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THE STIELTJES STRING AND ITS ASSOCIATED NODAL POINTS

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Abstract. Based on the theory of Stieltjes strings first introduced by Gantmakher and Krein in [4], we define the nodal points for a Stieltjes string. We show that when the eigenvalue is maximal, there are exactly $n+1$ nodal points for the D-D problem and n nodal points for the D-N problem, where n is the total number of non-zero point masses. We also find the position of these nodal points in terms of continued fractions involving the point masses $m_1,\dots,m_j$ and lengths $l_0,\dots,l_{j-1}$ in between the positions of these masses.

1. Introduction

Consider the problem of n point masses $(m_1,\dots,m_n)$ attached to a string of length L. Let $m_0 = m_{n+1} = 0$ be two point masses attached at the two endpoints of the string. For $j = 0,\dots,n-1$ the distance between the positions of masses m_j and m_{j+1} is denoted by l_j. So $L = \sum_{j=0}^{n} l_j$.

Now when the string is subjected to a small tension, the point masses will have vertical vibrations $w_j(t)$'s. Gantmakher and Krein (1960) performed an analysis of the relation between these vibrations and the m_j's, l_j's. Along the horizontal direction

$$T_{j-1}\cos\alpha_{j-1} = T_j\cos\alpha_j = \cdots = T_0\cos\alpha_0 := T,$$

where T_j is the tension in the segment of string between m_j and m_{j+1}, and α_j is the angle between this segment and the horizontal direction. (see figure)

Next apply Newton's second law of motion to see that

$$-m_j\frac{d^2w_j}{dt^2} = T_j\sin\alpha_j - T_{j-1}\sin\alpha_{j-1} = T\tan\alpha_j - T\tan\alpha_{j-1}.$$

Assume that $\alpha_j \sim 0$ for all $j \geqslant 1$ and T is fixed with $T \equiv 1$. Then,

$$-m_j\frac{d^2w_j}{dt^2} = \frac{\Delta w_{j+1}}{l_j} - \frac{\Delta w_j}{l_{j-1}} = \frac{w_{j+1}(t)-w_j(t)}{l_j} + \frac{w_{j-1}(t)-w_j(t)}{l_{j-1}}.$$

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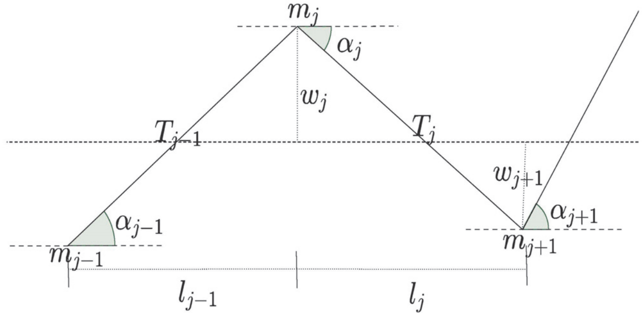

We consider two boundary conditions :

1. Dirichlet-Dirichlet condition (D-D problem): $w_0(t) = w_{n+1}(t) = 0$.
2. Dirichlet-Neumann condition (D-N problem) : $w_0(t) = 0,\ w_n(t) = w_{n+1}(t)$.

Using the discrete Fourier transform, $w_j(t) = u_j e^{i\lambda t}$, we obtain the difference equation

$$\frac{u_{j+1} - u_j}{l_j} + \frac{u_{j-1} - u_j}{l_{j-1}} + m_j \lambda^2 u_j = 0, \qquad (j = 1, \cdots, n). \tag{1}$$

Let $z = \lambda^2$, $u_j := R_{2j-2}(z)u_1$ and $R_{2j-1}(z) = \frac{1}{l_j}(R_{2j}(z) - R_{2j-2}(z))$. Assuming $u_1 \neq 0$, (1) is transformed into the following system.

$$\begin{cases} R_{2j-1}(z) = R_{2j-3}(z) - m_j z R_{2j-2}(z) \\ R_{2j}(z) = l_j R_{2j-1}(z) + R_{2j-2}(z) \end{cases}. \tag{2}$$

Using (2) and $u_0 = 0$ for both the conditions (D-D and D-N), we have

$$R_0 = 1, \qquad R_{-2} = 0, \qquad R_{-1} = \frac{1}{l_0}.$$

From (2), it is easy to see that for any $j, R_j(z)$ is a polynomial of degree $\lceil j/2 \rceil$.

