem 3.1 is an extension of Theorems 1.2 and 1.5, and Example 5.1 (3)-(4) illustrate the conditions "$f\in\mathscr{H}_{lnc}(\sigma(T))$", "$T$ is polaroid" and "T is isoloid" are inevitable.

Proof. (1) The proof is similar to [15, Lemma 1]: Let $\lambda\in\sigma(f(T))\backslash\pi_{00}(f(T))$.

If $\lambda\in\mathrm{acc}\,\sigma(f(T))$, it is easy to see that there exists $\mu\in\mathrm{acc}\,\sigma(T)\subseteq\sigma(T)\backslash\pi_{00}(T)$ such that $\lambda=f(\mu)$.

If $\lambda\in\mathrm{iso}\,\sigma(f(T))$ and $\lambda\notin\sigma_p(f(T))$, by $\sigma_p(f(T))\supseteq f(\sigma_p(T))$, there exists $\mu\in\sigma(T)\backslash\sigma_p(T)$ such that $\lambda=f(\mu)$. So $\lambda\in f(\sigma(T)\backslash\pi_{00}(T))$.

If $\lambda\in\mathrm{iso}\,\sigma(f(T))$ and $\lambda\in\sigma_p(f(T))$, then $\dim(\ker(f(T)-\lambda))=\infty$. By $f\in\mathscr{H}_{lnc}(\sigma(T))$, (2.1) and [1, Lemma 1.76], there exists $\mu_0\in\{\mu_i, i=1,\cdots,n\}$ such that $\dim(\ker(T-\mu_0))=\infty$. So $\lambda=f(\mu_0)\in f(\sigma(T)\backslash\pi_{00}(T))$.

(2) By the proof of (1), it is sufficient to prove the case that $\lambda\in\mathrm{iso}\,\sigma(f(T))$, $\dim(\ker(f(T)-\lambda))=\infty$ and $f\in\mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma(T))$.

Let $g(z)=f(z)-\lambda$ as in the proof of Theorem 2.2 (1), then $g(T)=0\oplus g(T_2)$ and $g\in\mathscr{H}_{lnc}(\sigma(T_2))$. By the proof of (1), we assume that $\dim(\ker g(T_1))=\infty$.

If $\sigma(T_1)$ is a finite set, there exists $\mu_0\in\sigma(T_1)$ such that $\dim(E(\{\mu_0\})\mathscr{X})=\infty$. Since T is polaroid, there exists an integer p such that

$$E(\{\mu_0\})\mathscr{X}=\ker(T_1-\mu_0)^p=\ker(T-\mu_0)^p.$$

So $\dim(\ker(T-\mu_0))=\infty$ and $\lambda=f(\mu_0)\in f(\sigma(T)\backslash\pi_{00}(T))$.

If $\sigma(T_1)$ is not a finite set, then it is easy to see that there exists $\mu_0\in\mathrm{acc}\,\sigma(T_1)$. So $\lambda=f(\mu_0)\in f(\sigma(T)\backslash\pi_{00}(T))$.

(3) It is sufficient to prove that $\lambda\in\pi_{00}(f(T))$ implies $\lambda\notin f(\sigma(T)\backslash\pi_{00}(T))$.

Suppose that $f\in\mathscr{H}_{lnc}(\sigma(T))$, $\lambda\in\pi_{00}(f(T))$ and $M:=\{\mu\in\sigma(T): f(\mu)-\lambda=0\}$. Then $M\subseteq\mathrm{iso}\,\sigma(T)$ and it is a finite set. By (2.1), [1, Lemma 1.76] and T is isoloid, we have $M\subseteq\pi_{00}(T)$. So $\lambda\notin f(\sigma(T)\backslash\pi_{00}(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma(T))$, $\lambda \in \pi_{00}(f(T))$ and $M = \{\mu \in \sigma(T) : f(\mu) - \lambda = 0\}$. Let $g(z) = f(z) - \lambda$ as in the proof of Theorem 2.2 (1), then $M = \sigma(T_1) \cup M_2$ where $M_2 := \{\mu \in \sigma(T_2) : f(\mu) - \lambda = 0\}$.