For the D-D problem, $u_{n+1} = 0$, implying $R_{2n}(z) = 0$. For the D-N problem, $u_n = u_{n+1}$ implies that $R_{2n-2}(z) = R_{2n}(z)$. Hence $R_{2n-1}(z) = 0$.

Zeros of R_{2n} (denoted by $\hat{z}$) are called eigenvalues of the D-D problem. Hence $R_{2n}(z)$ can be viewed as the characteristic function. Similarly, zeros of R_{2n-1} (denoted by $\bar{z}$) are called eigenvalue for the D-N problem, and $R_{2n-1}(z)$ is the associated characteristic function.

Furthermore, observe that

$$\frac{R_{2j}(z)}{R_{2j-1}(z)} = l_j + \frac{1}{\frac{R_{2j-1}(z)}{R_{2j-2}(z)}} = l_j + \frac{1}{-m_j z + \frac{R_{2j-3}(z)}{R_{2j-2}(z)}}.$$

Then we obtain a continued fraction by induction. Namely

$$\frac{R_{2n}(z)}{R_{2n-1}(z)} = [l_n;\ -m_n z, l_{n-1}, -m_{n-1} z, \ldots, l_0]. \tag{3}$$

Through this paper, we use the following notation to denote a finite continued fraction:

$$[a_0;\ a_1, \ldots, a_k] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_k}}}\ .$$

By (3), we have

$$\frac{R_{2n}(0)}{R_{2n-1}(0)} = l_n + l_{n-1} + l_{n-2} + \cdots + l_0 = L \text{ (Total length)}.$$

Therefore,

$$\frac{R_{2n}(z)}{R_{2n-1}(z)} = \frac{R_{2n}(0)}{R_{2n-1}(0)} \left(\frac{1 + a_1 z + \cdots + a_n z^n}{1 + b_1 z + \cdots + b_n z^n} \right) = \frac{L \prod_1^n (1 - z/\hat{z}_j)}{\prod_1^n (1 - z/\bar{z}_j)}\ ,$$

where $\{\hat{z}_j\}$ are the eigenvalues for the D-D problem, and $\{\bar{z}_j\}$ are the eigenvalues for the D-N problem.

THEOREM 1.1. ([6])

(a) Given $\{l_j\}$'s and $\{m_j\}$'s, the rational function $\frac{R_{2n}(z)}{R_{2n-1}(z)}$ is a continued fraction made up of constants and linear polynomials, as given in (3).

(b) Let $\{\bar{z}_j\}$ be the zeros of R_{2n-1} and $\{\hat{z}_j\}$ the zeros of R_{2n}. If L, $\{\hat{z}_j\}$'s and $\{\bar{z}_j\}$'s are all given, then the $\{l_j\}$'s and $\{m_j\}$'s can be recovered from (3).

Note that part (b) is in fact an inverse eigenvalue problem to solve for $2n+1$ quantities in terms of $2n+1$ known eigenvalues.

The above is the basic theory of Stieltjes strings (for the D-D and D-N problems), first introduced by Gantmakher and Krein [4] and also studied by Kac and Krein [6]. Recently Pivovarchik et al [1, 2] studied the corresponding problem for more general trees. In particular, they showed that from the $d+1$ eigenvalues, one can recover all the point masses $m_{k,j}$ and lengths $l_{k,j}$, by a method similar to Lagrange interpolation. Later they extended the result to a general tree of Stieltjes strings [7, 10]. Other related issues can be found in [9, 8].

In this paper, we investigate the nodal problem for Stieltjes strings on a finite interval. In (1), if for some j, u_j and u_{j+1} have opposite signs, then the position x when the interpolation of the string between (l_j, u_j) and (k_{j+1}, u_{j+1}) intersects the l-axis, so that the interpolating line segment passes through $(x, 0)$. We call this position x a *nodal point* of the solution defined by $\vec{u} = (u_1, \ldots, u_n)$. We shall show that there are exactly $n+1$ nodal points for the D-D problem associated with the maximal D-D eigenvalue $\hat{z}$, and exactly n nodal points for the D-N problem associated with the maximal D-N eigenvalue $\bar{z}$. Furthermore we shall give the position of the j^{th} nodal point x_j using a continued fraction. Denote by $\hat{z}_n$ the largest D-D eigenvalue and by $\bar{z}_n$ the largest D-N eigenvalue.

Our main theorem is

THEOREM 1.2. *There is a maximal number of $n+1$ (respectively n) nodal points for the D-D problem (respectively D-N problem). Furthermore,*

(a) The associated nodal points $\{\hat{x}_j\}$ for the D-D problem can be expressed in terms of $\hat{z}$, $\{l_j\}$'s and $\{m_j\}$'s as follows: $\hat{x}_0 = 0$, $\hat{x}_n = L$, and for $j = 1, \ldots, n-1$,

$$\hat{x}_j = [\sum_{i=0}^{j-1} l_i;\ m_j\hat{z}_n, -l_{j-1}, m_{j-1}\hat{z}_n, \ldots, m_1\hat{z}_n, -l_0]. \tag{4}$$

(b) The associated nodal points $\{\bar{x}_j\}$ for the D-N problem can be expressed in terms of $\bar{z}$, $\{l_j\}$'s and $\{m_j\}$'s as follows: $\bar{x}_0 = 0$, and for $j = 1, \ldots, n-1$,

$$\bar{x}_j = [\sum_{i=0}^{j-1} l_i;\ m_j\bar{z}_n, -l_{j-1}, m_{j-1}\bar{z}_n, \ldots, m_1\bar{z}_n, -l_0]\,. \tag{5}$$

Furthermore the D-D nodal points and D-N nodal points are interlacing. That is,

$$0 = \hat{x}_0 = \bar{x}_0 < \bar{x}_1 < \hat{x}_1 < \cdots < \hat{x}_{n-1} < \bar{x}_{n-1} < \hat{x}_n = L. \tag{6}$$

Note the similarity in the above formulas for nodal points $\hat{x}_j$ and $\bar{x}_j$, both being expressed in terms of $l_0, \ldots, l_{j-1}$ and $m_0, \ldots, m_j$ as continued fractions.

Theorem 1.2 will be proved in Section 2. In Section 3, we shall prove Lemma 2.1 which is instrumental in our proof of Theorem 1.2. The proof requires some intricate analysis. We need to use an interlacing theorem for matrix eigenvalues [5] for the proof.

We believe that our results are useful in applications.

2. Nodal problem

We now consider the nodal problem associated with (1). For the D-D problem, we let

$$\widetilde{A}_j(z) := \frac{-1}{l_{j-1}} + \frac{-1}{l_j} + m_j z\,; \qquad A_j(z) := l_j\widetilde{A}_j(z).$$

Then (1) becomes the following equation

$$\widehat{M}_n(z)\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} := \begin{pmatrix} \widetilde{A}_1(z) & \frac{1}{l_1} & & & \\ \frac{1}{l_1} & \widetilde{A}_2(z) & \frac{1}{l_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{l_{n-2}} & \widetilde{A}_{n-1}(z) & \frac{1}{l_{n-1}} \\ & & & \frac{1}{l_{n-1}} & \widetilde{A}_n(z) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \vec{0}. \tag{1}$$

We denote the above coefficient matrix by $\widehat{M}_n(z)$. Note that $\det\widehat{M}_n(z)$ is a characteristic function for the D-D problem. Similarly, the characteristic matrix for the D-N problem is given by

$$\overline{M}_n(z) := \begin{pmatrix} \widetilde{A}_1(z) & \frac{1}{l_1} & & & \\ \frac{1}{l_1} & \widetilde{A}_2(z) & \frac{1}{l_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{l_{n-2}} & \widetilde{A}_{n-1}(z) & \frac{1}{l_{n-1}} \\ & & & \frac{1}{l_{n-1}} & \frac{-1}{l_{n-1}} + m_n z \end{pmatrix} . \tag{2}$$

If $\vec{u} \neq \vec{0}$, then both matrices are singular and the characteristic equations $\det\widehat{M}_n(z) = 0$ and $\det\overline{M}_n(z) = 0$ have solutions. As we know these zeros are also the solutions of $R_{2n}(z) = 0$ and $R_{2n-1}(z) = 0$, respectively.