Since $g \in \mathscr{H}_{lnc}(\sigma(T_2))$ and $\sigma(T_1) \cap \sigma(T_2) = \phi$, $M_2 \subseteq \pi_{00}(T_2) \subseteq \pi_{00}(T)$ follows.

Meanwhile, $\lambda \in \pi_{00}(f(T))$ ensures $\dim(\mathscr{X}_1) < \infty$. Thus $\sigma(T_1)$ is a finite set and $\dim(E(\{\mu\})\mathscr{X}) < \infty$ for every $\mu \in \sigma(T_1)$. It is clear that $\sigma(T_1) \subseteq \pi_{00}(T)$. Therefore $M \subseteq \pi_{00}(T)$ and $\lambda \notin f(\sigma(T)\backslash\pi_{00}(T))$. $\square$

Denote $\pi_{00}^a(T) := \{\lambda \in \text{iso}\, \sigma_a(T) : 0 < \dim\ker(T-\lambda) < \infty\}$, $P_0^a(T) := \sigma_a(T)\backslash\sigma_{ub}(T)$ the set of all left poles of the resolvent with finite rank.

An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be a-isoloid if iso $\sigma_a(T) \subseteq \sigma_p(T)$.

An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be a-polaroid if iso $\sigma_a(T) \subseteq P(T)$.

THEOREM 3.2. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$.*

(1) *If $f \in \mathscr{H}_{lnc}(\sigma_a(T))$, then $\sigma_a(f(T))\backslash\pi_{00}^a(f(T)) \subseteq f(\sigma_a(T)\backslash\pi_{00}^a(T))$.*

(2) *If T is a-polaroid, then $\sigma_a(f(T))\backslash\pi_{00}^a(f(T)) \subseteq f(\sigma_a(T)\backslash\pi_{00}^a(T))$.*

(3) *If T is a-isoloid, then $\sigma_a(f(T))\backslash\pi_{00}^a(f(T)) \supseteq f(\sigma_a(T)\backslash\pi_{00}^a(T))$.*

The conditions "$f \in \mathscr{H}_{lnc}(\sigma_a(T))$", "$T$ is a-polaroid" and "T is a-isoloid" are crucial (see Example 5.1 (5)-(6)).

Proof. (1) Let $\sigma_a(f(T))\backslash\pi_{00}^a(f(T))$. If $\lambda \in \text{acc}\, \sigma_a(f(T))$, by $\sigma_a(f(T)) = f(\sigma_a(T))$, it is easy to see that there exists $\mu \in \text{acc}\, \sigma_a(T) \subseteq \sigma_a(T)\backslash\pi_{00}^a(T)$ such that $\lambda = f(\mu)$.

If $\lambda \in \text{iso}\, \sigma_a(f(T))$ and $\lambda \notin \sigma_p(f(T))$, by $\sigma_p(f(T)) \supseteq f(\sigma_p(T))$, there exists $\mu \in \sigma_a(T)\backslash\sigma_p(T)$ such that $\lambda = f(\mu)$. So $\lambda \in f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

If $\lambda \in \text{iso}\, \sigma_a(f(T))$ and $\lambda \in \sigma_p(f(T))$, then $\dim(\ker(f(T)-\lambda)) = \infty$. Since $f \in \mathscr{H}_{lnc}(\sigma_a(T))$, we have

$$f(T) - \lambda = \Pi_{i=1}^n (T-\mu_i)^{k_i} h(T) \tag{3.1}$$

where $\mu_1, \cdots, \mu_n$ are different elements of $\sigma_a(T)$ and $0 \notin \sigma_a(h(T))$. By (3.1) and [1, Lemma 1.76], there exists $\mu_0 \in \{\mu_i, i = 1, \cdots, n\}$ such that $\dim(\ker(T-\mu_0)) = \infty$. So $\lambda = f(\mu_0) \in f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

(2) By the proof of (1), it is sufficient to prove the case that $\lambda \in \text{iso}\, \sigma(f(T))$, $\dim(\ker(f(T)-\lambda)) = \infty$ and $f \in \mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma_a(T))$.

Obviously, $f \in \mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma(T))$. Let $g(z) = f(z) - \lambda$ as in the proof of Theorem 2.2 (1), then $g(T) = 0 \oplus g(T_2)$ and $g \in \mathscr{H}_{lnc}(\sigma(T_2)) \subseteq \mathscr{H}_{lnc}(\sigma_a(T_2))$. By the proof of (1), we assume that $\dim(\ker g(T_1)) = \infty$.