Obviously both $\det\widehat{M}_n(z)$ and $\det\overline{M}_n(z)$ are polynomials of degree n in z. By Theorem 3.1 below, both $\hat{z}_j$ and $\bar{z}_j$ are real and distinct. Let $\hat{z}_1 < \hat{z}_2 < \cdots < \hat{z}_{n-1} < \hat{z}_n$ be the zeros of $\det\widehat{M}_n(z) = 0$; and $\bar{z}_1 < \bar{z}_2 < \cdots < \bar{z}_{n-1} < \bar{z}_n$ be the zeros of $\det\overline{M}_n(z) = 0$. Then $\hat{z} = \hat{z}_n$ (respectively $\bar{z} = \bar{z}_n$) is the maximal eigenvalue of the D-D problem (respectively D-N problem).

Proof of theorem 1.2.

For simplicity, we let $\widehat{M}_n = \widehat{M}_n(\hat{z})$, $\widetilde{A}_j = \widetilde{A}_j(\hat{z})$, $A_j = A_j(\hat{z})$. From (1), we have

$$\begin{aligned} u_2 &= -l_1\widetilde{A}_1 u_1 = -A_1 u_1 \, . \\ u_3 &= \frac{-l_2}{l_1} u_1 - l_2\widetilde{A}_2 u_2 = -\Big(\frac{l_2}{l_1(-A_1)} + A_2\Big) u_2 \, . \end{aligned} \tag{3}$$

By induction, for all $j = 1, \ldots, n$,

$$\begin{aligned} u_j &= -\left(\frac{l_{j-1}}{l_{j-2}} \times \cfrac{1}{\frac{-l_{j-2}}{l_{j-3}} \times \cfrac{1}{\ddots \times \cfrac{1}{\frac{-l_2}{l_1} \times \frac{1}{-A_1} - A_2} - A_3} - A_{j-2}} + A_{j-1} \right) u_{j-1} \\ &:= -\mathscr{A}_{j-1}(\hat{z}) u_{j-1} \, . \end{aligned} \tag{4}$$

Now we need a lemma, the proof of which is deferred to the next section.

LEMMA 2.1. *For all* $j = 1, \ldots, n-1$,

(a) $\mathscr{A}_j(\hat{z}) > 0$.

(b) $\dfrac{1 + \mathscr{A}_j(\hat{z})}{l_j} = [m_j\hat{z}; -l_{j-1}, m_{j-1}\hat{z}, \ldots, m_1\hat{z}, -l_0]$.

Since $u_j = -\mathscr{A}_{j-1}(\hat{z})u_{j-1}$, we have $u_{j-1}u_j < 0$ for $j = 2,\ldots,n$ by Lemma 2.1. Thus when $\hat{z} = \hat{z}_n$, we obtain the maximum number of $n+1$ nodal points (including $\hat{x}_0 = 0$ and $\hat{x}_n = L$) for the D-D problem. Also as consecutive u_j's have the opposite signs. We know that $u_0 = 0$ and $\hat{x}_0 = 0$. By (4) and properties of similar triangles, $\dfrac{\hat{x}_1 - l_0}{u_1} = \dfrac{l_1}{(1+\mathscr{A}_1(\hat{z}))\,u_1}$, so that $\hat{x}_1 = l_0 + \dfrac{l_1}{1+\mathscr{A}_1(\hat{z})}$. Inductively, for $j = 1,\ldots,n-1$, we have

$$\hat{x}_j = \sum_{i=0}^{j-1} l_i + \frac{l_j}{1+\mathscr{A}_j(\hat{z})}\,.$$

The Dirichlet boundary condition $u_{n+1} = 0$ implies that $\mathscr{A}_n := 0$. Therefore Theorem 1.2(a) is valid.

Theorem 1.2(b) for the D-N problem can be established similarly. □

LEMMA 2.2. *The nodal points* $\{\hat{x}_j\}$ *and* $\{\bar{x}_j\}$ *are interlacing as in (6).*

Proof. For $j = 0,\ldots,n$, let $L_j = \sum_{i=0}^{j} l_i$. Then $L_n = L$. It is clear that

$$L_j > \hat{x}_j\,,\ \bar{x}_j > L_{j-1}.$$

Next we use the known quantities (4) and (5) to observe that when $j = 1$

$$\begin{cases} \hat{x}_1 = l_0 + \frac{1}{m_1\hat{z} - \frac{1}{l_0}} \\ \bar{x}_1 = l_0 + \frac{1}{m_1\bar{z} - \frac{1}{l_0}} \end{cases}.$$

This implies that

$$m_1\hat{z} - \frac{1}{l_0} = \frac{1}{\hat{x}_1 - l_0} > 0, \quad m_1\bar{z} - \frac{1}{l_0} = \frac{1}{\bar{x}_1 - l_0} > 0. \tag{5}$$

By Theorem 3.1 below, $\hat{z} > \bar{z}$. Thus

$$0 < m_1\bar{z} - \frac{1}{l_0} < m_1\hat{z} - \frac{1}{l_0},$$

so that $\hat{x}_1 < \bar{x}_1$. Then by (4), (5) and (5),

$$\begin{cases} \hat{x}_2 = [l_0+l_1;\ m_2\hat{z}, -l_1, m_1\hat{z}, -l_0] = l_0 + l_1 + \frac{1}{m_2\hat{z} + \frac{1}{\hat{x}_1 - l_0 - l_1}} = L_1 + \frac{1}{m_2\hat{z} - \frac{1}{L_1 - \hat{x}_1}} \\ \bar{x}_2 = [l_0+l_1,\ m_2\bar{z}, -l_1, m_1\bar{z}, -l_0] = l_0 + l_1 + \frac{1}{m_2\bar{z} + \frac{1}{\bar{x}_1 - l_0 - l_1}} = L_1 + \frac{1}{m_2\bar{z} - \frac{1}{L_1 - \bar{x}_1}} \end{cases}.$$

Hence

$$m_2\hat{z} - \frac{1}{L_1 - \hat{x}_1} = \frac{1}{\hat{x}_2 - L_1} > 0, \quad m_2\bar{z} - \frac{1}{L_1 - \bar{x}_1} = \frac{1}{\bar{x}_2 - L_1} > 0.$$

By Theorem 3.1 again, $\hat{z} > \bar{z}$, and $\bar{x}_1 > \hat{x}_1$. Thus we have $\hat{x}_2 < \bar{x}_2$. Inductively, for $j = 3, \dots, n-1$,

$$\begin{cases} \hat{x}_j = [L_{j-1};\ m_j\hat{z}, -L_{j-1} + \hat{x}_{j-1}] \\ \bar{x}_j = [L_{j-1};\ m_j\bar{z}, -L_{j-1} + \bar{x}_{j-1}] \end{cases} .$$

So

$$m_j\hat{z} - \frac{1}{L_{j-1} - \hat{x}_{j-1}} = \frac{1}{\hat{x}_j - L_{j-1}} > 0, \quad m_j\bar{z} - \frac{1}{L_{j-1} - \bar{x}_{j-1}} = \frac{1}{\bar{x}_j - L_{j-1}} > 0.$$

Using Theorem 3.1 again, we have $\hat{x}_j < \bar{x}_j$. Since $\hat{x}_0 = \bar{x}_0 = 0$, we conclude that (6) is valid. □

3. Proof of lemma 2.1

THEOREM 3.1. ([4]) *Given $\{l_j\}$'s and $\{m_j\}$'s in (1), then we can solve for $\{\hat{z}_j\}$ and $\{\bar{z}_j\}$, where $\{\hat{z}_j\}$ are the zeros of $R_{2n}(z) = 0$ and $\{\bar{z}_j\}$ are the zeros of $R_{2n-1}(z) = 0$. Moreover*

$$0 < \bar{z}_1 < \hat{z}_1 < \bar{z}_2 < \hat{z}_2 < \cdots < \bar{z}_n < \hat{z}_n.$$

We also need an Interlacing Theorem given in the classical book of Horn and Johnson [5, p.185].