If $\sigma_a(T_1)$ is a finite set, then $\sigma(T_1) = \sigma_a(T_1)$ for $\partial\sigma(T_1) \subseteq \sigma_a(T_1)$. Thus there exists $\mu_0 \in \sigma_a(T_1)$ such that $\dim(E(\{\mu_0\})\mathscr{X}) = \infty$. Since T is a-polaroid, there exists an integer p such that

$$E(\{\mu_0\})\mathscr{X} = \ker(T_1-\mu_0)^p = \ker(T-\mu_0)^p.$$

So $\dim(\ker(T-\mu_0)) = \infty$ and $\lambda = f(\mu_0) \in f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

If $\sigma_a(T_1)$ is not a finite set, then it is easy to see that there exists $\mu_0 \in \mathrm{acc}\,\sigma_a(T_1)$. So $\lambda = f(\mu_0) \in f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

(3) It is sufficient to prove that $\lambda \in \pi_{00}^a(f(T))$ implies $\lambda \notin f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

Suppose that $f \in \mathscr{H}_{lnc}(\sigma_a(T))$, $\lambda \in \pi_{00}^a(f(T))$ and $M^a := \{\mu \in \sigma_a(T) : f(\mu) - \lambda = 0\}$. Then $M^a \subseteq \mathrm{iso}\,\sigma_a(T)$ and it is a finite set. By (3.1), [1, Lemma 1.76] and T is a-isoloid, $M^a \subseteq \pi_{00}^a(T)$ follows. So $\lambda \notin f(\sigma_a(T)\backslash\pi_{00}^a(T))$.

Suppose that $f \in \mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma_a(T)) \subseteq \mathscr{H}(\sigma(T))\backslash\mathscr{H}_{lnc}(\sigma(T))$ and $\lambda \in \pi_{00}^a(f(T))$. Let $g(z) = f(z) - \lambda$ as in the proof of Theorem 2.2 (1), then $M^a = \sigma_a(T_1) \cup M_2^a$ where $M_2^a := \{\mu \in \sigma_a(T_2) : f(\mu) - \lambda = 0\}$.

Since $g \in \mathscr{H}_{lnc}(\sigma(T_2)) \subseteq \mathscr{H}_{lnc}(\sigma_a(T_2))$ and $\sigma(T_1) \cap \sigma(T_2) = \phi$, $M_2^a \subseteq \pi_{00}^a(T_2) \subseteq \pi_{00}^a(T)$ follows.

Meanwhile, $\lambda \in \pi_{00}^a(f(T))$ ensures $\dim(\mathscr{X}_1) < \infty$. Thus $\sigma(T_1)$ is a finite set, $\sigma(T_1) = \sigma_a(T_1)$. So $\dim(E(\{\mu\})\mathscr{X}) < \infty$ for every $\mu \in \sigma_a(T_1)$. Since T is a-isoloid, we have $\sigma_a(T_1) \subseteq \pi_{00}^a(T)$. Therefore $M^a \subseteq \pi_{00}^a(T)$ and $\lambda \notin f(\sigma_a(T)\backslash\pi_{00}^a(T))$. $\square$

THEOREM 3.3. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$.*

(1) $\sigma(f(T))\backslash\pi_0(f(T)) \subseteq f(\sigma(T)\backslash\pi_0(T))$.

(2) *If $f \in \mathscr{H}_{lnc}(\sigma(T))$ and T is isoloid, then*

$$\sigma(f(T))\backslash\pi_0(f(T)) \supseteq f(\sigma(T)\backslash\pi_0(T)).$$

Example 5.1 (7)-(8) imply that the conditions "$f \in \mathscr{H}_{lnc}(\sigma(T))$" and "$T$ is isoloid" are inevitable in (2) of Theorem 3.3, and [9, Lemma 2.9] and [11, Lemma 3.3] may fail without the condition "$f \in \mathscr{H}_{lnc}(\sigma(T))$".

Proof. (1) [9] and [11] proved the case $f \in \mathscr{H}_{lnc}(\sigma(T))$ of (1), now we show a proof of the general case. Let $\lambda \in \sigma(f(T))\backslash\pi_0(f(T))$. If $\lambda \in \mathrm{acc}\,\sigma(f(T))$, it is easy to see that there exists $\mu \in \mathrm{acc}\,\sigma(T) \subseteq \sigma(T)\backslash\pi_0(T)$ such that $\lambda = f(\mu)$.

If $\lambda \in \mathrm{iso}\,\sigma(f(T))$ and $\lambda \notin \sigma_p(f(T))$, by $\sigma_p(f(T)) \supseteq f(\sigma_p(T))$, there exists $\mu \in \sigma(T)\backslash\sigma_p(T)$ such that $\lambda = f(\mu)$. So $\lambda \in f(\sigma(T)\backslash\pi_0(T))$.

(2) See [9, Lemma 2.9] or [11, Lemma 3.3] for the proof. $\square$

Denote $\pi_0^a(T) := \{\lambda \in \mathrm{iso}\,\sigma_a(T) : 0 < \dim\ker(T-\lambda)\}$, $P^a(T) := \sigma_a(T)\backslash\sigma_{ubb}(T)$ the set of all left poles of the resolvent.

THEOREM 3.4. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$.*

(1) $\sigma_a(f(T))\backslash\pi_0^a(f(T)) \subseteq f(\sigma_a(T)\backslash\pi_0^a(T))$.

(2) *If $f \in \mathscr{H}_{lnc}(\sigma_a(T))$ and T is a-isoloid, then*

$$\sigma_a(f(T))\backslash\pi_0^a(f(T)) \supseteq f(\sigma_a(T)\backslash\pi_0^a(T)).$$

The conditions "$f \in \mathscr{H}_{lnc}(\sigma_a(T))$" and "$T$ is a-isoloid" are inevitable (see Example 5.1 (9)-(10)).

Proof. (1) Let $\sigma_a(f(T))\backslash\pi_0^a(f(T))$. If $\lambda \in \text{acc}\,\sigma_a(f(T))$, by $\sigma_a(f(T)) = f(\sigma_a(T))$, it is easy to see that there exists $\mu \in \text{acc}\,\sigma_a(T) \subseteq \sigma_a(T)\backslash\pi_0^a(T)$ such that $\lambda = f(\mu)$.

If $\lambda \in \text{iso}\,\sigma_a(f(T))$ and $\lambda \notin \sigma_p(f(T))$, by $\sigma_p(f(T)) \supseteq f(\sigma_p(T))$, there exists $\mu \in \sigma_a(T)\backslash\sigma_p(T)$ such that $\lambda = f(\mu)$. So $\lambda \in f(\sigma_a(T)\backslash\pi_0^a(T))$.

(2) It is sufficient to prove that $\lambda \in \pi_0^a(f(T))$ implies $\lambda \notin f(\sigma_a(T)\backslash\pi_0^a(T))$.

Suppose that $f \in \mathscr{H}_{nc}(\sigma_a(T))$, $\lambda \in \pi_0^a(f(T))$ and $M_a := \{\mu \in \sigma_a(T) : f(\mu) - \lambda = 0\}$. Then $M_a \subseteq \text{iso}\,\sigma_a(T)$ and it is a finite set. By [1, Lemma 1.76] and T is a-isoloid, we have $M_a \subseteq \pi_0^a(T)$. So $\lambda \notin f(\sigma_a(T)\backslash\pi_0^a(T))$. □

4. Some two-out-of-three results on Weyl type spectrums

We prove some two-out-of-three results on spectral mapping theorems for Weyl type spectrum, isolated spectral points and Weyl type theorems.

$T \in (aW)$ means a-Weyl theorem holds for T, that is,

$$\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T).$$

$T \in (gW)$ means generalized Weyl theorem holds for T, that is,

$$\sigma(T)\backslash\sigma_{bw}(T) = \pi_0(T).$$

$T \in (gaW)$ means generalized a-Weyl theorem holds for T, that is,

$$\sigma_a(T)\backslash\sigma_{ubw}(T) = \pi_0^a(T).$$

THEOREM 4.1. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$. If $T \in (W)$, then any two of the following three assertions imply the third one.*

(1) $\sigma_w(f(T)) = f(\sigma_w(T))$.