THEOREM 3.2. *If $B = B^T$, $B' = \begin{pmatrix} B & \vec{y} \\ \vec{y}^T & b \end{pmatrix}$, $\vec{y} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then*

$$\lambda_1' \leqslant \lambda_1 \leqslant \lambda_2' \leqslant \lambda_2 \leqslant \lambda_n' \leqslant \lambda_n \leqslant \lambda_{n+1}',$$

where $\{\lambda_j\}$ are the eigenvalues of B and $\{\lambda_j'\}$ are the eigenvalues of B'.

Proof of lemma 2.1(a).

From (1), $\det\widehat{M}_j(z)$ is a polynomial of order j. Its zeros, $z_1^{(j)}, \dots, z_j^{(j)}$, are exactly the eigenvalues of the D-D problem. We first claim that for $j = 1, \dots, n$, the zeros of $\det\widehat{M}_j(z)$ interlace with those of $\det\widehat{M}_{j-1}(z)$. That is,

$$z_1^{(j)} < z_1^{(j-1)} < z_2^{(j)} < \cdots < z_{j-1}^{(j)} < z_{j-1}^{(j-1)} < z_j^{(j)}, \tag{1}$$

and $\det\widehat{M}_j(z) > 0$ for all $z > z_j^{(j)}$.

Observe that for $j = 3, \dots, n$,

$$\det\widehat{M}_j(z) = \widetilde{A}_j(z)\det\widehat{M}_{j-1}(z) - (\frac{1}{l_{j-1}})^2 \det\widehat{M}_{j-2}(z)\ . \tag{2}$$

Solving $\det\widehat{M}_1(z) = 0$, we get $z_1^{(1)} = \dfrac{1}{m_1}(\dfrac{1}{l_0} + \dfrac{1}{l_1})$, and we observe that $\det\widehat{M}_1(z) > 0$ for all $z > z_1^{(1)}$. Also $z_1^{(2)} < z_2^{(2)}$ by Theorem 3.1. Then Theorem 3.2 implies that $z_1^{(2)} \leqq z_1^{(1)} \leqq z_2^{(2)}$. If $z_i^{(2)} = z_1^{(1)}$ for some i, then $(\frac{1}{l_1})^2 = 0$, which is impossible. So

$$z_1^{(2)} < z_1^{(1)} < z_2^{(2)}\ . \tag{3}$$

Furthermore,

$$\det \widehat{M}_2(z) = \widetilde{A}_2(z) \det \widehat{M}_1(z) - (\frac{1}{l_1})^2.$$

Hence $\det \widehat{M}_2(z_1^{(1)}) = -(\frac{1}{l_1})^2 < 0$. This means $\det \widehat{M}_2(z) > 0$ for all $z > z_2^{(2)}$.

Next, by Theorem 3.1, $z_1^{(3)} < z_2^{(3)} < z_3^{(3)}$. Using Theorem 3.2 again, we have

$$z_1^{(3)} \leqq z_1^{(2)} \leqq z_2^{(3)} \leqq z_2^{(2)} \leqq z_3^{(3)} .$$

If $z_i^{(3)} = z_j^{(2)}$, for some i, j, then $-(\frac{1}{l_1})^2 \det \widehat{M}_1(z) = 0$. Combining with (4), we have

$$z_1^{(3)} < z_1^{(2)} < z_2^{(3)} < z_2^{(2)} < z_3^{(3)}.$$

Then (1) follows by mathematical induction on j. Moreover, by (2),

$$\det \widehat{M}_j(z_{j-1}^{(j-1)}) = -\frac{1}{l_{j-1}^2} \det \widehat{M}_{j-2}(z_{j-1}^{(j-1)}) < 0.$$

This means that $\det \widehat{M}_j(z) > 0$ for all $z > z_j^{(j)}$.

We know that $\hat{z} = z_n^{(n)}$, the maximal zero of $\det \widehat{M}_n(z)$. Thus $\hat{z} > z_i^{(j)}$, for all $n \geqslant j \geqslant i$ (where at least one of the inequalities is a strict one). By above argument, $\det \widehat{M}_j(\hat{z}) > 0$ for all $j = 1, \ldots, n-1$ but $\det \widehat{M}_n(\hat{z}) = 0$. On the other hand, $\det \widehat{M}_1(\hat{z}) = \widetilde{A}_1(\hat{z}) > 0$. Therefore by (2), $\widetilde{A}_j(\hat{z}) > 0$ for $j = 1, \ldots, n$.

Now, we claim that $\mathscr{A}_j(\hat{z}) > 0 \quad \forall j = 1, \ldots, n-1$. First $\mathscr{A}_1(\hat{z}) = A_1(\hat{z}) > 0$. Then by (3),

$$\begin{aligned}
\mathscr{A}_2(\hat{z}) &= \frac{l_2}{l_1(-l_1\widetilde{A}_1(\hat{z}))} + l_2\widetilde{A}_2(\hat{z}) = \frac{l_2}{\widetilde{A}_1(\hat{z})}[-(\frac{1}{l_1})^2 + \widetilde{A}_2(\hat{z})\widetilde{A}_1(\hat{z})] \\
&= \frac{l_2}{\det \widehat{M}_1(\hat{z})} \det \widehat{M}_2(\hat{z}) > 0.
\end{aligned}$$

Inductively, for $j = 3, \ldots, n-1$, by (4),

$$\begin{aligned}
\mathscr{A}_j(\hat{z}) &= \frac{l_j}{l_{j-1}(-\mathscr{A}_{j-1}(\hat{z}))} + A_j(\hat{z}) = \frac{l_j \det \widehat{M}_{j-2}(\hat{z})}{-l_{j-1}^2 \det \widehat{M}_{j-1}(\hat{z})} + A_j(\hat{z}) \\
&= \frac{l_j}{\det \widehat{M}_{j-1}(\hat{z})} \left(\frac{-\det \widehat{M}_{j-2}(\hat{z})}{l_{j-1}^2} + \widetilde{A}_j(\hat{z}) \det \widehat{M}_{j-1}(\hat{z}) \right) \\
&= \frac{l_j}{\det \widehat{M}_{j-1}(\hat{z})} \det \widehat{M}_j(\hat{z}) > 0.
\end{aligned}$$

Thus $\mathscr{A}_j(\hat{z}) > 0$ for all $j = 1, \ldots, n-1$, and

$$\mathscr{A}_n(\hat{z}) = \frac{l_n}{\det \widehat{M}_{n-1}(\hat{z})} \det \widehat{M}_n(\hat{z}) = 0. \quad \square$$

Before we prove Lemma 2.1(b), we let for $j = 1, \ldots, n-1$,

$$F_j(\hat{z}) := [0;\, m_j\hat{z}, -l_{j-1}, m_{j-1}\hat{z}, l_{j-1}, \ldots, m_1\hat{z}, -l_0]\ .$$

Proof of lemma 2.1(b).

We need to show that for all $j = 1, \ldots, n-1$, $\dfrac{l_j}{1+\mathscr{A}_j(\hat{z})} = F_j(\hat{z})$. For simplicity, we let $\mathscr{A}_j = \mathscr{A}_j(\hat{z})$, $\quad F_j = F_j(\hat{z})$, and

$$A_j = l_j\widetilde{A}_j(\hat{z}) = -l_j(\frac{1}{l_{j-1}} - m_j\hat{z}) - 1.$$