(2) $\sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T))$.

(3) $f(T) \in (W)$.

Proof. (1) and (2)$\Rightarrow$(3): Let $T \in (W)$, then

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)).$$

So (3) holds.

(2) and (3)$\Rightarrow$(1): Let $T \in (W)$, then

$$\sigma_w(f(T)) = \sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T)) = f(\sigma_w(T)).$$

So (1) holds.

(3) and (1)$\Rightarrow$(2): Let $T \in (W)$, then

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = \sigma_w(f(T)) = f(\sigma_w(T)) = f(\sigma(T)\backslash\pi_{00}(T)).$$

So (2) holds. □

Since T is polaroid ensures T is isoloid, Theorem 4.1 and Theorem 3.1 deduce the following result.

COROLLARY 4.1. *Let $T \in (W)$ and $f \in \mathscr{H}(\sigma(T))$. If (i) T is isoloid and $f \in \mathscr{H}_{lnc}(\sigma(T))$ or (ii) T is polaroid, then the following two assertions are equivalent to each other.*

(1) $\sigma_w(f(T)) = f(\sigma_w(T))$.

(2) $f(T) \in (W)$.

Corollary 4.1 is a generalization of Theorem 1.6. Corollary 4.1 together with Theorem 1.4 and Theorem 3.1 implies that [4, Theorem 3.14 (ii)] holds for all $f \in \mathscr{H}(\sigma(T))$.

Theorems 4.2-4.4 hold in a similar manner to Theorem 4.1, so we write down them without proofs.

THEOREM 4.2. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$. If $T \in (aW)$, then any two of the following three assertions imply the third one.*

(1) $\sigma_{uw}(f(T)) = f(\sigma_{uw}(T))$.

(2) $\sigma_a(f(T)) \backslash \pi_{00}^a(f(T)) = f(\sigma_a(T) \backslash \pi_{00}^a(T))$.

(3) $f(T) \in (aW)$.

Theorem 4.2 and Theorem 3.2 deduce the result below.

COROLLARY 4.2. *Let $T \in (aW)$ and $f \in \mathscr{H}(\sigma(T))$. If (i) T is a-polaroid or (ii) T is a-isoloid and $f \in \mathscr{H}_{lnc}(\sigma_a(T))$, then the following two assertions are equivalent to each other.*

(1) $\sigma_{uw}(f(T)) = f(\sigma_{uw}(T))$.

(2) $f(T) \in (aW)$.

By [2, Theorem 3.6], Corollary 4.2 implies that [4, Theorem 3.12 (i)] holds for all $f \in \mathscr{H}(\sigma(T))$.

THEOREM 4.3. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$. If $T \in (gW)$, then any two of the following three assertions imply the third one.*

(1) $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$.

(2) $\sigma(f(T)) \backslash \pi_0(f(T)) = f(\sigma(T) \backslash \pi_0(T))$.

(3) $f(T) \in (gW)$.

Theorem 4.3 and Theorem 3.3 deduce the following result.

COROLLARY 4.3. *Let $T \in (gW)$ and $f \in \mathscr{H}(\sigma(T))$. If T is isoloid and $f \in \mathscr{H}_{lnc}(\sigma(T))$, then the following two assertions are equivalent to each other.*

(1) $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$.

(2) $f(T) \in (gW)$.

Example 5.1 (11) implies that the condition $f \in \mathscr{H}_{lnc}(\sigma(T))$ in Corollary 4.3 is inevitable, and, for $f \notin \mathscr{H}_{lnc}(\sigma(T))$, Corollary 4.3 may fail even if T is polaroid.

Example 5.1 (11) also implies that [9, Theorem 2.10] and [11, Theorem 3.4] may fail if $f \notin \mathscr{H}_{lnc}(\sigma(T))$.

THEOREM 4.4. *Let $T \in \mathscr{L}(\mathscr{X})$ and $f \in \mathscr{H}(\sigma(T))$. If $T \in (gaW)$, then any two of the following three assertions imply the third one.*

(1) $\sigma_{ubw}(f(T)) = f(\sigma_{ubw}(T))$.