Obviously $\mathscr{A}_1 = A_1$, and $\mathscr{A}_2 = \dfrac{l_2}{l_1(-\mathscr{A}_1)} + A_2$. So

$$\mathscr{A}_3 = \frac{l_3}{l_2} \times \frac{1}{\frac{-l_2}{l_1(-\mathscr{A}_1)} - A_2} + A_3 = \frac{l_3}{l_2(-\mathscr{A}_2)} + A_3.$$

It is easy to see that

$$\mathscr{A}_j = \frac{l_j}{l_{j-1}(-\mathscr{A}_{j-1})} + A_j,\ \forall j = 2, \ldots, n-1\ .$$

Next,

$$\frac{l_1}{1+\mathscr{A}_1} = \frac{l_1}{1+A_1} = \frac{l_1}{-l_1(\frac{1}{l_0} - m_1\hat{z})} = \frac{1}{m_1\hat{z} - \frac{1}{l_0}} = F_1\ .$$

Therefore we have

$$\frac{1}{l_1}(\frac{1}{\mathscr{A}_1} + 1) = \frac{1}{l_1 - F_1}.$$

By induction on j,

$$\frac{l_j}{1+\mathscr{A}_j} = \frac{l_j}{1 + \frac{l_j}{l_{j-1}(-\mathscr{A}_{j-1})} + A_j} = \frac{l_j}{l_j(\frac{1}{l_{j-1}(-\mathscr{A}_{j-1})} + \frac{-1}{l_{j-1}} + m_j\hat{z})} = \frac{1}{m_j\hat{z} - \frac{1}{l_{j-1}}(\frac{1}{\mathscr{A}_{j-1}} + 1)}$$

$$= \frac{1}{m_j\hat{z} + \frac{1}{-l_{j-1}+F_{j-1}}} = F_j. \quad \square$$

4. Further discussion

So we have solved the direct problem of finding the nodal points of $\hat{x}_i$ (resp. $\bar{x}_i$) from the point masses m_j's and lengths l_j's and eigenvalues $\hat{z}_n$ (resp. $\bar{z}_n$). Can one solve the inverse problem of finding m_j's and l_j's (in total $2n+1$ quantities), using the knowledge of another $(2n+1)$ quantities: $\hat{z}_n$, $\bar{z}_n$, $\hat{x}_i$, $\bar{x}_i$ $(i = 1, \ldots, n-1)$ and total length L? We call this an inverse nodal problem for the Stieltjes string. This problem is still open.

We would also like to remark that our problem (1) can be reformulated as a problem involving Jacobi matrices:

$$J\vec{u} = \lambda^2 X\vec{u},$$

where J is the $n \times n$ tridiagonal matrix of the form

$$J = \begin{pmatrix} \frac{1}{l_1} & \frac{-1}{l_1} & & & \\ \frac{-1}{l_1} & \frac{1}{l_1} + \frac{1}{l_2} & \frac{-1}{l_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{-1}{l_{n-2}} & \frac{-1}{l_{n-2}} + \frac{-1}{l_{n-1}} & \frac{-1}{l_{n-1}} \\ & & & \frac{1}{l_{n-1}} & b_n \end{pmatrix},$$

where $b_n = \dfrac{1}{l_{n-1}} + \dfrac{1}{l_n}$ for D-D problems; $b_n = \frac{1}{l_n}$ for D-N problems. And $X = \text{diag}[m_1, m_2, \ldots, m_n]$, a diagonal matrix. Let $\vec{v} = X^{1/2}\vec{u}$. Then (4) is transformed to

$$H\vec{v} = X^{-1/2} J X^{-1/2} \vec{v} = \lambda^2 \vec{v},$$

where H is still a Jacobi matrix. The topic of Jacobi matrices has been extensively studied. See, for example, [3, 11] and references therein. It would be interesting to apply the theory to this nodal problem. In fact, our matrices $\hat{M}_n$ and $\bar{M}_n$ both satisfy $M_n = zX - J$. Moreover, it is shown in [11, p.78] that the number of nodal points of $\vec{u}(\bar{z_k})$ in $(0, L)$ is exactly k, and so is that of $\vec{u}(\hat{z_{k+1}})$. However, the definition of nodes in [11] is different from our definition of nodal points. Our results should be new in literature.

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