(2) $\sigma_a(f(T))\backslash\pi_0^a(f(T)) = f(\sigma_a(T)\backslash\pi_0^a(T))$.

(3) $f(T) \in (gaW)$.

Theorem 4.4 and Theorem 3.4 deduce the following result.

COROLLARY 4.4. *Let $T \in (gaW)$ and $f \in \mathscr{H}(\sigma(T))$. If T is a-isoloid and $f \in \mathscr{H}_{lnc}(\sigma_a(T))$, then the following two assertions are equivalent to each other.*

(1) $\sigma_{ubw}(f(T)) = f(\sigma_{ubw}(T))$.

(2) $f(T) \in (gaW)$.

Example 5.1 (12) implies that the condition $f \in \mathscr{H}_{lnc}(\sigma_a(T))$ is crucial, and, for $f \notin \mathscr{H}_{lnc}(\sigma_a(T))$, Corollary 4.4 may fail even if T is a-polaroid.

5. An example

EXAMPLE 5.1. Let U be the unilateral right shift operator on the Hilbert space $l_2(N)$ defined by $U(x_0,x_1,x_2,\cdots) = (0,x_0,x_1,x_2,\cdots)$, S the weighted unilateral right shift operator on the Hilbert space $l_2(N)$ defined by $U(x_0,x_1,x_2,\cdots) = (0,x_0,\frac{1}{2}x_1,\frac{1}{3}x_2,\cdots)$, $\mathscr{D} := \{z : |z| \leqslant 1\}$ and $\partial\mathscr{D} := \{z : |z| = 1\}$.

(1) If $T := U$ and $f \equiv 0 \notin \mathscr{H}_{lnc}(\sigma(T))$, then $\sigma_{bw}(f(T)) \not\supseteq f(\sigma_{bw}(T))$. In fact, T is hyponormal and of stable sign index on ρ_{bf}, $\sigma(T) = \sigma_{bw}(T) = \mathscr{D}$, $\sigma(f(T)) = \{0\}$, $\sigma_{bw}(f(T)) = \phi$ and $f(\sigma_{bw}(T)) = \{0\}$.

(2) If $T := U^*$ and $f \equiv 0 \notin \mathscr{H}_{lnc}(\sigma(T))$, then $\sigma_{ubw}(f(T)) \not\supseteq f(\sigma_{ubw}(T))$. In fact, T is co-hyponormal and of stable sign index on $\rho_{ubf}(T)$, $\sigma(T) = \sigma_{bw}(T) = \sigma_{ubw}(T) = \mathscr{D}$, $\sigma(f(T)) = \{0\}$, $\sigma_{ubw}(f(T)) \subseteq \sigma_{bw}(f(T)) = \phi$ and $f(\sigma_{ubw}(T)) = \{0\}$.

(3) If $T := S^*$ and $f \equiv 0$, then T is not polaroid and $\sigma(f(T))\backslash\pi_{00}(f(T)) = \{0\} \not\subseteq f(\sigma(T)\backslash\pi_{00}(T))$. In fact, $\sigma(T) = \sigma_w(T) = \pi_{00}(T) = \{0\}$, $\sigma(f(T)) = \{0\}$, $\pi_{00}(f(T)) = \phi$.

(4) If $T := I \oplus \frac{1}{2}U \oplus (S-I)$ on $\mathscr{H} = \mathscr{C} \oplus l_2(N) \oplus l_2(N)$ and $f(z) = z^2$. Then T is not isoloid, and $\sigma(f(T)) \backslash \pi_{00}(f(T)) = \{z : |z| \leqslant \frac{1}{4}\} \not\supseteq f(\sigma(T) \backslash \pi_{00}(T))$. In fact, $\sigma(T) = \{1\} \cup \{z : |z| \leqslant \frac{1}{2}\} \cup \{-1\}$, $\pi_{00}(T) = \{1\}$, $\pi_{00}(f(T)) = \{1\}$.

(5) If $T := S^*$ and $f \equiv 0$, then T is not a-polaroid and

$$\sigma_a(f(T)) \backslash \pi_{00}^a(f(T)) \not\subseteq f(\sigma_a(T) \backslash \pi_{00}^a(T)).$$

In fact, $\sigma(T) = \sigma_a(T) = \pi_{00}^a(T) = \{0\}$, $\sigma_a(f(T)) = \{0\}$, $\pi_{00}^a(f(T)) = \phi$.

(6) If $T := I \oplus \frac{1}{2}U \oplus (S-I)$ on $\mathscr{H} = \mathscr{C} \oplus l_2(N) \oplus l_2(N)$ and $f(z) = z^2$. Then T is not a-isoloid, and $\sigma_a(f(T)) \backslash \pi_{00}^a(f(T)) = \{z : |z| = \frac{1}{4}\} \not\supseteq f(\sigma_a(T) \backslash \pi_{00}^a(T))$. In fact, $\sigma_a(T) = \{1\} \cup \{z : |z| = \frac{1}{2}\} \cup \{-1\}$, $\pi_{00}^a(T) = \{1\}$, $\pi_{00}^a(f(T)) = \{1\}$.

(7) If $T := U$ and $f \equiv 0$. Then $\sigma(T) = \mathscr{D}$, $\sigma_p(T) = \phi$, $\sigma(f(T)) = \{0\}$ and $\pi_0(f(T)) = \{0\}$. So T is isoloid and polaroid, but $\sigma(f(T)) \backslash \pi_0(f(T)) \not\supseteq f(\sigma(T) \backslash \pi_0(T))$.

(8) If $T := I \oplus \frac{1}{2}S_1 \oplus (S_2 - I)$ on $\mathscr{H} = l_2(N) \oplus l_2(N) \oplus l_2(N)$ and $f(z) = z^2$.

Then $\sigma(T) = \{1\} \cup \{z : |z| \leqslant \frac{1}{2}\} \cup \{-1\}$, $\pi_0(T) = \{1\}$, $\sigma(f(T)) = \{1\} \cup \{z : |z| \leqslant \frac{1}{4}\}$, $\pi_0(f(T)) = \{1\}$. So T is not isoloid, and

$$\sigma(f(T)) \backslash \pi_0(f(T)) = \{z : |z| \leqslant \frac{1}{4}\} \not\supseteq f(\sigma(T) \backslash \pi_0(T)).$$

(9) If $T := U$ and and $f \equiv 0$. Then $\sigma(T) = \mathscr{D}$, $\sigma_a(T) = \partial\mathscr{D}$, $\pi_0^a(T) = \phi$, $\sigma_a(f(T)) = \pi_0^a(f(T)) = \{0\}$. So T is a-isoloid and a-polaroid, but $\sigma_a(f(T)) \backslash \pi_0^a(f(T)) \not\supseteq f(\sigma_a(T) \backslash \pi_0^a(T))$.

(10) If $T := I \oplus \frac{1}{2}S_1 \oplus (S_2 - I)$ on $\mathscr{H} = l_2(N) \oplus l_2(N) \oplus l_2(N)$ and $f(z) = z^2$. Then $\sigma_a(T) = \{1\} \cup \{z : |z| = \frac{1}{2}\} \cup \{-1\}$, $\pi_0^a(T) = \{1\}$, $\sigma_a(f(T)) = \{1\} \cup \{z : |z| = \frac{1}{4}\}$, $\pi_0^a(f(T)) = \{1\}$. So T is not a-isoloid, and $\sigma_a(f(T)) \backslash \pi_0^a(f(T)) = \{z : |z| = \frac{1}{4}\} \not\supseteq f(\sigma_a(T) \backslash \pi_0^a(T))$.

(11) If $T := U$ and $f \equiv 0$. Then $\sigma(T) = \sigma_{bw}(T) = \mathscr{D}$, and $\sigma_{bw}(f(T)) = \phi$. So T is isoloid and polaroid, $\sigma_{bw}(f(T)) \neq f(\sigma_{bw}(T)) = \{0\}$ and $f(T) = 0 \in (gW)$.

(12) If $T := U$ and $f \equiv 0$. Then $\sigma_a(T) = \sigma_{ubw}(T) = \partial\mathscr{D}$, and $\sigma_{ubw}(f(T)) = \phi$. So T is a-isoloid and a-polaroid, $\sigma_{ubw}(f(T)) \neq f(\sigma_{ubw}(T)) = \{0\}$ and $f(T) = 0 \in (gaW)$.

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